TWMS J. App. and Eng. Math. V.14, N.1, 2024, pp. 247-258

EXTREMAL POINTS FOR A $(n, \mathbf{p})$-TYPE RIEMANN–LIOUVILLE FRACTIONAL-ORDER BOUNDARY VALUE PROBLEMS

B. M. B. KRUSHNA[1], §

Abstract. The main objective of this work is to use the Krein–Rutman theorem to characterize extremal points for a $(n, \mathbf{p})$-type Riemann–Liouville fractional-order boundary value problem. The key premise is that a mapping from a linear, compact operator to its spectral radius, which depends on $\Im$, is continuous and strictly increasing as a function of $\Im$. A nonlinear problem is also treated as an application of the result for the linear case's extremal point.

Keywords: Fractional derivative, Boundary value problem, Extremal point.

AMS Subject Classification: 26A33, 34B08, 47A30.

Fractional-order differential equations (FDEqs) have emerged as an important tool for modeling a wide range of physical phenomena. Besides that, substantial progress has been achieved in the theory of fractional calculus [12, 13, 16, 19, 20, 22, 23]. The Krein-Rutman theorem [15] has been used to prove the existence of extremal points for second-order DEqs, higher-order DEqs, and systems of DEqs, see Coppel [1], Schmitt and Smith [24].

The existence of a nontrivial solution that lies in a cone is a standard approach for describing the extremal point of boundary value problems (BVPs), see [3, 8, 9, 10]. Cone theoretic arguments are applied to linear, monotone, compact operators that are developed to support the traditional Green's function technique. The sign properties of a Green's function, which exists to serve as the Kernel of the operators, are being used to show that the mapping preserves the cone. According to Kerin and Rootman's operator theory, the existence of the largest eigenvalues of the operator with the corresponding eigenfunction occurs in a cone. Eloe et al [3, 4, 7], and Eloe and Henderson [5, 6] extended these methods to a different BVPs. The authors recently worked on first extremal points (FEPs) for a variety of FBVPs [11, 25]. Neugebauer [17] investigated the classification of first extremal points for a FBVP. In [21], Prasad et al utilized the Guo–Krasnosel'skii fixed point theorem to determine the eigenvalue intervals for which the iterative system of (n, p)-type FBVP has at least one positive solution. Inspired and motivated by above works, in this article, we consider the FDEqs

$$\mathfrak{D}_{0+}^{\mathbf{q}}\varpi(t) + \mathsf{g}(t)\varpi(t) = 0, \quad t \in (0, \Im), \tag{1}$$

[1] Department of Mathematics, MVGR College of Engineering, Vizianagaram, 535 005, India.
e-mail: muraleebalu@yahoo.com; ORCID: https://orcid.org/ 0000-0001-7426-7451.

§ Manuscript received: January 17, 2022; accepted: April 18, 2022

associated with conditions

$$\left.\begin{aligned} &\varpi^{(i)}(0) = 0, \quad i = \overline{0, n-2}, \\ &\varpi^{(\mathbf{p})}(\Im) = 0, \end{aligned}\right\} \tag{2}$$

where $\mathbf{q} \in (n-1, n]$ for $n \geq 3$, $\mathbf{p} \in [1, \mathbf{q}-1]$ is a fixed integer, $\Im > 0$ and $\mathfrak{D}^{\mathbf{q}}_{0+}$ is the Riemann–Liouville derivative. The goal of this article is to prove the existence of a largest interval, $[0, \Im_0)$, s.t. on any subinterval $[\mathrm{m}_1, \mathrm{m}_2]$ of $[0, \Im_0)$, there is only one trivial solution of FBVP (1) and (2). The value $\Im_0$ is defined as the FEP of (1), which corresponds to the conditions (2). We'll refer to the FBVP($\Im$), (1) and (2), seeing as $\Im$ is a variable in this article.

Throughout the paper, we consider the following assumptions:

($\mathcal{H}_1$) $\Delta = \Gamma(\mathbf{q})\Im^{\mathbf{q}-\mathbf{p}-1}$.

($\mathcal{H}_2$) $\mathrm{g}(t)$ is a nonnegative continuous function and does not vanish identically on any compact subinterval of $[0, \infty)$.

