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THE RESTRAINED MONOPHONIC NUMBER OF A GRAPH

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ABSTRACT. A set S of vertices of a connected graph G is a monophonic set of G if each vertex v of G lies on a $x-y$ monophonic path for some x and y in S. The minimum cardinality of a monophonic set of G is the monophonic number of G and is denoted by $m(G)$. A restrained monophonic set S of a graph G is a monophonic set such that either $S = V$ or the subgraph induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained monophonic set of G is the restrained monophonic number of G and is denoted by $m_r(G)$. We determine bounds for it and determine the same for some special classes of graphs. Further, several interesting results and realization theorems are proved.

Keywords: monophonic set, monophonic number, restrained monophonic set, restrained monophonic number.

AMS Subject Classification: 05C12.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology we refer to Harary [16]. For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called $u - v$ geodesic. It is known that d is a metric on the vertex set V of G. The neighborhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The closed neighborhood of a vertex v is the set $N[v] = N(v) \bigcup \{v\}$. A vertex v is an extreme vertex if the subgraph induced by its neighbors is complete. The *closed interval* $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G, while for

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$S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices of G is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set*. The geodetic number of a graph was introduced in [6, 17] and further studied in [1, 3, 4, 7, 8]. A geodetic set S of a graph G is a restrained geodetic set if the subgraph $G[V - S]$ has no isolated vertex. The minimum cardinality of a restrained geodetic set of G is the restrained geodetic number. The restrained geodetic number of a graph was introduced and studied in [2, 5].

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A set S of vertices of G is a *monophonic set* of G if each vertex v of G lies on a $x - y$ monophonic path for some x and y in S. The minimum cardinality of a monophonic set of G is the *monophonic number* of G and is denoted by $m(G)$. The monophonic number of a graph and its related concepts have been studied in [9, 10, 11, 12, 13, 14, 15, 18, 21, 22, 23]. For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max \{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, $rad_m(G)$ of G is $rad_m(G) = \min \{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_m(G)$ of G is $diam_m(G) = \max \{e_m(v) : v \in V(G)\}$. A vertex u in G is *monophonic eccentric vertex* of a vertex v in G if $e_m(u) = d_m(u, v)$. The monophonic distance was introduced and studied in [19, 20]. These concepts have interesting applications in Channel Assignment Problem in FM radio technologies. The monophonic matrix is used to discuss different aspects of certain molecular graphs associated to the molecules arising in special situations of molecular problems in theoretical Chemistry.

The following theorems will be used in the sequel.

Theorem 1.1. [5] *Each extreme vertex of a connected graph G belongs to every restrained geodetic set of G.*

Theorem 1.2. [21] *Each extreme vertex of a connected graph G belongs to every monophonic set of G.*

Theorem 1.3. [21] *Let G be a connected graph with a cutvertex v and let S be a monophonic set of G. Then every component of $G - v$ contains an element of S.*

Throughout this paper G denotes a connected graph with at least two vertices.

2. Restrained monophonic number

Definition 2.1. *A restrained monophonic set S of a graph G is a monophonic set such that either $S = V$ or the subgraph induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained monophonic set of G is the restrained monophonic number of G and is denoted by $m_r(G)$.*

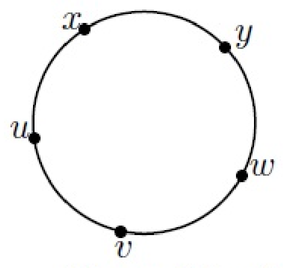

Figure 2.1: G

Example 2.1. *For the graph G given in Figure 2.1, it is easily verified that $S = \{x, w\}$ is a minimum monophonic set of G and so $m(G) = 2$. Since the subgraph induced by $V - S$ has an isolated vertex y, S is not a restrained monophonic set of G. It is clear that, $S \cup \{y\}$ is a minimum restrained monophonic set of G so that $m_r(G) = 3$. Thus the monophonic number and the restrained monophonic number of a graph are different.*

Definition 2.2. *A vertex v of a connected graph G is said to be a restrained monophonic vertex of G if v belongs to every minimum restrained monophonic set of G.*

If G has the unique minimum restrained monophonic set S, then every vertex in S is a restrained monophonic vertex. In the next theorem, we show that there are certain vertices in a non-trivial connected graph G that are restrained monophonic vertices of G. Observe that, every restrained monophonic set of G is a monophonic set of G. This together with Theorem 1.2 gives the next result.

Theorem 2.1. *Each extreme vertex of a connected graph G belongs to every restrained monophonic set of G.*

Corollary 2.1. *For the complete graph $K_p (p \geq 2)$, $m_r(K_p) = p$.*

Since the complement of each restrained monophonic set has cardinality greater than or equal 2, the next result follows.

