International Journal of Non-Linear Mechanics Volume 69, March 2015,
Pages 66–70

Re-visiting the head-on collision problem between two solitary waves in shallow water

A.Erinc Ozden and Hilmi Demiray
Isik University, Department of Mathematics
34980 Sile - Istanbul - Turkey
e-mail: eozden@isikun.edu.tr; demiray@isikun.edu.tr
Fax: 0090 216-7121474

Abstract

Upon discovering the wrongness of the statement "*although this term does not cause any secularity for this order it will cause secularity at higher order expansion, therefore, that term must vanish*" by Su and Mirie [4], in the present work, we studied the head- on collision of two solitary waves propagating in shallow water by introducing a set of stretched coordinates in which the trajectory functions are of order of ϵ^2, where ϵ is the smallness parameter measuring nonlinearity. Expanding the field variables and trajectory functions into power series in ϵ, we obtained a set of differential equations governing various terms in the perturbation expansion. By solving them under non-secularity condition we obtained the evolution equations and also the expressions for phase functions. By seeking a progressive wave solution to these evolution equations we have determined the speed correction terms and the phase shifts. As opposed to the result of Su and Mirie [4] and similar works, our calculations show that the phase shifts depend on both amplitudes of the colliding waves.

Keywords: Head-on collision, solitary waves in water, phase shifts
AMSC: 35Q51, 35Q53

1 Introduction

It is well-known that long-time asymptotic behavior of two dimensional unidirectional shallow water waves in the case of weak nonlinearity is described by the Korteweg-de Vries (KdV) equation [1]. Since the inverse scattering transform (IST) for exactly solving the KdV equation was found by Gardner, Green, Kruskal and Miura [2], the interesting features of the collision between solitary waves has been revealed: When two solitary waves approach closely, they interact, exchange their energies and positions with one another, and then separate off, regaining their original wave forms. Throughout the whole process of the collision, the solitary waves are remarkably stable entities preserving their identities through interaction. The unique effect due to the collision is their phase shifts [3]. It is believed that this striking colliding property of solitary waves can only be preserved in an integrable system.

According to IST, all KdV solitary waves travel in the same direction, under the boundary conditions vanishing at infinity [2, 3], so for overtaking collision between solitary waves, one can use the IST to obtain the overtaking colliding effect of solitary waves. However, for the head-on collision between solitary waves one must employ some kind of asymptotic expansion to solve the original field equations. In this regard a fundamental approach to study head-on collision problems had been performed by Su and Mirie [4], in which the Poincaré-Lighthill-Kuo (PLK) method had been employed to study the asymptotic analysis of such collision problems. Several researchers, including myself (HD), utilizing their method studied the head-on collision of solitary waves in various media [5-12]. The most attractive point of the method of Su and Mirie [4] is the statement that "*although this term does not cause any secularity at this order but it will cause to secularity at higher order expansion, therefore, that term must vanish*". But our calculations for higher order expansion show that the term mentioned in their work does not cause any secularity in the solution; it rather occurs in the next order equation. This means the order of trajectory functions should be ϵ^2, not ϵ.

In the present work, based on the above argument, we studied the head-on collision of two solitary waves propagating in the shallow water by introducing a set of stretched coordinates in which the trajectory functions are of order of ϵ^2. Taking the non-dimensional form of the field equations used by Su and Mirie [4] and expanding the field variables and trajectory functions into power series of ϵ we obtained a set of differential equations governing the various terms in the perturbation expansion. By solving these equations under the non-secularity conditions we obtained the evolution equations which give the solitary wave solutions for both right and left going waves. Moreover, by deriving non-secular solutions for ϵ^3 order equations we obtained some restrictions which makes it possible to determine the trajectory functions of order ϵ^2. Using the conventional definition of phase shifts we determined the expressions of phase shifts of right and left going waves. As opposed to the results of previous studies our calculation shows that the phase shifts depend on both amplitudes of colliding waves.

