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INVERSE DOMINATION INTEGRITY OF GRAPHS

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Abstract. With the growing demand for information transport, networks and network architecture have grown increasingly vital. Nodes and the connections that connect them make up a communication network. When the communication network's nodes or links are destroyed, the network's efficiency reduces. If a network is modeled by a graph, then there are various graph theoretical parameters used to express the vulnerability of communication networks such as connectivity, integrity, weak integrity, neighbor integrity, hub integrity, domination integrity, toughness, tenacity etc. In this paper, we introduce a new vulnerability parameter known as an inverse domination integrity which is defined as $IDI(G) = \min_{S \subseteq V(G)} \{|S| + m(G-S)\}$, where S is an inverse dominating set and $m(G-S)$ denotes the order of largest component of $G-S$. We derive few bounds of an inverse domination integrity of graphs. Also, we determine an inverse domination integrity of some families of graphs. Finally, we compute different types of measures of vulnerabilities of probabilistic neural network which are useful in classification and pattern recognition problems.

Keywords: Communication network, network vulnerability, integrity, inverse domination integrity.

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1. Introduction

In this paper, we consider simple, finite, undirected graphs. Let G be a graph with a vertex set $V(G)$ and an edge set $E(G)$ such that $|V(G)| = n$ and $|E(G)| = m$. The open neighbourhood of a vertex $u \in V(G)$ is defined as the set $N_G(u)$ consisting of all vertices v which are adjacent with u. $N_G[u] = N(u) \cup \{u\}$ denotes the closed neighbourhood of u. The degree of a vertex $d_G(v)$ is the number of edges incident to it in G. We refer to Harary [21] for notations and terminologies not defined here.

Network designers place a premium on the stability of a communication network made of processing nodes (vertices) and communication links (edges). As linkages or nodes are lost, the network's effectiveness will deteriorate. As a result, communication networks should be built as stable as possible, not only in terms of preventing early damage, but

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also in terms of possible network reformation. Many parameters have been introduced for the measurement of vulnerability. Some of them are connectivity, integrity, weak integrity, neighbor integrity, hub integrity, domination integrity, toughness, tenacity etc.
The connectivity of a graph G is defined by

$$\kappa(G) = \min_{S \subseteq V(G)} |S|,$$

for which $G \setminus S$ is disconnected or trivial.

The concept of integrity was introduced by Barefoot et al. in [10] as a measure of the stability of a graph. The integrity of a graph G is defined in [10] as

$$I(G) = \min_{S \subseteq V(G)} \{|S| + m(G \setminus S)\},$$

where $m(G \setminus S)$ denotes the order of the largest component of $G \setminus S$.
For more information about integrity refer to [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 16, 18, 19, 20].
The weak integrity of a graph G is defined in [23] as

$$I_w(G) = \min_{S \subseteq V(G)} \{|S| + m_e(G \setminus S)\},$$

where $m_e(G \setminus S)$ denotes the number of edges of the largest component of $G \setminus S$.

A vertex u in G is said to be subverted if the closed neighborhood $N[u]$ is deleted from G. A set of vertices $S = \{u_1, u_2, ..., u_m\}$ is called *vertex subversion stratergy* of G if each of the vertices in S has been subverted from G. Let $G \setminus S$ be the *survival subgraph* left when S has been a vertex subversion stratergy of G. The vertex neighbor integrity [13, 14] of a graph G, $VNI(G)$, is defined as

$$VNI(G) = \min_{S \subset V(G)} \{|S| + m(G \setminus S)\},$$

where S is any vertex subversion stratergy of G and $m(G \setminus S)$ denotes the order of the largest component of $G \setminus S$.