Theorem 2.2. *There is no graph G of order p with $m_r(G) = p - 1$.*

Corollary 2.2. *If T is a tree of order p with k endvertices and $p-k \geq 2$, then $m_r(T) = k$.*

Proof. This follows from Theorems 2.1 and 2.2. □

The next theorem follows from Theorem 1.3.

Theorem 2.3. *Let G be a connected graph with a cutvertex v and let S be a restrained monophonic set of G. Then every component of $G - v$ contains an element of S.*

Theorem 2.4. *For any connected graph G, $2 \leq m(G) \leq m_r(G) \leq p$, and $m_r(G) \neq p - 1$.*

Proof. Any monophonic set needs at least two vertices and so $m(G) \geq 2$. Since every restrained monophonic set of G is also a monophonic set of G, it follows that $m(G) \leq m_r(G)$. Also, since $V(G)$ induces a restrained monophonic set of G, it is clear that $m_r(G) \leq p$. Theorem 2.2 gives $m_r(G) \neq p - 1$. □

It is observed that, the bounds in the above theorem are sharp. For the complete graph $K_p (p \geq 2)$, $m(K_p) = m_r(K_p) = p$. The set of two endvertices of a path $P_n (n \geq 4)$ is its unique minimum monophonic set and unique minimum restrained monophonic set so that $m(P_n) = m_r(P_n) = 2$. For the graph G given in Example 2.1, $m(G) < m_r(G)$.

Corollary 2.3. *Let G be a connected graph. If $m_r(G) = 2$, then $m(G) = 2$.*

The converse of Corollary 2.3 need not be true. For the graph G given in Figure 2.1, the monophonic number of G is 2 and the restrained monophonic number of G is 3.

Theorem 2.5. *If G is a connected graph of order p with $m(G) = p - 1$, then $m_r(G) = p$.*

Proof. This follows from Theorems 2.2 and 2.4. □

Theorem 2.6. *For any connected graph G, $2 \leq m_r(G) \leq g_r(G) \leq p$, $m_r(G) \neq p - 1$, $g_r(G) \neq p - 1$.*

Proof. Any restrained monophonic set needs at least two vertices and so $m_r(G) \geq 2$. Since every restrained geodetic set of G is also a restrained monophonic set of G, it follows that $m_r(G) \leq g_r(G)$. Also, since $V(G)$ induces a restrained geodetic set of G, it is clear that $g_r(G) \leq p$. From the definitions of the restrained monophonic number and the restrained geodetic number, we have $m_r(G) \neq p-1$, $g_r(G) \neq p-1$. □

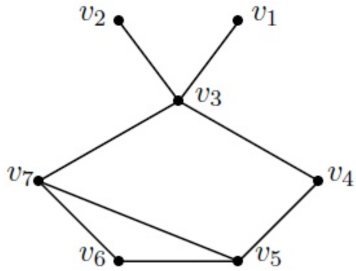

Figure 2.2: G

Remark 2.1. *The bounds in Theorem 2.6 are sharp. For the star graph $K_{1,p-1}$, $m_r(K_{1,p-1}) = g_r(K_{1,p-1}) = p$. For a non-trivial path $P_n(n \geq 4)$, $m_r(P_n) = g_r(P_n) = 2$. Also, if G is an even cycle of order at least 6, then $m_r(G) = g_r(G) = 2$. All the inequalities in Theorem 2.6 can be strict. For the graph G given in Figure 2.2, $S = \{v_1, v_2, v_6\}$ is a minimum restrained monophonic set of G so that $m_r(G) = 3$ and no 3-element subset of the vertex set is a restrained geodetic set of G. Since $S \cup \{v_5\}$ is a restrained geodetic set of G, it follows that $g_r(G) = 4$. Thus we have $2 < m_r(G) < g_r(G) < p$.*

In view of this remark, we leave the following problem as an open question.

Problem 2.1. *Characterize graphs G for which $m_r(G) = g_r(G)$.*

Theorem 2.7. *For the path P_n $(n \geq 4)$ or a cycle $C_n(n \geq 6)$ or $G = \overline{K}_2 + H$, where H is a connected graph of order $p-2$, $(p \geq 4)$, $m_r(G) = 2$.*

Proof. It is easily verified that the end vertices of the path P_n is a minimum restrained monophonic set of P_n so that $m_r(P_n) = 2$. For any cycle $C_n(n \geq 6)$, a set $S = \{u, v\}$ on C_n with $d(u, v) \geq 3$, is obviously a restrained monophonic set so that $m_r(C_n) = 2$ if $n \geq 6$. Next, suppose that $G = \overline{K}_2 + H$, where H is a connected graph of order $p-2$. Let $V(\overline{K}_2) = \{u_1, u_2\}$. It is easily verified that the set $S = \{u_1, u_2\}$ is a minimum restrained monophonic set of G and so $m_r(G) = 2$. □

Remark 2.2. *The last part of the Theorem 2.7 shows that it is always possible to construct a graph G of given order $p \geq 4$ with $g_r(G) = m_r(G) = 2$.*

This leads to the following problem.