($\mathcal{H}_3$) There exist two constants Υ and $\wp > 0$ s.t.
$$\left[\frac{\Gamma(\mathbf{q}-\mathbf{p})^2}{(2\mathbf{q}-2\mathbf{p})} + \frac{\Gamma(\mathbf{q})\Gamma(\mathbf{q}-\mathbf{p})}{(2\mathbf{q}-\mathbf{p})}\right]\frac{\wp^{\mathbf{q}}\Upsilon}{\Gamma(\mathbf{q})} = 1.$$

($\mathcal{H}_4$) There exists a constant $k \in [0, \mathbf{p}]$ s.t.
$$\frac{\Gamma(\mathbf{p}-k+2)}{\Gamma(\mathbf{p}-k+\mathbf{q})\Gamma(\mathbf{p}-\mathbf{q}-k+2)} < 1.$$

This article is organized as follows. Section 2 consists some auxiliary results. The main theorems are presented in Section 3, and Section 4 makes significant progress in discussing nonlinear eigenvalue problems for FBVPs using fixed point theory.

1. Auxiliary Results

Definition 1.1. We say $\Im_0$ is the FEP of the FBVP($\Im$), (1) and (2), if

$$\Im_0 = \inf\Big\{\Im > 0 \ : \ (1) \text{ and } (2) \text{ has a nontrivial solution}\Big\}.$$

Lemma 1.1. *Suppose that ($\mathcal{H}_1$) holds. If $\jmath(t) \in \mathcal{C}[0, \Im]$, then the FDEq*

$$D^{\boldsymbol{q}}_{0+}\varpi(t) + \jmath(t) = 0, \quad t \in (0, \Im), \tag{3}$$

with (2) *has a unique solution* $\varpi(t) = \displaystyle\int_0^{\Im} \aleph(\Im; t, \varrho)\jmath(\varrho)d\varrho$, *where*

$$\aleph(\Im; t, \varrho) = \frac{1}{\Delta}\begin{cases} t^{\boldsymbol{q}-1}(\Im-\varrho)^{\boldsymbol{q}-\boldsymbol{p}-1}, & 0 \leq t \leq \varrho \leq \Im, \\ t^{\boldsymbol{q}-1}(\Im-\varrho)^{\boldsymbol{q}-\boldsymbol{p}-1} - \Im^{\boldsymbol{q}-\boldsymbol{p}-1}(t-\varrho)^{\boldsymbol{q}-1}, & 0 \leq \varrho < t \leq \Im. \end{cases} \tag{4}$$

Proof. Let $\varpi(t) \in \mathcal{C}^{[\mathbf{q}]+1}[0, \Im]$ be a solution of FBVP (3), (2). Then (3) can be expressed as

$$\varpi(t) = \frac{-1}{\Gamma(\mathbf{q})}\int_0^t (t-\varrho)^{\mathbf{q}-1}\jmath(\varrho)d\varrho + k_1 t^{\mathbf{q}-1} + k_2 t^{\mathbf{q}-2} + \cdots + k_n t^{\mathbf{q}-\mathbf{p}-1}.$$

Using $\varpi^{(j)}(0) = 0$, $j = \overline{0, n-2}$, one has $k_n = k_{n-1} = \cdots = k_2 = 0$. Then

$$\begin{cases} \varpi(t) = \dfrac{-1}{\Gamma(\mathbf{q})}\displaystyle\int_0^t (t-\varrho)^{\mathbf{q}-1}\jmath(\varrho)d\varrho + k_1 t^{\mathbf{q}-1}, \\ \varpi^{(\mathbf{p})}(t) = k_1\displaystyle\prod_{i=1}^{\mathbf{p}}(\mathbf{q}-i)t^{\mathbf{q}-\mathbf{p}-1} - \prod_{i=1}^{\mathbf{p}}\frac{(\mathbf{q}-i)}{\Gamma(\mathbf{q})}\int_0^t (t-\varrho)^{\mathbf{q}-\mathbf{p}-1}\jmath(\varrho)d\varrho. \end{cases}$$

From $\varpi^{(\mathbf{p})}(\Im)=0$, we get $k_1=\int_0^{\Im}\left[\frac{(\Im-\varrho)^{\mathbf{q}-\mathbf{p}-1}}{\Gamma(\mathbf{q})\Im^{\mathbf{q}-\mathbf{p}-1}}\right]\jmath(\varrho)d\varrho$. Hence the unique solution of (3), (2) is