Definition 1.1. [32] *Let $H \subseteq V(G)$ and $x, y \in V(G)$. An H-path between x and y is a path where all intermediate vertices are from H. The set $H \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) \setminus H$, there is an H-path in G between x and y. This includes the degenerate cases where the path consists of the single edge xy or a single vertex x if $x = y$.*

The concept of hub integrity was introduced by Sultan et al. [26] as a measure of the vulnerability of a graph. The hub integrity of a graph G is defined in [26] as

$$HI(G) = \min_{S \subseteq V(G)} \{|S| + m(G \setminus S)\},$$

where S is a hub set of G and $m(G \setminus S)$ denotes the order of the maximum component of $G - S$.

Definition 1.2. [22] *A set D of vertices in a graph G is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D.*

Sundareswaran and Swaminathan [27] defined the domination integrity of a graph. The domination integrity of a graph G is defined as

$$DI(G) = \min_{S \subseteq V(G)} \{|S| + m(G \setminus S)\},$$

where S is a dominating set of G and $m(G \setminus S)$ denotes the order of the maximum component of $G - S$. For more information related to domination integrity refer to [29, 30].

The toughness of a graph was introduced by Chvátal [12]. The toughness of a graph G is defined by

$$t(G) = \min_{S \subseteq V(G)} \frac{|S|}{\omega(G \setminus S)},$$

where $\omega(G \setminus S)$ denotes the number of components of $G - S$.

In [15], the tenacity of a graph G is defined by

$$T(G) = \min_{S \subseteq V(G)} \frac{|S| + m(G \setminus S)}{\omega(G \setminus S)},$$

where $\omega(G \setminus S)$ denotes the number of components of $G \setminus S$ and $m(G \setminus S)$ denotes the order of the maximum component of $G \setminus S$.

Inverse domination is one of the important domination concepts and it was introduced by Kulli and Singarkatti [24] as follows.

Definition 1.3. [24] *Let D be a minimum dominating set in a graph $G = (V, E)$. If $V - D$ contains a dominating set D^{-1} of G, then D^{-1} is called an inverse dominating set with respect to D. The inverse domination number $\gamma^{-1}(G)$ of G is the cardinality of a smallest inverse dominating set of G. A γ^{-1}-set is a minimum inverse dominating set.*

The following results are useful to prove our main results.

Proposition 1.1. [24] *If a graph G has no isolated vertices, then $\gamma(G) \leq \gamma^{-1}(G)$.*

Theorem 1.1. [24] *If a (n, m) graph G has no isolated vertices, then $\frac{2n-m}{3} \geq \gamma^{-1}(G)$.*

Proposition 1.2. [24] *For any path P_n with n vertices,*

$$\gamma^{-1}(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 1, & if\ n \equiv 0(\mod 3), \\ \lceil \frac{n}{3} \rceil, & otherwise. \end{cases}$$

Proposition 1.3. [24] *For a cycle C_n with n vertices, $\gamma^{-1}(C_n) = \lceil \frac{n}{3} \rceil$.*

Proposition 1.4. [28] *Let G be a connected graph. Then $I(G) \leq DI(G)$.*

2. Inverse domination integrity of graphs

We introduce a new measure of vulnerability of a graph G and it is called an inverse domination integrity.

Definition 2.1. *The inverse domination integrity of a graph G is defined as $IDI(G) = \min_{S \subseteq V(G)} \{|S| + m(G \setminus S)\}$, where S is an inverse dominating set of G and $m(G \setminus S)$ denotes the order of the maximum component of $G \setminus S$. Any set $S \subseteq V(G)$ with property that $|S| + m(G \setminus S) = IDI(G)$ is called as an IDI-set of G.*

If the integrity and domination integrity values of two graphs are the same, these factors are insufficient to distinguish them. As a result, a new parameter is required to distinguish these graphs. Then, the following questions arise: How can a network designer determine which network is more stable than the other? Is the inverse domination integrity a vulnerability parameter that compare these graphs in resistance? Let's have a look at this with a basic comparison of two graphs. Assume that G and H are graphs with same order as follows in Fig. 1.