Problem 2.2. *Characterize graphs G for which (i) $g_r(G) = 2$ (ii) $m_r(G) = 2$.*

Theorem 2.8. *If G is a non-trivial connected graph of order p and monophonic diameter $d_m \geq 3$, then $m_r(G) \leq p - d_m + 1$.*

Proof. Let u and v be vertices of G such that $d_m(u, v) = d_m$ and let $P : u = v_0, v_1, ..., v_{d_m} = v$ be a $u - v$ monophonic path of length d_m. Let $S = V - \{v_1, v_2, ..., v_{d_{m-1}}\}$. Then, it is clear that S is a restrained monophonic set of G so that $m_r(G) \leq |S| = p - d_m + 1$. □

Corollary 2.4. *If G is a connected graph with order $p \geq 2$ and $m_r(G) = p$, then its monophonic diameter $d_m \leq 2$.*

This shows that for a connected graph G of order $p \geq 2$ and $m_r(G) = p$, the monophonic diameter satisfies $d_m = 1$ and $d_m = 2$. Both bounds are attained for the complete graph $K_p(p \geq 2)$ and the cycle C_4.

Theorem 2.9. *For any cycle C_p,* $m_r(C_p) = \begin{cases} 2 & \textit{if } p \geq 6 \\ 3 & \textit{if } p = 3 \textit{ and } p = 5 \\ 4 & \textit{if } p = 4. \end{cases}$

Proof. If $p = 3$, then $G = C_3$ is a complete graph, by Corollary 2.1, we have $m_r(C_3) = 3$.

If $p = 4$, no 2-element subset or no 3-element subset of $V(C_4)$ forms a restrained monophonic set of C_4 and so $m_r(C_4) = 4$.

If $p = 5$, it is easily observed that, any three consecutive vertices of $G = C_p$ form a minimum restrained monophonic set of $G = C_p$ and so $m_r(C_p) = 3$.

If $p \geq 6$, any two antipodal vertices of $G = C_p$ form a minimum restrained monophonic set of $G = C_p$ and so $m_r(C_p) = 2$. □

Theorem 2.10. *For any wheel $W_p = K_1 + C_{p-1}$, $(p \geq 4)$,*

$m_r(W_p) = \begin{cases} 4 & \textit{if } p = 4 \\ 2 & \textit{if } p \geq 5. \end{cases}$

Proof. If $p = 4$, then $G = W_4$ is a complete graph, and so by Corollary 2.1, $m_r(W_p) = 4$.

If $p \geq 5$, it is easily observed that, any two non-adjacent vertices of C_{p-1} form a minimum restrained monophonic set of $G = W_p$ and so $m_r(W_p) = 2$. □

Theorem 2.11. *For the star graph $K_{1,p-1}(p \geq 2)$, $m_r(K_{1,p-1}) = p$.*

Proof. Let $S = \{v_1, v_2, \cdots, v_{p-1}\}$ be the end vertices of $G = K_{1,p-1}$. By Theorem 2.1, every restrained monophonic set of G contains S. Clearly S is not a restrained monophonic set of G and so $S = V(K_{1,p-1})$ is the unique restrained monophonic set of $K_{1,p-1}$, so that $m_r(K_{1,p-1}) = p$. □

Theorem 2.12. *If $G = K_1 + \bigcup m_j K_j$, where $j \geq 2, \sum m_j \geq 2$, then $m_r(G) = p$.*

Proof. Let $G = K_1 + \bigcup m_j K_j$, where $j \geq 2, \sum m_j \geq 2$. Let $K_1 = \{v\}$ and S be the set of all extreme vertices of G. Since every vertex of G is an extreme vertex except the vertex v and v is the only cutvertex of G, by Theorem 2.1, every restrained monophonic set of G contains S. It is clear that, S is not a restrained monophonic set of G. Hence $V(G)$ is the unique minimum restrained monophonic set of G and so $m_r(G) = p$. □

Remark 2.3. *Thus there are a number of classes of graphs G (complete and non-complete) of order p with $m_r(G) = p$.*

This leads to the following open problem.

Problem 2.3. *Characterize the class of graphs G of order p for which $m_r(G) = p$.*

A *caterpillar* is a tree for which the removal of all the endvertices gives a path.