2 Basic Equations

We consider a plane irrotational flow of an incompressible fluid. Let $\psi^\star(x^\star, y^\star, t^\star)$ be the velocity potential related to the velocity components $u^\star$ and $v^\star$ in the $x^\star$ and $y^\star$ directions, respectively, by

$$u^\star = \frac{\partial \psi^\star}{\partial x^\star}, \qquad v^\star = \frac{\partial \psi^\star}{\partial y^\star}. \tag{1}$$

The incompressibility of the fluid requires that $\psi^\star$ must satisfy the Laplace equation

$$\frac{\partial^2 \psi^\star}{\partial x^{\star 2}} + \frac{\partial^2 \psi^\star}{\partial y^{\star 2}} = 0. \tag{2}$$

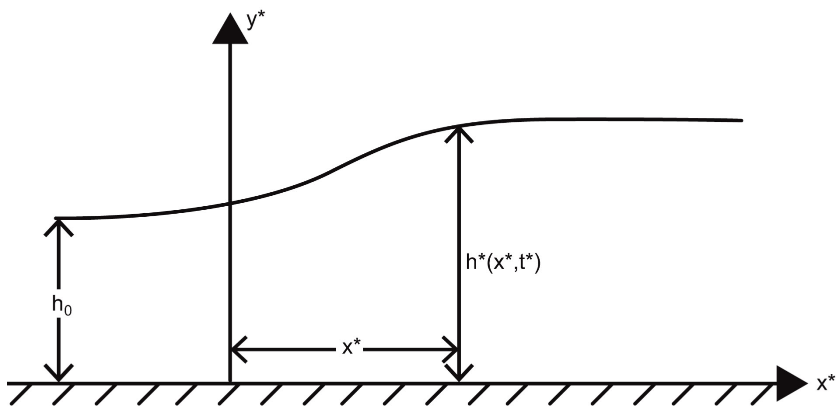

Figure 1: The geometry of the shallow water wave problem.

The boundary conditions to be satisfied are:

$$\begin{aligned}
&\frac{\partial \psi^\star}{\partial y^\star} = 0 \quad \text{at } y^\star = 0,\\
&\frac{\partial h^\star}{\partial t^\star} + \frac{\partial \psi^\star}{\partial x^\star}\frac{\partial h^\star}{\partial x^\star} - \frac{\partial \psi^\star}{\partial y^\star} = 0 \quad \text{on } y^\star = h^\star,\\
&\frac{\partial \psi^\star}{\partial t^\star} + \frac{1}{2}\left[\left(\frac{\partial \psi^\star}{\partial x^\star}\right)^2 + \left(\frac{\partial \psi^\star}{\partial y^\star}\right)^2\right] + g(h^\star - h_0) = 0 \quad \text{on } y^\star = h^\star,
\end{aligned} \tag{3}$$

where g is gravity acceleration of the earth. At this stage it is convenient to introduce the following non-dimensional quantities

$$\begin{aligned}
&x^\star = h_0 x, \quad y^\star = h_0 y, \quad t^\star = \left(\frac{h_0}{g}\right)^{1/2} t,\\
&h^\star = h_0(1+\zeta), \quad \psi^\star = \left(g h_0^3\right)^{1/2} \psi
\end{aligned} \tag{4}$$

where h_0 is the still water level from the horizontal bottom. Introducing (4) into (2)-(3), the following non-dimensional equations are obtained

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0, \tag{5}$$

$$\frac{\partial\psi}{\partial y} = 0 \quad \text{at } y = 0,$$

$$\frac{\partial\zeta}{\partial t} + \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial x} - \frac{\partial\psi}{\partial y} = 0 \quad \text{at } y = 1 + \zeta,$$

$$\frac{\partial\psi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] + \zeta = 0 \quad \text{at } y = 1 + \zeta. \tag{6}$$

Here we seek a power series solution for ψ of the form

$$\psi = \sum_{n=0}^{\infty} a_n(x,t) y^{2n}. \tag{7}$$

Introducing (7) into the Laplace equation (5) we obtain

$$a_1 = -\frac{1}{2!}\frac{\partial^2 a_0}{\partial x^2}, \quad a_2 = \frac{1}{4!}\frac{\partial^4 a_0}{\partial x^4}, \ldots. \tag{8}$$

Denoting the value of $\psi(x, y, t)$ at $y = 0$ by $\Psi(x, t)$, the solution (7) can be written as follows

$$\psi = \sum_{n=0}^{\infty} (-1)^n \frac{\partial^{2n}\Psi}{\partial x^{2n}} y^{2n}. \tag{9}$$