For G and H, $I(G) = I(H) = 3$ and $DI(G) = DI(H) = 3$. So, integrity and domination integrity do not distinguish between G and H. The inverse domination integrity values of these graphs are computed as follows.

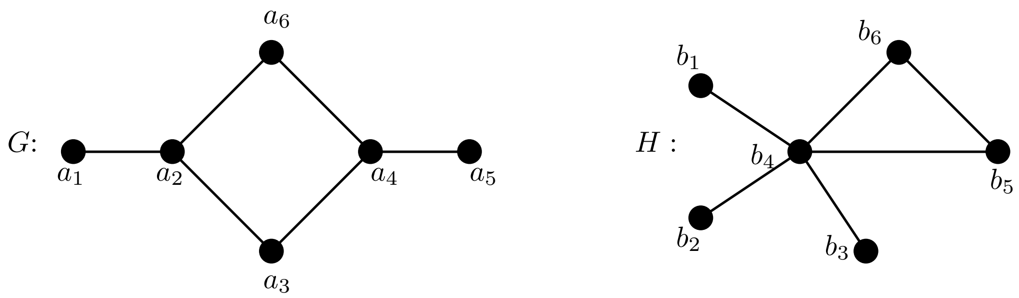

FIGURE 1. Two graphs G and H.

Consider $S_1 = \{a_1, a_3, a_5, a_6\}$ as an inverse dominating set of G and $m(G - S_1) = 1$. So, $IDI(G) = 5$.

Consider $S_2 = \{b_1, b_2, b_3, b_5\}$ as an inverse dominating set of H and $m(G - S_2) = 2$. So, $IDI(H) = 6$.

For G and H, $IDI(G) = 5$ and $IDI(H) = 6$. So, it can be said that H is more stable than G. Then, an inverse domination integrity is a suitable measure of vulnerability which distinguishes between these graphs.

In this section, we obtain the inverse domination integrity of graphs.

Observation 2.1. *For any graph G without isolated vertices, $IDI(G) > \gamma^{-1}(G)$.*

Proposition 2.1. *For any graph G without isolated vertices, $I(G) \leq DI(G) \leq IDI(G)$.*

By Proposition 1.4, $I(G) \leq DI(G)$. By Proposition 1.1, we know that $\gamma(G) \leq \gamma^{-1}(G)$, then $DI(G) \leq IDI(G)$. Therefore, $I(G) \leq DI(G) \leq IDI(G)$.

Proposition 2.2. *For any graph G without isolated vertices, $IDI(G) > \gamma(G)$*

Proof. Since $IDI(G) > \gamma^{-1}(G)$ and by Proposition 1.1, $\gamma^{-1}(G) \geq \gamma(G)$. Hence $IDI(G) > \gamma(G)$ □

Observation 2.2. $2 \leq IDI(G) \leq n$. *$IDI(G) = 2$, if and only if $G = K_2$. If $G \neq K_2$, then* $3 \leq IDI(G) \leq n$.

Theorem 2.1. *Let G be a graph of order $n \geq 2$ with no isolated vertices. Then $IDI(G) \geq \gamma^{-1}(G) + 1$*

Proof. Let G be a graph of order $n \geq 2$ with no isolated vertices and S be a γ^{-1}-set of G. So, $|S| = \gamma^{-1}(G)$ and $m(G - S) \geq 1$. Then

$$\begin{aligned} IDI(G) &= \min_{S \subseteq V(G)} \{|S| + m(G - S)\} \\ &\geq min\{\gamma^{-1}(G) + 1\} \\ &\geq \gamma^{-1}(G) + 1 \end{aligned}$$

□

Theorem 2.2. *Let every nonend vertex of a tree T be adjacent to atleast two end vertices, then $IDI(T) = n$*