Theorem 2.13. *For every non-trivial tree T of order p and monophonic diameter $d_m \geq 3$, $m_r(T) = p - d_m + 1$ if and only if T is a caterpillar.*

Proof. Let T be any non-trivial tree. Let $P : u = v_0, v_1, ..., v_{d_m}$ be a monophonic diametral path. Let k be the number of endvertices of T and l be the number of internal vertices of T other than $v_1, v_2, ..., v_{d_{m-1}}$. Then $d_m - 1 + l + k = p$. By Corollary 2.2, $m_r(T) = k$ and so $m_r(T) = p - d_m - l + 1$. Hence $m_r(T) = p - d_m + 1$ if and only if $l = 0$, if and only if all the internal vertices of T lie on the monophonic diametral path P, if and only if T is a caterpillar. □

Theorem 2.14. *For the complete bipartite graph $G = K_{m,n}(2 \leq m \leq n)$,*

$$m_r(G) = \begin{cases} n+2 & \text{if } 2 = m \leq n \\ 4 & \text{if } 3 \leq m \leq n. \end{cases}$$

Proof. Let $G = K_{m,n}$ and let $V_1 = \{x_1, x_2, \cdots, x_m\}$ and $V_2 = \{y_1, y_2, \cdots, y_n\}$ be the partite sets of G. If $m = n = 2$, then $G = K_{2,2}$ is a cycle of order 4 so that by Theorem 2.9, $m_r(G) = 4$. If $m = 2 < n$, then V_1 is a minimum monophonic set of G, and since the subgraph induced by $G - V_1$ has isolated vertices, V_1 is not a restrained monophonic set of G. It is clear that $S = V_1 \cup V_2$ is the unique minimum restrained monophonic set of G and so $m_r(K_{2,n}) = n + 2$.

Now, if $m \geq 3$ and let $S = \{x_1, x_2, y_1, y_2\}$. Clearly S is a restrained monophonic set of G and so $m_r(G) \leq 4$. It remains to show that if X is a 3 - element subset of $V(G)$, then X is not a restrained monophonic set of G. If $m = 3$ and X contains all the elements from V_1 then the subgraph induced by $G - X$ has an isolated vertex. Hence X is not a restrained monophonic set of G. Then $X \cap V_1 = \{x_i, x_j\}$ and $X \cap V_2 = \{y_k\}$. Since X contains two elements from V_1, there exist an element $x_l \in V_1$ and $x_l \notin X$, it is clear that x_l is not an internal vertex of any $u - v$ monophonic path, for some $u, v \in X$ and so X is not a monophonic set of G. Hence $m_r(G) = 4$. □

3. Some Realization Results

Theorem 3.1. *For any integer k such that $2 \leq k \leq p$ and $k \neq p - 1$, there is a connected graph G of order $p \geq 4$ such that $m_r(G) = k$.*

Proof. For $k = p$, the theorem follows from Theorem 2.11 and by taking $G = K_{1,p-1}$. For $2 \leq k \leq p - 2$. Let $P_3 : u_1, u_2, u_3$ be a path of order 3. Add $p - 3$ new vetices $v_1, v_2, \cdots, v_{k-2}, w_1, w_2, \cdots, w_{p-k-1}$ to P_3 by joining each $v_i (1 \leq i \leq k - 2)$ to u_2 and joining each $w_j (1 \leq j \leq p - k - 1)$ to u_1, u_2, u_3; and joining each $w_i (1 \leq i \leq p - k - 2)$ with $w_j (i + 1 \leq j \leq p - k - 1)$, thereby producing the graph G in Figure 3.1. Let $S = \{v_1, v_2, \cdots, v_{k-2}, u_1, u_3\}$ be the set of all extreme vertices of G. By Theorem 2.1, every restrained monophonic set of G contains S. It is clear that S is a monophonic set of G and the subgraph induced by S has no isolated vertex, S is the unique minimum restrained monophonic set of G and so $m_r(G) = k$. □

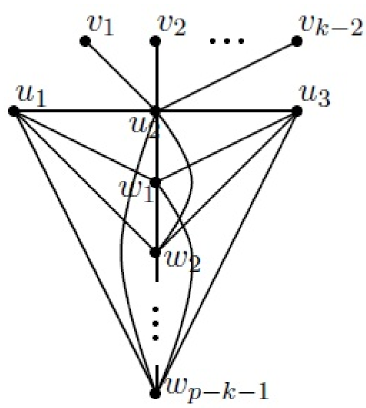

Figure 3.1: G

In view of Theorem 2.4, we have the following realization theorem.