The solution (9) also satisfies the boundary condition at $y = 0$. Using the other boundary conditions we obtain

$$\frac{\partial\zeta}{\partial t} + \frac{\partial}{\partial x}\left\{(1+\zeta)w + \sum_{n=1}^{\infty}(-1)^n \frac{(1+\zeta)^{2n+1}}{(2n+1)!}\frac{\partial^{2n} w}{\partial x^{2n}}\right\} = 0, \tag{10}$$

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}\left\{\zeta + \frac{w^2}{2} + \sum_{n=1}^{\infty}(-1)^n \frac{(1+\zeta)^{2n}}{(2n)!}\right.$$

$$\left.\left[\frac{\partial^{2n} w}{\partial t \partial x^{2n-1}} + \frac{1}{2}\sum_{m=0}^{2n}(-1)^m \binom{2n}{m}\frac{\partial^m w}{\partial x^m}\frac{\partial^{2n-m} w}{\partial x^{2n-m}}\right]\right\} = 0, \tag{11}$$

where $w = \dfrac{\partial\psi}{\partial x}$ and $\dbinom{2n}{m}$ are the binomial coefficients.

Following Su and Mirie [4], we introduce the following stretched coordinates

$$\epsilon^{\frac{1}{2}}k(x - C_R t) = \xi - \epsilon k\theta(\xi, \eta), \tag{12}$$

$$\epsilon^{\frac{1}{2}}l(x + C_L t) = \eta - \epsilon l\phi(\xi, \eta), \tag{13}$$

where ϵ is the smallness parameter representing the order of nonlinearity, k and l are the dimensionless wave numbers of order unity for the right and left going waves, respectively, and C_R and C_L, are the speeds of right and left going waves, $\theta(\xi, \eta)$ and $\phi(\xi, \eta)$ are two unknown functions to be determined from the solution. Then, the following differential operators can be introduced:

$$\frac{\partial}{\partial t} + C_R\frac{\partial}{\partial x} = \frac{\epsilon^{\frac{1}{2}}}{D}(C_R + C_L)\left[l\frac{\partial}{\partial \eta} + \epsilon kl\left(\frac{\partial \theta}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{\partial \theta}{\partial \xi}\frac{\partial}{\partial \eta}\right)\right], \tag{14}$$

$$\frac{\partial}{\partial t} - C_L\frac{\partial}{\partial x} = -\frac{\epsilon^{\frac{1}{2}}}{D}(C_R + C_L)\left[k\frac{\partial}{\partial \xi} + \epsilon kl\left(\frac{\partial \phi}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{\partial \phi}{\partial \eta}\frac{\partial}{\partial \xi}\right)\right], \tag{15}$$

where

$$D = \left(1 - \epsilon k\frac{\partial \theta}{\partial \xi}\right)\left(1 - \epsilon l\frac{\partial \phi}{\partial \eta}\right) - \epsilon^2 kl\frac{\partial \theta}{\partial \eta}\frac{\partial \phi}{\partial \xi}. \tag{16}$$

Introducing (14) and (15) into (10) and (11) we obtain

$$\left[\frac{\partial}{\partial t} \pm C_{R,L}\frac{\partial}{\partial x}\right][w \pm \zeta] + \frac{\partial}{\partial x}F_{\pm} = 0, \tag{17}$$

where $F_{\pm}$ is defined by

$$\begin{aligned} F_{\pm} &= \pm(1 - C_{R,L})(w \pm \zeta) + \frac{w^2}{2} \pm \zeta w \\ &+ \sum_{n=1}^{\infty}(-1)^n\frac{(1+\zeta)^{2n}}{(2n)!}\left[\frac{\partial^{2n}w}{\partial t\partial x^{2n-1}} \pm \frac{(1+\zeta)}{2n+1}\frac{\partial^{2n}w}{\partial x^{2n}}\right. \\ &\left.+\frac{1}{2}\sum_{m=0}^{2n}(-1)^m\binom{2n}{m}\frac{\partial^m w}{\partial x^m}\frac{\partial^{2n-m}w}{\partial x^{2n-m}}\right]. \end{aligned} \tag{18}$$