Proof. If every nonend vertex of T is adjacent to at least two end vertices, then the set of all nonend vertices is a minimum dominating set and the set of all end vertices is a minimum inverse dominating set. Let D and D^{-1} denote the minimum dominating and inverse dominating sets, respectively. So, $|S| = |V \setminus D|$ and $m(G \setminus S) = |D|$. Therefore, $IDI(G) = |V \setminus D| + |D| = n$ □

Theorem 2.3. *For any graph G order n containing no isolated vertices, if $IDI(G) = n$, then $diam(G) \leq 2$.*

Proof. Let G be a graph of order n such that it contains no isolated vertices. Assume that $diam(G) \geq 3$, then G contains a path P_4. Thus, $IDI(G) \leq n-1$ which is a contradiction. Hence, $diam(G) \leq 2$. □

Proposition 2.3. *For any complete graph $K_n (n \geq 2)$, $IDI(K_n) = n$.*

Proof. Let $V(K_n) = \{x_1, x_2, ..., x_n\}$ and $S \subseteq V(K_n)$ is an inverse dominating set of complete graph $K_n (n \geq 2)$. The only minimum dominating set of K_n is any one vertex of K_n that is $D = \{x_1\}$. Since K_n is a complete graph, $S = \{x_2, x_3, ..., x_i\}$ for any i, $2 \leq i \leq n$ so that $|S| = i - 1$. Correspondingly, $m(K_n - S) = n - i + 1$ and hence $IDI(K_n) = n$. □

Proposition 2.4. *For any path P_n with $n \geq 3$,*

$$IDI(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 3, & if\ n \equiv 0(\mod 3), \\ \lceil \frac{n}{3} \rceil + 2, & otherwise. \end{cases}$$

Proof. Let P_n be any path $n \geq 3$. From Proposition 1.2, it is known that $\gamma^{-1}(P_n) = \lceil \frac{n}{3} \rceil + 1$ if $n \equiv 0(\mod 3)$ and $\gamma^{-1}(P_n) = \lceil \frac{n}{3} \rceil$ if $n \not\equiv 0(\mod 3)$. For any minimum inverse dominating set S of P_n, $m(P_n \setminus S) = 2$. Therefore, $IDI(P_n) \leq \gamma^{-1}(P_n) + 2$. If X is any inverse dominating set of P_n, then $|X| + m(P_n \setminus X) \geq \gamma^{-1}(P_n) + 2$. Hence the proof. □

Proposition 2.5. *For any cycle C_n, $n \geq 4$*

$$IDI(C_n) = \begin{cases} 3, & if\ n = 4, \\ \lceil \frac{n}{3} \rceil + 2, & if\ n \geq 5. \end{cases}$$

Proof. Let C_n be any cycle $n \geq 4$.
Case 1. Suppose $n = 4$. Let S be an inverse dominating set of C_4. $|S| = \beta_0(C_4) = 2$ and $m(C_4 \setminus S) = 1$. Therefore, $IDI(C_4) = 3$.
Case 2. From Proposition 1.3, it is known that $\gamma^{-1}(C_n) = \lceil \frac{n}{3} \rceil$ if $n \geq 5$. For any minimum invesre dominating set S of C_n, $m(C_n - S) = 2$. Therefore, $IDI(C_n) \leq \gamma^{-1}(C_n) + 2$. If X is any inverse dominating set of C_n, then $|X| + m(C_n \setminus X) \geq \gamma^{-1}(C_n) + 2$. Hence the proof. □

Proposition 2.6. *For any complete bipartite graph $K_{a,b}$ with $2 \leq a \leq b$,*

$$IDI(K_{a,b}) = a + b,$$

Proof. Let $V(K_{a,b}) = V_1 \cup V_2$, where $V_1 = \{x_1, x_2, ..., x_a\}$ and $V_2 = \{y_1, y_2, ..., y_b\}$. It is easy to say that $IDI(K_{2,2}) = 3$. Let us consider the other case except $a = b = 2$. The minimum dominating set of a complete bipartite graph contains one vertex from V_1 and another vertex from V_2. Then $S_i = \{x_2, ..., x_a, y_2, ..., y_b\}$ contains an inverse dominating set of $K_{a,b}$. If $|S_i| = k$, then $m(K_{a,b} \setminus S_i) = a + b - k$ and hence $IDI(K_{a,b}) = a + b$. □