Theorem 3.2. *If p, a and b are positive integers such that $2 \leq a \leq b \leq p - 3$, then there exists a connected graph G of order p, $m(G) = a$ and $m_r(G) = b$.*

Proof. We prove this theorem by considering four cases.
Case 1. $a = b = p - 3$. By Theorem 1.2 and Corollary 2.2, any tree of order p with three internal vertices has the desired properties.
Case 2. $a = b < p - 3$. For the graph G given in Figure 3.1 (put $a = b = k$), it is proved that, there is a connected graph of order p with $m(G) = m_r(G) = b$.
Case 3. $a < b = p - 3$. Let $P_3 : v_1, v_2, v_3$ be a path of order 3. Now, add $p - a - 3$ new vertices $u_1, u_2, \cdots, u_{a-2}, w_1, w_2, ..., w_{p-a-1}$ to P_3 by joining each $w_i (1 \le i \le p - a - 1)$ to v_1 and v_3 and joining each $u_j (1 \le j \le a - 2)$ to v_2, thereby producing the graph G of order p, which is shown in Figure 3.2. Let $S = \{u_1, u_2, \cdots, u_{a-2}\}$ be the set of all extreme vertices of G.

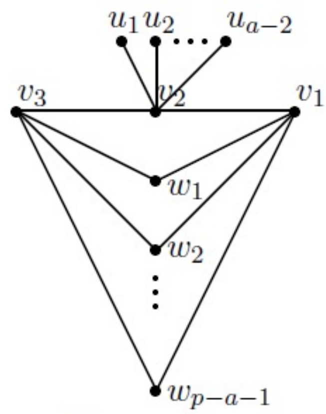

Figure 3.2: G

By Theorems 1.2 and 2.1, every monophonic set and every restrained monophonic set of G contain S. Clearly, S is not a monophonic set of G and also for any $x \in V(G) - S$, $S \cup \{x\}$ is not a monophonic set of G. Let $S_1 = S \cup \{v_1, v_3\}$. It is easily verified that S_1 is a monophonic set of G and so $m(G) = a$. Since the subgraph induced by $V - S_1$ has the isolated vertices $v_2, w_1, w_2, \cdots, w_{p-a-1}$, S_1 is not a restrained monophonic set of G. Observe that every restrained monophonic set of G contains $\{w_1, w_2, \cdots, w_{p-a-1}\}$. Let $S_2 = S \cup \{w_1, w_2, \cdots, w_{p-a-1}\}$. Clearly, S_2 is a monophonic set and the subgraph induced by $V - S_2$ has no isolated vertex, S_2 is a minimum restrained monophonic set of G and so $m_r(G) = b = p - 3$.

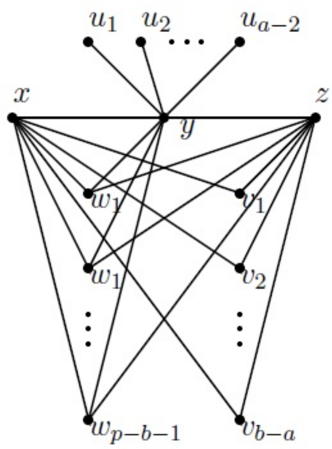

Figure 3.3: G

Case 4. $a < b < p - 3$. Let $P_3 = x, y, z$ be a path of order 3. Add $p - 3$ new vertices $u_1, u_2, \cdots, u_{a-2}, v_1, v_2, \ldots, v_{b-a}, w_1, w_2, \cdots, w_{p-b-1}$ to P_3 by joining each u_i $(1 \le i \le a-2)$

to the vertex y; joining each vertex w_j $(1 \leq j \leq p-b-1)$ to the vertices x, y and z; and joining each vertex v_k $(1 \leq k \leq b-a)$ to the vertices x and z, thereby producing the graph G of order p which shown in Figure 3.3. Let $S = \{u_1, u_2, \ldots, u_{a-2}\}$ be the set of endvertices of G. By Theorems 1.2 and 2.1, every monophonic set and every restrained monophonic set of G contain S. Also for any $u \in V(G) - S$, $S \cup \{u\}$ is not a monophonic and restrained monophonic set of G. Let $S' = S \cup \{x, z\}$. It is easily verified that S' is a monophonic set of G and so $m(G) = a$. Since the induced subgraph $G[V - S']$ has an isolated vertex, S' is not a restrained monophonic set of G. Clearly, every restrained monophonic set of G contains $\{v_1, v_2, \cdots, v_{b-a}\}$. Hence $S'' = S' \cup \{v_1, v_2, \cdots, v_{b-a}\}$ is a restrained monophonic set of G and so $m_r(G) = b$. □

In view of Theorem 2.6, we have the following realization theorem.