$$\begin{aligned}\varpi(t)&=\frac{-1}{\Gamma(\mathbf{q})}\int_0^t(t-\varrho)^{\mathbf{q}-1}\jmath(\varrho)d\varrho+\frac{t^{\mathbf{q}-1}}{\Gamma(\mathbf{q})}\int_0^{\Im}\frac{(\Im-\varrho)^{\mathbf{q}-\mathbf{p}-1}}{\Im^{\mathbf{q}-\mathbf{p}-1}}\jmath(\varrho)d\varrho\\&=\int_0^{\Im}\aleph(\Im;t,\varrho)\jmath(\varrho)d\varrho,\end{aligned}$$

where $\aleph(\Im;t,\varrho)$ is given in (4). □

Lemma 1.2. *Suppose that $(\mathcal{H}_1)$ holds. Then the Kernel $\aleph(\Im;t,\varrho)$ given by (4) has the properties*

$$(i)\ \aleph(\Im;t,\varrho)>0,\ \forall\ t,\varrho\in(0,\Im),$$
$$(ii)\ \frac{\partial}{\partial\Im}\Big\{\aleph(\Im;t,\varrho)\Big\}>0,\ \forall\ t,\varrho\in(0,\Im).$$

Proof. The Kernel $\aleph(\Im;t,\varrho)$ is given in (4). Let $0<t\le\varrho<\Im$. Then

$$\aleph(\Im;t,\varrho)=\frac{1}{\Delta}\Big[t^{\mathbf{q}-1}(\Im-\varrho)^{\mathbf{q}-\mathbf{p}-1}\Big]>\frac{1}{\Delta}\Big[t^{\mathbf{q}-1}(\Im-\varrho)^{-\mathbf{p}}\Big]\Im^{\mathbf{q}-1}(1-\varrho)^{\mathbf{q}-1}>0.$$

Let $0<\varrho\le t<\Im$. Then

$$\begin{aligned}\aleph(\Im;t,\varrho)&=\frac{1}{\Delta}\Big[t^{\mathbf{q}-1}(\Im-\varrho)^{\mathbf{q}-\mathbf{p}-1}-\Im^{\mathbf{q}-\mathbf{p}-1}(t-\varrho)^{\mathbf{q}-1}\Big]\\&\ge\begin{cases}\left[\frac{t^{\mathbf{q}-1}(\Im-\varrho)^{-\mathbf{p}}}{\Im^{1-\mathbf{q}}}\right]\left[1-\frac{(\mathbf{q}-1)\varrho}{\Im}+\frac{(\mathbf{q}-1)(\mathbf{q}-2)\varrho^2}{2\Im^2}+O(\varrho^3)\right]\\-\frac{t^{\mathbf{q}-1}}{\Im^{\mathbf{p}-\mathbf{q}+1}}\left[1-\frac{(\mathbf{q}-1)\varrho}{t}+\frac{(\mathbf{q}-1)(\mathbf{q}-2)\varrho^2}{2t^2}+O(\varrho^3)\right]\end{cases}\\&>0.\end{aligned}$$

Hence $\aleph(\Im;t,\varrho)>0$. Let $0<t\le\varrho<\Im$. Then

$$\frac{\partial}{\partial\Im}\Big\{\aleph(\Im;t,\varrho)\Big\}=\frac{\partial}{\partial\Im}\left\{\frac{t^{\mathbf{q}-1}(\Im-\varrho)^{\mathbf{q}-\mathbf{p}-1}}{\Gamma(\mathbf{q})\Im^{\mathbf{q}-\mathbf{p}-1}}\right\}=\frac{t^{\mathbf{q}-1}(\mathbf{q}-\mathbf{p}-1)\varrho(\Im-\varrho)^{\mathbf{q}-\mathbf{p}-2}}{\Delta\Im}>0.$$

Let $0<\varrho\le t<\Im$. Then

$$\begin{aligned}\frac{\partial}{\partial\Im}\Big\{\aleph(\Im;t,\varrho)\Big\}&=\frac{\partial}{\partial\Im}\left\{\frac{t^{\mathbf{q}-1}(\Im-\varrho)^{\mathbf{q}-\mathbf{p}-1}}{\Gamma(\mathbf{q})\Im^{\mathbf{q}-\mathbf{p}-1}}-\frac{(t-\varrho)^{\mathbf{q}-1}}{\Gamma(\mathbf{q})}\right\}\\&=\frac{t^{\mathbf{q}-1}(\mathbf{q}-\mathbf{p}-1)\varrho(\Im-\varrho)^{\mathbf{q}-\mathbf{p}-2}}{\Delta\Im}>0.\end{aligned}$$