Proposition 2.7. *For the star graph $K_{1,b}$, $IDI(K_{1,b}) = b + 1$,*

Proof. Let $V(K_{1,b}) = \{x_1, x_2, ..., x_b\}$ and $d(x_1) = \Delta(K_{1,b})$. $D = \{x_1\}$ is a minimum dominating set and $S = \{V(K_{1,b}) \setminus \{x_1\}\}$ is an inverse dominating set. No proper subset of S is an inverse dominating set. So, $|S| = b$ and $m(K_{1,b} \setminus S) = 1$. Thus, $IDI(K_{1,b}) = 1 + b$. Therefore, $IDI(K_{1,b}) = b + 1$. □

Proposition 2.8. *For the wheel graph $W_n (n \geq 4)$, $IDI(W_n) = n$.*

Proof. Let x be the central vertex of a wheel and $V(W_n) = \{x, y_1, y_2, ..., y_{n-1}\}$. $D = \{x\}$ is a minimum dominating set and $S = \{V(W_n) \setminus \{x\}\}$ is an inverse dominating set. No proper subset of S is an inverse dominating set. So, $|S| = n - 1$ and $m(W_n \setminus S) = 1$. Thus, $IDI(W_n) = n$. □

3. Vulnerability measures in probabilistic neural network

A probabilistic neural network (PNN(n, k, m)) is a feedforward neural network, which is widely used in classification and pattern recognition problems. In the PNN(n, k, m) algorithm, the parent probability distribution function (PDF) of each class is approximated by a Parzen window and a non-parametric function. Then, using PDF of each class, the class probability of a new input data is estimated and Bayes' rule is then employed to allocate the class with highest posterior probability to new input data. By this method, the probability of mis-classification is minimized. This type of artificial neural network (ANN) was derived from the Bayesian network and a statistical algorithm called Kernel Fisher discriminant analysis. It was introduced by D.F. Specht in 1966. In a PNN(n, k, m), the operations are organized into a multilayered feedforward network with four layers: namely, Input layer, Pattern layer, Summation layer and Output layer.