Theorem 3.3. *For every pair a, b of positive integers with $2 \leq a \leq b$, there is a connected graph G with $m_r(G) = a$ and $g_r(G) = b$.*

Proof. For $2 \leq a = b$, the graph $K_{1,a-1}$ of order a has the desired properties, by Theorems 1.1 and 2.11. So, assume that $2 \leq a < b$. Let $P_i : x_i, w_i, y_i$ $(1 \leq i \leq b-a)$ be $b-a$ copies of a path of length 2 and $P : v_1, v_2, v_3, v_4$ a path of length 3. Let G be the graph obtained by joining each $x_i (1 \leq i \leq b-a)$ in P_i and v_2 in P, joining each $y_i (1 \leq i \leq b-a)$ in P_i and v_4 in P; and adding $a-1$ new vertices $u_1, u_2, ..., u_{a-1}$ and joining each $u_i (1 \leq i \leq a-1)$ to v_4. The graph G is shown in Figure 3.4. Let $S = \{v_1, u_1, ..., u_{a-1}\}$ be the set of all extreme vertices of G. It is easily verified that S is a restrained monophonic set of G and so by Theorem 2.1, $m_r(G) = |S| = a$. Next, we show that $g_r(G) = b$. By Theorem 1.1, every restrained geodetic set of G contains S. Clearly, S is not a geodetic set of G. It is easily verified that at least one vertex of each $P_i (1 \leq i \leq b-a)$ must belong to every restrained geodetic set of G. Since $T = S \cup \{w_1, w_2, ..., w_{b-a}\}$ is a restrained geodetic set of G, it follows from Theorem 1.1 that T is a minimum restrained geodetic set of G and so $g_r(G) = b$. □

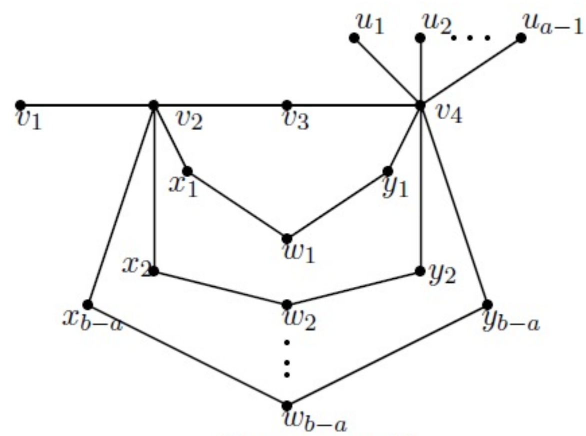

Figure 3.4: G

For any connected graph G, $rad_m(G) \leq diam_m(G)$. It is shown in [19] that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. This theorem can also be extended so that the restrained monophonic number can be prescribed when $rad_m(G) < diam_m(G)$.

Theorem 3.4. *For positive integers r, d and $k \geq 4$ with $r < d$, there exists a connected graph G such that $rad_m(G) = r$, $diam_m(G) = d$ and $m_r(G) = k$.*

Proof. We prove this theorem by considering two cases.

Case 1. $r = 1$. Then $d \geq 2$. Let $C_{d+2} : v_1, v_2, ..., v_{d+2}, v_1$ be a cycle of order $d + 2$. Let G be the graph obtained by adding $k - 2$ new vertices $u_1, u_2, ..., u_{k-2}$ to C_{d+2} and joining each of the vertices $u_1, u_2, ..., u_{k-2}, v_3, v_4, ..., v_{d+1}$ to the vertex v_1. The graph G is shown in Figure 3.5. It is easily verified that $1 \leq e_m(x) \leq d$ for any vertex x in G and $e_m(v_1) = 1, e_m(v_2) = d$. Then $rad_m(G) = 1$ and $diam_m(G) = d$. Let $S = \{u_1, u_2, ..., u_{k-2}, v_2, v_{d+2}\}$ be the set of all extreme vertices of G. Since S is a restrained monophonic set of G, it follows from Theorem 2.1 that $m_r(G) = k$.

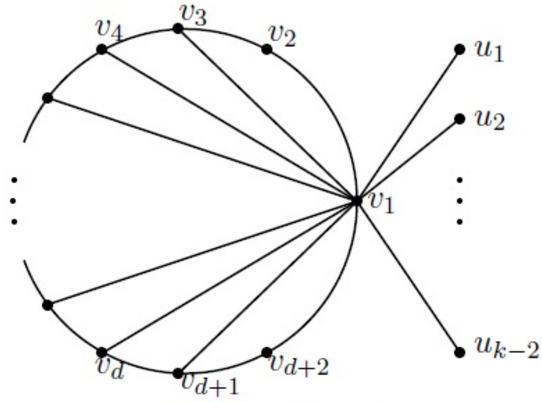

Figure 3.5: G

Case 2. $r \geq 2$. Let $C : v_1, v_2, ..., v_{r+2}, v_1$ be a cycle of order $r+2$ and let $W = K_1+C_{d+2}$ be the wheel with $V(C_{d+2}) = \{u_1, u_2, ..., u_{d+2}\}$, $K_1 = \{x\}$ and all other vertices distinct. Let H be the graph obtained from C and W by identifying v_1 of C and the central vertex K_1 of W. Now, add $k - 3$ new vertices $w_1, w_2, ..., w_{k-3}$ to the graph H and join each $w_i (1 \leq i \leq k - 3)$ to the vertex v_1 and obtain the graph G of Figure 3.6. It is easily verified that $r \leq e_m(x) \leq d$ for any vertex x in G and $e_m(v_1) = r$ and $e_m(u_1) = d$. Thus $rad_m(G) = r$ and $diam_m(G) = d$. Let $S = \{w_1, w_2, ..., w_{k-3}\}$ be the set of all extreme vertices of G. By Theorem 2.1, every restrained monophonic set of G contains S. It is clear that S is not a monophonic set of G. Let $T = S \bigcup \{u_1, u_3, v_3\}$. It is easily verified that T is a minimum restrained monophonic set of G and so $m_r(G) = k$. □