For our future purposes it is convenient to introduce the following change of dependent variables

$$w + \zeta = 2\epsilon\alpha, \qquad w - \zeta = -2\epsilon\beta. \tag{19}$$

Then, the equation (17) takes the following form

$$2\epsilon(C_R + C_L)\left[l\frac{\partial\alpha}{\partial\eta} + \epsilon kl\left(\frac{\partial\theta}{\partial\eta}\frac{\partial\alpha}{\partial\xi} - \frac{\partial\theta}{\partial\xi}\frac{\partial\alpha}{\partial\eta}\right)\right]$$
$$+\left\{k\frac{\partial}{\partial\xi} + l\frac{\partial}{\partial\eta} + \epsilon kl\left[\frac{\partial}{\partial\eta}(\theta-\phi)\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\xi}(\theta-\phi)\frac{\partial}{\partial\eta}\right]\right\}F_+ = 0. \quad (20)$$

A similar expression is valid for β provided that (α,β), (ξ,η), (k,l), (θ,ϕ) and (F_+,F_-) are replaced with each other. We shall assume that the field quantities may be expanded into asymptotic series in ϵ as follows

$$\begin{aligned}
&\alpha(\xi,\eta) = \alpha_0 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + \dots\ ,\\
&\beta(\xi,\eta) = \beta_0 + \epsilon\beta_1 + \epsilon^2\beta_2 + \dots\ ,\\
&\theta(\xi,\eta) = \epsilon\theta_1 + \epsilon^2\theta_2 + \dots\ ,\\
&\phi(\xi,\eta) = \epsilon\phi_1 + \epsilon^2\phi_2 + \dots\ ,\\
&C_R = 1 + \epsilon a R_1 + \epsilon^2 a^2 R_2 + \dots\ ,\\
&C_L = 1 + \epsilon b L_1 + \epsilon^2 b^2 L_2 + \dots\ .
\end{aligned} \quad (21)$$

Here it is to be noted that the terms θ_0 and ϕ_0 in Su and Mirie's [4] work are set equal to zero. This means that in the present work the order of trajectory functions is assumed to be of order ϵ^2.

Introducing (21) into (20) and setting the coefficients of like powers of ϵ equal to zero the following sets of differential equations are obtained

$O(\epsilon)$ *equations:*

$$\frac{\partial\alpha_0}{\partial\eta} = 0, \quad \frac{\partial\beta_0}{\partial\xi} = 0 \quad (22)$$

the solution of which yields

$$\alpha_0 = af(\xi), \quad \beta_0 = bg(\eta), \quad (23)$$

where $f(\xi)$ and $g(\eta)$ are two unknown functions to be determined from the solution.

$O(\epsilon^2)$ *equations:*

$$4l\frac{\partial\alpha_1}{\partial\eta} + \frac{1}{3}k^3\alpha_0^{'''} + \frac{2}{3}l^3\beta_0^{'''} - l(\alpha_0+\beta_0)\beta_0^{'}$$
$$+ (3k\alpha_0 - k\beta_0 - 2akR_1)\,\alpha_0^{'} = 0. \quad (24)$$

Integrating equation (24) with respect to η and setting the secular terms equal to zero we obtain

$$R_1 = \frac{1}{2}, \quad k^2 = 3a, \quad f^{'''} + 3ff^{'} - f^{'} = 0, \tag{25}$$

and

$$\alpha_1 = \frac{7}{8}b^2g^2 + \frac{ab}{4}fg - \frac{b^2}{2}g + a^2F_1(\xi) + \frac{abk}{4l}f^{'}M(\eta), \tag{26}$$

where $M(\eta)$ is defined by

$$M(\eta) = \int^{\eta} g(\eta^{'})d\eta^{'}. \tag{27}$$

Similar expressions are valid for β_1 by making proper changes between $\alpha_1 \leftrightarrow \beta_1$, $f \leftrightarrow g$, etc. The result will be as follows

$$L_1 = \frac{1}{2}, \quad l^2 = 3b, \quad g^{'''} + 3gg^{'} - g^{'} = 0, \tag{28}$$

and

$$\beta_1 = \frac{7}{8}a^2f^2 + \frac{ab}{4}fg - \frac{a^2}{2}f + b^2G_1(\eta) + \frac{abl}{4k}g^{'}N(\xi), \tag{29}$$

where $N(\xi)$ is defined by

$$N(\xi) = \int^{\xi} f(\xi^{'})d\xi^{'}. \tag{30}$$

Here $F_1(\xi)$ and $G_1(\eta)$ are two unknown functions whose governing equations are to be obtained from the higher order expansions, R_1 and L_1 are the speed correction terms of order ϵ for the right and left going waves, respectively.