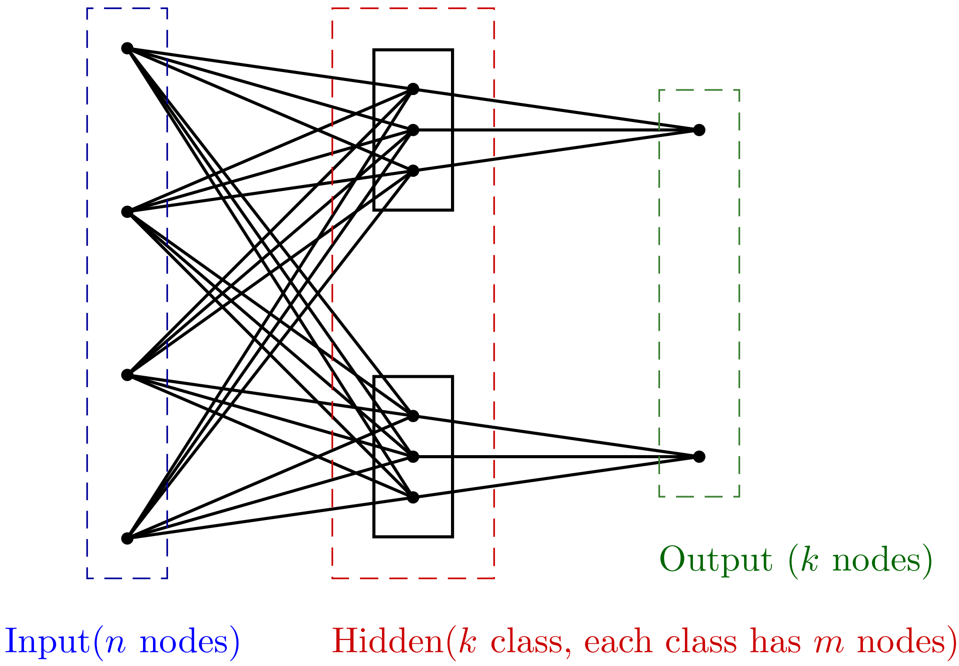

Figure 2. The probabilistic neural network PNN[4, 2, 3].

Theorem 3.1. *Let $PNN(n, k, m)$ be the probabilistic neural network. Then*

$$\kappa(PNN(n, k, m)) = min\{n, mk\}.$$

Proof. Let $PNN(n, k, m)$ be the probabilistic neural network. There are two cases to choose the set S.

Case 1. Choose set S in such a way that it should contain all the vertices in input layer of $PNN(n, k, m)$. So, $|S| = n$. The removal of the vertices of set S from $PNN(n, k, m)$ results in k disconnected star graphs each of order $m + 1$.

Case 2. Let $S \subset V(PNN(n, k, m))$ be a set containing all the vertices of hidden layer.

So, $|S| = mk$. The removal of vertices from $PNN(n, k, m)$ leaves a totally disconnected graph with $n + k$ vertices.
Combining the above two cases we get, $\kappa(PNN(n, k, m)) = min\{n, mk\}$. □

Theorem 3.2. *Let $PNN(n, k, m)$ be the probabilistic neural network. Then*

$$I(PNN(n, k, m)) = min\{n + m + 1, mk + 1\}.$$

Proof. Let $PNN(n, k, m)$ be the probabilistic neural network. There are two cases to choose the set S.
Case 1. Choose set S in such a way that it should contain all the vertices in input layer of $PNN(n, k, m)$. So, $|S| = n$. The deletion of the vertices of set S from $PNN(n, k, m)$ results in k disconnected star graphs each of order $m+1$. So, $m(PNN(n, k, m)\setminus S) = m+1$. $|S| + m(PNN(n, k, m) \setminus S)$ is minimum for above chosen S.
Case 2. Let $S \subset V(PNN(n, k, m))$ be a set containing all the vertices of hidden layer. So, $|S| = mk$. The removal of vertices from $PNN(n, k, m)$ leaves a disconnected graph with $n+k$ vertices. Hence, $m(PNN(n, k, m) \setminus S) = 1$. Therefore, $|S|+m(PNN(n, k, m) \setminus S) = mk + 1$ is minimum for above set S.
Combining the above two cases we get, $I(PNN(n, k, m)) = min\{n + m + 1, mk + 1\}$. □

Theorem 3.3. *Let $PNN(n, k, m)$ be the probabilistic neural network. Then*

$$VNI(PNN(n, k, m)) = 2.$$

Proof. Let S be a subversion stratergy of $V(PNN(n, k, m))$. Choose any one vertex from input layer, $|S| = 1$. If we remove the set S and all its adjacent vertices, then there exist $n - 1 + k$ components which are of order 1. So, $VNI(PNN(n, k, m)) = 2$. □

Theorem 3.4. *Let $PNN(n, k, m)$ be the probabilistic neural network. Then*

$$HI(PNN(n, k, m)) = mk + 1.$$

Proof. Let $PNN(n, k, m)$ be the probabilistic neural network. The set of vertices in hidden layer of $PNN(n, k, m)$ forms a hub set S, it follows that $|S| = mk$. The removal of S from $PNN(n, k, m)$ results a disconnected graph. Therefore, $HI(PNN(n, k, m) \setminus S) = mk + 1$. □