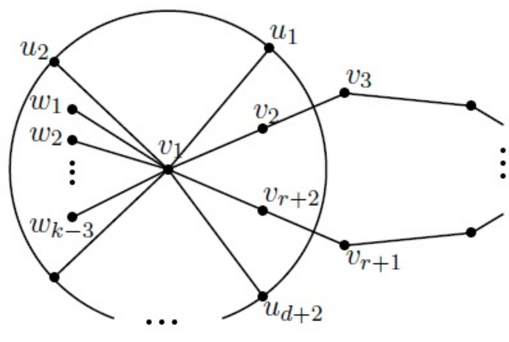

Figure 3.6: G

Problem 3.1. *For any three positive integers r, d and $k \geq 4$ with $r = d$, does there exist a connected graph G with $rad_m(G) = r$, $diam_m(G) = d$ and $m_r(G) = k$?*

Theorem 3.5. *For each triple d, k, p of integers with $2 \leq k \leq p - d + 1$ and $d \geq 2$, there is a connected graph G of order p such that $diam_m(G) = d$ and $m_r(G) = k$.*

Proof. Let $P_{d+1} : u_1, u_2, ..., u_{d+1}$ be a path of length d. Add $p - d - 1$ new vertices, $v_1, v_2, ..., v_{k-2}, w_1, w_2, ..., w_{p-d-k+1}$ to P_{d+1} and join each $w_i (1 \leq i \leq p - d - k + 1)$ to u_1, u_2 and u_3, and also join each $v_j (1 \leq j \leq k - 2)$ to u_2, thereby producing the graph G of Figure 3.7. Then G has order p and monophonic diameter d. If $p - d - k + 1 \leq 1$, then $S = \{v_1, v_2, ..., v_{k-2}, u_1, u_{d+1}\}$ is the set of all extreme vertices of G. Since S is a restrained monophonic set of G, it follows from Theorem 2.1 that $m_r(G) = k$. So, let $p - d - k + 1 \geq 2$. If $d = 2$, then $S_1 = \{v_1, v_2, ..., v_{k-2}\}$ is the set of all extreme vertices of G. It is clear that neither S_1 nor $S_1 \cup \{x\}$, where $x \notin S_1$, is a restrained monophonic set of G. Since $S_2 = S_1 \cup \{u_1, u_3\}$ is a restrained monophonic set of G, it follows from Theorem 2.1 that $m_r(G) = k$. If $d \geq 3$, then $S_3 = \{v_1, v_2, ..., v_{k-2}, u_{d+1}\}$ is the set of all extreme vertices of G. Now, S_3 is not a restrained monophonic set of G. Since $S_4 = S_3 \cup \{u_1\}$ is a restrained monophonic set of G, it follows from Theorem 2.1 that $m_r(G) = k$. □

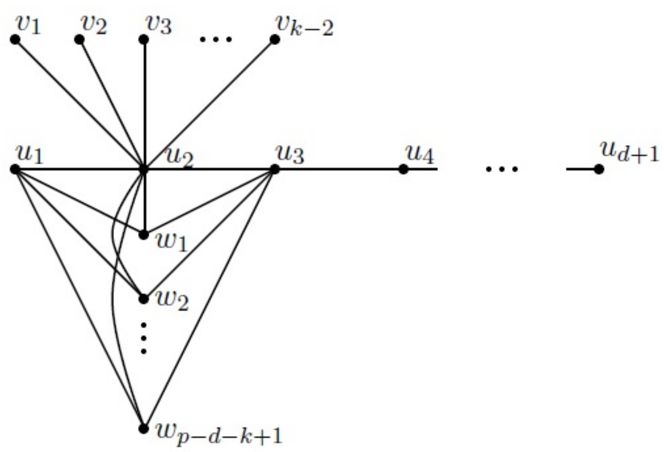

Figure 3.7: G

4. Conclusions

In this paper, the concept of restrained monophonic number of a graph is introduced and certain general properties satisfied by this parameter are studied. This parameter is determined for several standard graphs. Also, certain realization results of this parameter are proved with regard to the parameters monophonic number and restrained geodetic number of a graph. As a future work of this paper, new parameters like connected restrained monophonic number of a graph, forcing restrained monophonic number of a graph etc., can be developed and investigated.