Su and Mirie [4] stated that the terms $f'M(\eta)$ in equation (26) and $g'N(\xi)$ in equation (29) do not cause any secularity at this order but they will cause secularity in the next order equations. Therefore, these terms should be eliminated by introducing the functions $\epsilon\theta_0(\eta)$ and $\epsilon\phi_0(\xi)$ in trajectory functions. But as will be shown in the solution of the next order differential equations these terms do not cause any secularity; therefore, $\epsilon\theta_0(\eta)$ and $\epsilon\phi_0(\xi)$ must vanish.

$O\left(\epsilon^3\right)$ equations:

From the master equation (10), for this order, the following equation is obtained

$$\begin{aligned}
&4l\frac{\partial\alpha_2}{\partial\eta}+ak\frac{\partial^3}{\partial\xi^3}(\alpha_1-\beta_1)-3bk\frac{\partial^3}{\partial\xi\partial\eta^2}(\alpha_1-\beta_1)+k\alpha_0\frac{\partial}{\partial\xi}(3\alpha_1-\beta_1)\\
&-2bl\frac{\partial^3}{\partial\eta^3}(\alpha_1-\beta_1)-l\frac{\partial}{\partial\eta}\left[\beta_0(\alpha_1+\beta_1)\right]-k\beta_0\frac{\partial}{\partial\xi}(\alpha_1+\beta_1)-ak\frac{\partial\alpha_1}{\partial\xi}\\
&+(bl+3l\alpha_0)\frac{\partial\alpha_1}{\partial\eta}+3k\alpha_0^{'}\alpha_1-l\alpha_0\frac{\partial\beta_1}{\partial\eta}-k\alpha_0^{'}\beta_1-\frac{3}{10}a^2k\alpha_0^{(v)}\\
&-\frac{9}{20}b^2l\beta_0^{(v)}+\left(\frac{3}{4}a^2k+3ak\beta_0\right)\alpha_0^{'''}+\left(\frac{3}{4}b^2l+6bl\alpha_0+3bl\beta_0\right)\beta_0^{'''}\\
&+3ak\alpha_0^{'}\alpha_0^{''}+\left(3bk\alpha_0^{'}+6bl\beta_0^{'}\right)\beta_0^{''}+4kl\frac{\partial\theta_1}{\partial\eta}\alpha_0^{'}-2a^2kR_2\alpha_0^{'}=0. \qquad (31)
\end{aligned}$$

When the equation (31) is integrated with respect to η there might be two types of secularities. The first type of secularity is of the form $\int^{\eta}M(\eta^{'})d\eta^{'}$ and the second type is proportional to η. Luckily, the coefficient of $\int^{\eta}M(\eta^{'})d\eta^{'}$ vanishes identically and the coefficient of η gives the following evolution equation for $F_1(\xi)$

$$F_1^{''}+(3f-1)F_1=(2R_2-\frac{19}{20})f+\frac{9}{8}f^2+\frac{1}{4}f^3. \qquad (32)$$

$F_1=f^{'}$ is one of the solution of homogeneous equation in (32). Therefore, the first term on the right-hand side causes to secularity in the solution of F_1 and the coefficient of f must vanish

$$R_2=\frac{19}{40}. \qquad (33)$$

The solution of the remaining parts gives

$$F_1=f-\frac{1}{8}f^2. \qquad (34)$$

Similarly, for the left going wave one obtains

$$L_2=\frac{19}{40},\quad G_1=g-\frac{1}{8}g^2. \qquad (35)$$

Here R_2 and L_2 are the speed correction terms of order ϵ^2. Introducing (34) and (35) into the expressions of α_1 and β_1 we have

$$\alpha_1 = \frac{1}{8}(7b^2g^2 - a^2f^2) - \frac{1}{2}(b^2g - 2a^2f) + \frac{ab}{4}fg + \frac{abk}{4l}f^{'}M, \tag{36}$$

$$\beta_1 = \frac{1}{8}(7a^2f^2 - b^2g^2) - \frac{1}{2}(a^2f - 2b^2g) + \frac{ab}{4}fg + \frac{abl}{4k}g^{'}N. \tag{37}$$