Theorem 3.5. *Let $PNN(n, k, m)$ be the probabilistic neural network. Then*

$$DI(PNN(n, k, m)) = min\{n + k + 1, mk + 1\}.$$

Proof. Let $PNN(n, k, m)$ be the probabilistic neural network. There are two cases to choose the set S.
Case 1. Choose set S in such a way that it should contain all the vertices in input layer and output layer of $PNN(n, k, m)$, which is a dominating set of $PNN(n, k, m)$. So, $|S| = n+k$. The deletion of the vertices of set S from $PNN(n, k, m)$ results in disconnected graphs each of order 1. So, $m(PNN(n, k, m) \setminus S) = 1$. $|S| + m(PNN(n, k, m) \setminus S)$ is minimum for above chosen S.
Case 2. Let $S \subset V(PNN(n, k, m))$ be a set containing all the vertices of hidden layer which is a dominating set for $PNN(n, k, m)$. So, $|S| = mk$. The removal of vertices from $PNN(n, k, m)$ leaves a disconnected graph. Hence, $m(PNN(n, k, m) \setminus S) = 1$. Therefore, $|S| + m(PNN(n, k, m) \setminus S) = mk + 1$ is minimum for above set S.
Combining the above two cases we get, $DI(PNN(n, k, m)) = min\{n+k+1, mk+1\}$. □

Theorem 3.6. *Let $PNN(n, k, m)$ be the probabilistic neural network. Then*

$$I_w(PNN(n, k, m)) = min\{n + m, mk\}.$$

Proof. Let $PNN(n,k,m)$ be the probabilistic neural network. There are two cases to choose the set S.
Case 1. Choose set S in such a way that it should contain all the vertices in input layer of $PNN(n,k,m)$. So, $|S| = n$. The deletion of the vertices of set S from $PNN(n,k,m)$ results in disconnected graph each of size m. So, $m_e(PNN(n,k,m) \setminus S) = m$. $|S| + m(PNN(n,k,m) \setminus S)$ is minimum for above chosen S.
Case 2. Let $S \subset V(PNN(n,k,m))$ be a set containing all the vertices of hidden layer of $PNN(n,k,m)$. So, $|S| = mk$. The removal of vertices from $PNN(n,k,m)$ leaves a disconnected graph each of size 0. Hence, $m(PNN(n,k,m) \setminus S) = 0$. Therefore, $|S| + m(PNN(n,k,m) \setminus S) = mk$ is minimum for above set S.
Combining the above two cases we get, $I_w(PNN(n,k,m)) = min\{n+m, mk\}$. □

Theorem 3.7. *Let $PNN(n,k,m)$ be the probabilistic neural network. Then*

$$t(PNN(n,k,m)) = min\left\{\frac{n}{k}, \frac{mk}{n+k}\right\}.$$

Proof. Let $PNN(n,k,m)$ be the probabilistic neural network. There are two cases to choose the set S.
Case 1. Choose set S in such a way that it should contain all the vertices in input layer of $PNN(n,k,m)$. So, $|S| = n$. The removal of S from $PNN(n,k,m)$ results in disconnected graph where number of components is k. So, $\omega(PNN(n,k,m) \setminus S) = k$. $\frac{|S|}{\omega(PNN(n,k,m)\setminus S)}$ is minimum for above chosen S.
Case 2. Let S be a set containing all the vertices of hidden layer of $PNN(n,k,m)$. So, $|S| = mk$. The removal of vertices from $PNN(n,k,m)$ results in a disconnected graph where number of components is $n+k$. Hence, $\omega(PNN(n,k,m) \setminus S) = n+k$. $\frac{|S|}{\omega(PNN(n,k,m)\setminus S)}$ is minimum for above chosen S.
Combining the above two cases we get, $t(PNN(n,k,m)) = min\left\{\frac{n}{k}, \frac{mk}{n+k}\right\}$. □