References

[1] Abdollahzadeh Ahangar, H., (2017), Graphs with Large Total Geodetic Number, Filomat, 31(13), pp.4297-–4304.

[2] Abdollahzadeh Ahangar, H. and Najimi, M., (2017), Total Restrained Geodetic Number of Graphs. Iran. J. Sci. Technol. Trans. A Sci., 41(2), pp.473–480.

[3] Abdollahzadeh Ahangar, H., Fujie-Okamoto, F. and Samodivkin, V., (2016), On the Forcing Connected Geodetic Number and the Connected Geodetic Number of a Graph, Ars Combin., 126(3), pp.323–335.

[4] Abdollahzadeh Ahangar, H., Kousari, S., Sheikholeslami, S.M. and Volkmann, L., (2015), Graphs with Large Geodetic Number, Filomat, 29(6), pp.1361–1368.

[5] Abdollahzadeh Ahangar, H., Samodivkin, V., Sheikholeslami, S.M. and Abdollah Khodkar, (2015), The Restrained Geodetic Number of a Graph, Bull. Malays. Math. Sci. Soc., 38, pp.1143 -1155.

[6] Buckley, F. and Harary, F., (1990), Distance in Graphs, Addison-Wesley, Redwood City, CA.

[7] Chartrand, G., Harary, F. and Zhang, P., (2002), On the Geodetic Number of a Graph, Networks, 39(1), pp.1-6.
[8] Chartrand, G., Johns, G.L. and Zhang, P., (2004), On the Detour Number and Geodetic Number of a Graph, Ars Combin., 72, pp.3-15.
[9] Costa, E.R., Dourado, M.C. and Sampaio, R.M., (2015), Inapproximability Results Related to Monophonic Convexity, Discrete Appl. Math., 197, pp.70-74.
[10] Dourado, M.C., Protti, F. and Szwarcfiter, J.L., (2008), Algorithmic Aspects of Monophonic Convexity, Electron. Notes Discrete Math., 30, pp.177-182.
[11] Dourado, M.C., Protti, F. and Szwarcfiter, J.L., (2010), Complexity Results Related to Monophonic Convexity, Discrete Appl. Math., 158, pp.1268-1274.
[12] Ganesamoorthy, K. and Lakshmi Priya, S., (2020), The Outer Connected Monophonic Number of a Graph, Ars Combin., 153, pp.149-160.
[13] Ganesamoorthy, K. and Lakshmi Priya, S., (2021), Extreme Outer Connected Monophonic Graphs, Communications in Combinatorics and Optimization, DOI: 10.22049/CCO.2021.27042.1184.
[14] Ganesamoorthy, K. and Lakshmi Priya, S., (2021), Further Results on the Outer Connected Monophonic Number of a Graph, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.Mathematics, 41(4), pp.51-59.
[15] Ganesamoorthy, K., Murugan, M. and Santhakumaran, A.P., (2021), Extreme-Support Total Monophonic Graphs, Bull. Iran. Math. Soc., 47, pp.159-170.
[16] Harary, F., (1969), Graph Theory, Addison-Wesley.
[17] Harary, F., Loukakis, E. and Tsouros, C., (1993), The Geodetic Number of a Graph, Math. Comput. Modeling., 17(11), pp.87-95.
[18] Palugaa, E.M. and Canoy, S.R., (2007), Monophonic Numbers of the Join and Composition of Connected Graphs, Discrete Math., 307, pp. 1146-1154.
[19] Santhakumaran, A. P. and Titus, P., (2011), Monophonic Distance in Graphs, Discrete Mathematics, Algorithms and Applications, 3(2), pp. 159-169.
[20] Santhakumaran, A. P. and Titus, P., (2012), A Note on "Monophonic Distance in Graphs", Discrete Mathematics, Algorithms and Applications, 4(2), DOI: 10.1142/S1793830912500188.
[21] Santhakumaran, A.P., Titus, P. and Ganesamoorthy, K., (2014), On the Monophonic Number of a Graph, J. Appl. Math. & Informatics, 32(1-2), pp. 255 - 266.
[22] Santhakumaran, A.P., Titus, P., Ganesamoorthy, K. and Murugan, M., (2021), The Forcing Total Monophonic Number of a Graph, Proyecciones, 40(2), pp.561 - 571.
[23] Santhakumaran, A.P., Venkata Raghu, T. and Ganesamoorthy, K., (2021), Minimal Restrained Monophonic Sets in Graphs, TWMS J. App. and Eng. Math., 11(3), pp. 762-771.

A. P. Santhakumaran for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.6, N.1.

P. Titus for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.6, N.1.

K. Ganesamoorthy for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.6, N.1.