Hence $\frac{\partial}{\partial\Im}\Big\{\aleph(\Im;t,\varrho)\Big\}>0$. □

Let us define

$$\aleph(\Im;t,\varrho)=t^{\mathbf{q}-\mathbf{p}-1}\mathfrak{K}(\Im;t,\varrho),$$

where

$$\mathfrak{K}(\Im;t,\varrho)=\frac{1}{\Delta}\begin{cases}t^{\mathbf{p}}(\Im-\varrho)^{\mathbf{q}-\mathbf{p}-1}, & 0\le t\le\varrho\le\Im,\\t^{\mathbf{p}}(\Im-\varrho)^{\mathbf{q}-\mathbf{p}-1}-\Big(\frac{t}{\Im}\Big)^{\mathbf{p}-\mathbf{q}+1}(t-\varrho)^{\mathbf{q}-1}, & 0\le\varrho\le t\le\Im.\end{cases}\qquad(5)$$

Lemma 1.3. *Suppose that* $(\mathcal{H}_1)$ *holds. Then the Kernel* $\mathfrak{K}(\Im; t, \varrho)$ *given by* (5) *has the properties*

$$(i)\ \mathfrak{K}(\Im; t, \varrho) > 0,\ \ for\ t, \varrho \in (0, \Im).$$
$$(ii)\ \frac{\partial}{\partial \Im}\Big\{\mathfrak{K}(\Im; t, \varrho)\Big\} > 0,\ \ for\ t, \varrho \in (0, \Im).$$
$$(iii)\ \mathfrak{K}(\Im; 0, \varrho) = 0,\ \ for\ \varrho \in (0, \Im).$$
$$(iv)\ \frac{\partial^i}{\partial t^i}\Big\{\mathfrak{K}(\Im; 0, \varrho)\Big\} = 0,\ \ i = \overline{1, n-2}.$$
$$(v)\ \frac{\partial^p}{\partial t^p}\Big\{\mathfrak{K}(\Im; 0, \varrho)\Big\} > 0,\ \ for\ \varrho \in (0, \Im).$$
$$(vi)\ \frac{\partial}{\partial \Im}\left[\frac{\partial^p}{\partial t^p}\Big\{\mathfrak{K}(\Im; 0, \varrho)\Big\}\right] > 0,\ \ for\ \varrho \in (0, \Im).$$

Proof. The Kernel $\mathfrak{K}(\Im; t, \varrho)$ is given in (5). For $0 < t \le \varrho < \Im$,

$$\mathfrak{K}(\Im; t, \varrho) = \frac{1}{\Delta}\Big[t^{\mathbf{p}}(\Im - \varrho)^{\mathbf{q}-\mathbf{p}-1}\Big] > \frac{1}{\Delta}\Big[t^{\mathbf{p}}(\Im - \varrho)^{-\mathbf{p}}\Big]\Im^{\mathbf{q}-1}(1-\varrho)^{\mathbf{q}-1} > 0.$$

For $0 < \varrho \le t < \Im$,

$$\begin{aligned}
\mathfrak{K}(\Im; t, \varrho) &= \frac{1}{\Delta}\left[t^{\mathbf{p}}(\Im - \varrho)^{\mathbf{q}-\mathbf{p}-1} - \left(\frac{t}{\Im}\right)^{\mathbf{p}-\mathbf{q}+1}(t-\varrho)^{\mathbf{q}-1}\right] \\
&= \frac{1}{\Delta}\left\{\left(1 - \frac{\varrho}{\Im}\right)^{\mathbf{q}-1}\left[\frac{t^{\mathbf{p}}(\Im-\varrho)^{-\mathbf{p}}}{\Im^{1-\mathbf{q}}}\right] - \left(1 - \frac{\varrho}{t}\right)^{\mathbf{q}-1}\left[\frac{t^{\mathbf{p}}}{\Im^{\mathbf{p}-\mathbf{q}+1}}\right]\right\} \\
&\ge \left\{\begin{array}{l}
\left[1 - \frac{(\mathbf{q}-1)\varrho}{\Im} + \frac{(\mathbf{q}-1)(\mathbf{q}-2)\varrho^2}{2\Im^2} + O(\varrho^3)\right]\left[\frac{t^{\mathbf{p}}(\Im-\varrho)^{-\mathbf{p}}}{\Im^{1-\mathbf{q}}}\right] \\
-\left[1 - \frac{(\mathbf{q}-1)\varrho}{t} + \frac{(\mathbf{q}-1)(\mathbf{q}-2)\varrho^2}{2t^2} + O(\varrho^3)\right]\left[\frac{t^{\mathbf{p}}}{\Im^{\mathbf{p}-\mathbf{q}+1}}\right]
\end{array}\right. \\
&> 0.
\end{aligned}$$