Inserting (36) and (37) into the equation (31) the function α_2 is found to be

$$\begin{aligned}\alpha_2 = &\frac{3}{16}a^2bfg - \frac{9}{8}ab^2fg + 2ab^2fg^2 + \frac{1}{32}a^2bf^2g - \frac{7}{10}b^3g + \frac{43}{32}b^3g^2 \\ &+ \frac{1}{32}b^3g^3 + \frac{3ab^2k^2}{16l^2}fg - \frac{9ab^2k^2}{32l^2}f^2g + \frac{abk}{16l}\left[35af - 9a - b\right]f^{'}M \\ &+ \frac{abl}{16k}\left[afg^{'} - 2bg^{'} + 7bgg^{'} + \frac{ak}{l}f^{'}g\right]N + \frac{7ab^2k}{16l}f^{'}\int g^2d\eta^{'} \\ &+ \frac{ab^2k^2}{16l^2}\left[f - \frac{3}{2}f^2\right]\int gMd\eta^{'} + \frac{ab^2k}{16l}f^{'}\int g^{'}Md\eta^{'} - akf^{'}\theta_1 \\ &+ a^3F_2(\xi).\end{aligned} \tag{38}$$

A similar expression may be given for β_2. Recalling the expression of $g(\eta)$, i.e., $g = \operatorname{sech}^2(\frac{\eta}{2})$ and $M = \int^{\eta} g(\eta^{'})d\eta^{'}$, the following relations may be obtained

$$\begin{aligned}&\int gMd\eta^{'} = -2g, \quad \int g^{'}Md\eta^{'} = \frac{2}{3}M(g-1), \\ &\int g^2d\eta^{'} = \frac{2}{3}M(g+2).\end{aligned} \tag{39}$$

Since the coefficients of the above terms in (38) are all functions of ξ, the products of them with the integration constants can be inserted into the function $F_2(\xi)$. Substituting (39) into (38) and using the relations $k^2 = 3a$ and $l^2 = 3b$ we have

$$\begin{aligned}\alpha_2 = &\frac{1}{4}a^2bfg - \frac{9}{8}ab^2fg + 2ab^2fg^2 - \frac{1}{16}a^2bf^2g - \frac{7}{10}b^3g \\ &+ \frac{43}{32}b^3g^2 + \frac{1}{32}b^3g^3 + \frac{abk}{16l}\left[35af - 9a + 3b + 3bg\right]f^{'}M - akf^{'}\theta_1 \\ &+ \frac{abl}{16k}\left[afg^{'} - 2bg^{'} + 7bgg^{'} + \frac{ak}{l}f^{'}g\right]N + a^3F_2(\xi).\end{aligned} \tag{40}$$

By making a proper substitution a similar expression may be given for β_2 as

$$\begin{aligned}\beta_2 = &\frac{1}{4}ab^2fg - \frac{9}{8}a^2bfg + 2a^2bf^2g - \frac{1}{16}ab^2fg^2 - \frac{7}{10}a^3f \\ &+ \frac{43}{32}a^3f^2 + \frac{1}{32}a^3f^3 + \frac{abl}{16k}\left[35bg - 9b + 3a + 3af\right]g^{'}N - blg^{'}\phi_1 \\ &+ \frac{abk}{16l}\left[bf^{'}g - 2af^{'} + 7aff^{'} + \frac{bl}{k}fg^{'}\right]M + b^3G_2(\xi). \end{aligned} \tag{41}$$

Now if we set θ_1 and ϕ_1 in (40) and (41) equal to zero and try to obtain solution for α_3 from $O(\epsilon^4)$ equation we have the following type of secularity

$$\frac{a^3bk^2}{64l^2}\left(-243f^4 + 324f^3 - 108f^2 + 324f(f^{'})^2 - 108(f^{'})^2\right)\int^{\eta} Md\eta^{'}. \tag{42}$$