Theorem 3.8. *Let $PNN(n,k,m)$ be the probabilistic neural network. Then*

$$T(PNN(n,k,m)) = min\left\{\frac{n+m+1}{k}, \frac{mk+1}{n+k}\right\}.$$

Proof. Let $PNN(n,k,m)$ be the probabilistic neural network. There are two cases to choose the set S.
Case 1. Choose set S in such a way that it should contain all the vertices in input layer of $PNN(n,k,m)$. So, $|S| = n$. The deletion of the vertices of set S from $PNN(n,k,m)$ results in k disconnected star graphs each of order $m+1$. So, $m(PNN(n,k,m)\setminus S) = m+1$ and $\omega(PNN(n,k,m) \setminus S) = k$. $\frac{|S|+m(PNN(n,k,m)\setminus S)}{\omega(PNN(n,k,m))\setminus S}$ is minimum for above chosen S.
Case 2. Let $S \subset V(PNN(n,k,m))$ be a set containing all the vertices of hidden layer. So, $|S| = mk$. The removal of vertices from $PNN(n,k,m)$ leaves a disconnected graph with $n+k$ vertices. Hence, $m(PNN(n,k,m) \setminus S) = 1$ and $\omega(PNN(n,k,m) \setminus S) = n+k$. Therefore, $\frac{|S|+m(PNN(n,k,m)\setminus S)}{\omega(PNN(n,k,m))\setminus S}$ is minimum for above set S.
Combining the above two cases we get, $T(PNN(n,k,m)) = min\left\{\frac{n+m+1}{k}, \frac{mk+1}{n+k}\right\}$. □

Theorem 3.9. *Let $PNN(n,k,m)$ be the probabilistic neural network. Then*

$$IDI(PNN(n,k,m)) = \begin{cases} n+k+m, & if\ mk \leq n+k, \\ mk+1, & if\ mk > n+k. \end{cases}$$

Proof. Let $PNN(n,k,m)$ be the probabilistic neural network. There are two cases to choose the set S.

Case 1. Suppose $mk \leq n+k$. Let D be a minimum dominating set which consists of all the vertices of hidden layer. So, choose set S in such a way that it should contain all the vertices in input layer and output layer of $PNN(n,k,m)$, which is an inverse dominating set of $PNN(n,k,m)$. So, $|S| = n+k$. The deletion of the vertices of set S from $PNN(n,k,m)$ results in totally disconnected graphs of order 1. So, $m(PNN(n,k,m) \setminus S) = 1$. $|S| + m(PNN(n,k,m) \setminus S)$ is minimum for above chosen S. Thus, $IDI(PNN(n,k,m)) = n+k+1$ if $m \leq n+k$.

Case 2. Suppose $mk > n+k$. Choose set D in such a way that it should contain all the vertices in input layer and output layer of $PNN(n,k,m)$, which is a minimum dominating set of $PNN(n,k,m)$. Let $S \subset V(PNN(n,k,m))$ be a set containing all the vertices of hidden layer which is a dominating set for $PNN(n,k,m)$. So, $|S| = mk$. The removal of vertices from $PNN(n,k,m)$ leaves a totally disconnected graph. Hence, $m(PNN(n,k,m) \setminus S) = 1$. Therefore, $|S| + m(PNN(n,k,m) \setminus S) = mk+1$ is minimum for above set S. Thus, $IDI(PNN(n,k,m)) = mk+1$ if $m > n+k$.

□

4. Conclusions

In this paper, we have introduced the inverse domination integrity, a new measure of network vulnerability. We have computed the inverse domination integrity of standard class of graphs. Also, we have studied some properties and bounds for inverse domination integrity. The probabilistic neural network is taken to model the network system and the vulnerability parameter values of them reveal how network can be made more stable than the earlier. These results can help the network designers to choose a suitable topology for the network. This study can be very useful in the investigation of complex network robustness.

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