However, by choosing the unknown function θ_1 in equation (40) as

$$\theta_1 = \frac{9ab}{4l}f\int_{-\infty}^{\eta} g(\eta^{'})d\eta^{'} \tag{43}$$

this secularity can be removed. Similarly, from (41) the unknown function ϕ_1 may be given by

$$\phi_1 = \frac{9ab}{4k}g\int_{+\infty}^{\xi} f(\xi^{'})d\xi^{'}. \tag{44}$$

In order to remove the secularity of type two in the solution of α_3, the following equation must be satisfied for $F_2(\xi)$

$$\begin{aligned}F_2^{''} + (3f-1)F_2 =&(2R_3 - \frac{55}{56})f - \frac{591}{64}f^4 + \left(\frac{201}{16} + \frac{3}{8a}\right)f^3 \\ &- \left(\frac{393}{160} + \frac{3}{8a}\right)f^2. \end{aligned} \tag{45}$$

From the solution of this equation we obtain

$$R_3 = \frac{55}{112}, \qquad F_2 = \frac{197}{160}f^3 - \left(\frac{217}{160} + \frac{3}{16a}\right)f^2 + \left(\frac{43}{40} + \frac{1}{8a}\right)f. \tag{46}$$

Similarly, for other unknowns we have

$$L_3 = \frac{55}{112}, \qquad G_2 = \frac{197}{160}g^3 - \left(\frac{217}{160} + \frac{3}{16b}\right)g^2 + \left(\frac{43}{40} + \frac{1}{8b}\right)g. \tag{47}$$

Here R_3 and L_3 correspond to ϵ^3 order speed correction terms. Then, the final solution for α_2 and β_2 take the following form

$$\begin{aligned}\alpha_2 = &\frac{1}{4}a^2bfg - \frac{9}{8}ab^2fg + 2ab^2fg^2 - \frac{1}{16}a^2bf^2g + \frac{1}{32}b^3g^3 + \frac{43}{32}b^3g^2 \\ &- \frac{7}{10}b^3g + \frac{197}{160}a^3f^3 - \left(\frac{217}{160}a^3 + \frac{3}{16}a^2\right)f^2 + \left(\frac{43}{40}a^3 + \frac{1}{8}a^2\right)f \\ &+ \frac{abk}{16l}\left(-af - 9a + 3b + 3bg\right)f^{'}M + \frac{abl}{16k}\left(afg^{'} - 2bg^{'} + 7bgg^{'}\right. \\ &\left.+\frac{ak}{l}f^{'}g\right)N,\end{aligned} \tag{48}$$

$$\begin{aligned}\beta_2 = &\frac{1}{4}ab^2fg - \frac{9}{8}a^2bfg + 2a^2bf^2g - \frac{1}{16}ab^2fg^2 + \frac{1}{32}a^3f^3 + \frac{43}{32}a^3f^2 \\ &- \frac{7}{10}a^3f + \frac{197}{160}b^3g^3 - \left(\frac{217}{160}b^3 + \frac{3}{16}b^2\right)g^2 + \left(\frac{43}{40}b^3 + \frac{1}{8}b^2\right)g \\ &+ \frac{abl}{16k}\left(-bg - 9b + 3a + 3af\right)g^{'}N + \frac{abk}{16l}\left(bf^{'}g - 2af^{'} + 7aff^{'}\right. \\ &\left.+\frac{al}{k}fg^{'}\right)M.\end{aligned} \tag{49}$$

Thus, for this order the trajectories of the solitary waves become

$$\begin{aligned}\epsilon^{\frac{1}{2}}k(x - C_Rt) &= \xi - \epsilon^2k\theta_1 + \mathcal{O}(\epsilon^3), \\ \epsilon^{\frac{1}{2}}l(x + C_Lt) &= \eta - \epsilon^2l\phi_1 + \mathcal{O}(\epsilon^3).\end{aligned} \tag{50}$$

To obtain the phase shifts after a head-on collision of solitary waves characterized by a and b are asymptotically far from each other at the initial time ($t = -\infty$), the solitary wave a is at $\xi = 0$, $\eta = -\infty$, and the solitary wave b is at $\eta = 0$, $\xi = +\infty$, respectively. After the collision ($t = +\infty$), the solitary wave b is far to the right of solitary wave a, i.e., the solitary wave a is at $\xi = 0$, $\eta = +\infty$, and the solitary wave b is at $\eta = 0$, $\xi = -\infty$. Using (43), (44) and (50) one can obtain the corresponding phase shifts Δ_a and Δ_b as follows:

$$
\begin{aligned}
\Delta_a &= \epsilon^{1/2} k(x - C_R t) \mid_{\xi=0,\eta=\infty} - \epsilon^{1/2} k(x - C_R t) \mid_{\xi=0,\eta=-\infty} \\
&= -\epsilon^2 \frac{kab}{4l} 9 f(0) \int_{-\infty}^{+\infty} g(\eta^{'}) d\eta^{'} \\
&= -\epsilon^2 \frac{9kab}{4l} \int_{-\infty}^{+\infty} g(\eta^{'}) d\eta^{'}, \qquad (51)
\end{aligned}
$$

$$
\begin{aligned}
\Delta_b &= \epsilon^{1/2} k(x + C_L t) \mid_{\eta=0,\xi=-\infty} - \epsilon^{1/2} k(x + C_L t) \mid_{\eta=0,\xi=\infty} \\
&= \epsilon^2 \frac{lab}{4k} 9 g(0) \int_{-\infty}^{+\infty} f(\xi^{'}) d\xi^{'} \\
&= \epsilon^2 \frac{9lab}{4k} \int_{-\infty}^{+\infty} f(\xi^{'}) d\xi^{'}. \qquad (52)
\end{aligned}
$$

Using the explicit expressions of $f(\xi)$ and $g(\eta)$ the phase shifts are obtained as

$$
\Delta_a = -\epsilon^2 \frac{9kab}{l}, \quad \Delta_b = \epsilon^2 \frac{9lab}{k}. \qquad (53)
$$

Here, as opposed to the results of previous works on the same subject the phase shifts depend on the amplitudes of both waves.

3 Conclusion

Starting with non-dimensional field equations derived in Su and Mirie [4], introducing the stretched coordinates with trajectory functions of order ϵ^2 and expanding the field variables and trajectory functions into power series in ϵ we obtained a set of differential equations governing the various terms in perturbation expansion. By solving these differential equations under the restriction of non-secular solution we obtained evolution equations governing the colliding solitary waves and trajectory functions. Using the conventional definition of phase shifts we obtained the explicit expressions of them. As opposed to the result of previous works on the same subject in our case the phase shifts are found to be depend on amplitudes of both waves. We further noticed that the order of phase shift is ϵ^2 rather than ϵ.

References

[1] Korteweg, D. J. and de Vries, G. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, *Phil. Mag.*, **39**(1895), 422–443

[2] Gardner, C. S., Greene, J. M., Kruskal, M. D. and Miura, R. M. Method for solving the Korteweg-de Vries equation, *Phys. Rev. Lett.*, **19**(1967), 1095–1097

[3] Zakharov, V., Manakov, S., Novikov, S. and Pitaevsky, L. *Theory of Soliton: The Inverse Problem Method.* Nauka, Moscow, 1980 [English translation(Plenium, New York, 1984)]

[4] Su, C.H. and Mirie, R. M. On head-on collisions between two solitary waves, *Journal of Fluid Mechanics*, **98**(1980), 509–525

[5] Huang, G., Lou, S. and Xu, Z. Head-on collision between two solitary waves in a rayleigh-bénard convecting fluid, *Physical Review E*, **47**(1993), R3830

[6] Huang, G. and Velarde, M. G. Head-on collision of two concentric cylindrical ion acoustic solitary waves, *Physical Review E*, **53**(1996), 2988

[7] Zou, Q. and Su, C. H. Overtaking collision between two solitary waves, *Physics of Fluids (1958-1988)*, **29**(1986), 2113–2123

[8] Xue, J. K. Head-on collision of the blood solitary waves, *Physics Letters A*, **331**(2004), 409–413

[9] Demiray, H. Interactions of nonlinear ion-acoustic waves in a collisionless plasma, *Journal of Computational and Applied Mathematics*, **206**(2007), 826–831

[10] Demiray, H. Head-on collision of solitary waves in fluid-filled elastic tubes, *Applied Mathematics Letters*, **18**(2005), 941–950

[11] El-Tantawy, S., Moslem, W., Sabry, R., El-Labany, S., El-Metwally, M. and Schlickeiser, R. Nonplanar solitons collision in ultracold neutral plasmas, *Physics of Plasmas*, **20**(2013), 092126

[12] Demiray, H. Head-on-collision of nonlinear waves in a fluid of variable viscosity contained in an elastic tube, *Chaos, Solitons & Fractals*, **41**(2009), 1578–1586