Hence $\mathfrak{K}(\Im; t, \varrho) > 0$. Let $0 < t \le \varrho < \Im$. Then

$$\begin{aligned}
\frac{\partial}{\partial \Im}\Big\{\mathfrak{K}(\Im; t, \varrho)\Big\} = \frac{\partial}{\partial \Im}\left\{\frac{t^{\mathbf{p}}(\Im - \varrho)^{\mathbf{q}-\mathbf{p}-1}}{\Gamma(\mathbf{q})\Im^{\mathbf{q}-\mathbf{p}-1}}\right\} &= \left[\frac{t^{\mathbf{p}}(\mathbf{q}-\mathbf{p}-1)}{\Gamma(\mathbf{q})}\right]\frac{\varrho(\Im-\varrho)^{\mathbf{q}-\mathbf{p}-2}}{\Im^{\mathbf{q}-\mathbf{p}}} \\
&= \frac{(\mathbf{q}-\mathbf{p}-1)t^{\mathbf{p}}\varrho(\Im-\varrho)^{\mathbf{q}-\mathbf{p}-2}}{\Delta\Im} > 0.
\end{aligned}$$

Let $0 < \varrho \le t < \Im$. Then

$$\begin{aligned}
\frac{\partial}{\partial \Im}\Big\{\mathfrak{K}(\Im; t, \varrho)\Big\} &= \frac{\partial}{\partial \Im}\left\{\frac{t^{\mathbf{p}}(\Im - \varrho)^{\mathbf{q}-\mathbf{p}-1}}{\Gamma(\mathbf{q})\Im^{\mathbf{q}-\mathbf{p}-1}} - \frac{t^{\mathbf{p}}}{\Gamma(\mathbf{q})}\left(\frac{t-\varrho}{t}\right)^{\mathbf{q}-1}\right\} \\
&= \frac{(\mathbf{q}-\mathbf{p}-1)t^{\mathbf{p}}\varrho(\Im-\varrho)^{\mathbf{q}-\mathbf{p}-2}}{\Delta\Im} > 0.
\end{aligned}$$

Hence $\dfrac{\partial}{\partial \Im}\Big\{\mathfrak{K}(\Im; t, \varrho)\Big\} > 0$. We can easily establish the properties (iii) and (iv) utilizing simple algebraic calculations. Let $0 < t \leq \varrho < \Im$. Then

$$\frac{\partial^{\mathbf{p}}}{\partial t^{\mathbf{p}}}\Big\{\mathfrak{K}(\Im; t, \varrho)\Big\} = \left\{ \begin{array}{c} \dfrac{p!(\Im-\varrho)^{\mathbf{q}-\mathbf{p}-1}}{\Delta} - \displaystyle\sum_{k=0}^{\mathbf{p}} \left(\begin{array}{c}\mathbf{p}\\ k\end{array}\right) t^{\mathbf{p}-\mathbf{q}-k+1} \times \\ \dfrac{\Gamma(p-k+2)}{\Gamma(\mathbf{p}-\mathbf{q}-k+2)} \dfrac{(t-\varrho)^{\mathbf{p}+\mathbf{q}-k-1}}{\Gamma(\mathbf{p}-k+\mathbf{q})} \end{array} \right.$$
$$> 0.$$

Hence $\dfrac{\partial^{\mathbf{p}}}{\partial t^{\mathbf{p}}}\Big\{\mathfrak{K}(\Im; t, \varrho)\Big\} > 0$. Finally, Let $0 < \varrho \leq t < \Im$. Then

$$\frac{\partial}{\partial \Im}\left[\frac{\partial^{\mathbf{p}}}{\partial t^{\mathbf{p}}}\Big\{\mathfrak{K}(\Im; 0, \varrho)\Big\}\right] = \frac{\partial}{\partial \Im}\left[\frac{p!(\Im-\varrho)^{\mathbf{q}-\mathbf{p}-1}}{\Gamma(\mathbf{q})\Im^{\mathbf{q}-\mathbf{p}-1}}\right] = \frac{(\mathbf{q}-\mathbf{p}-1)(\mathbf{p})!(\Im-\varrho)^{\mathbf{q}-\mathbf{p}-2}\varrho}{\Gamma(\mathbf{q})\Im^{\mathbf{q}-\mathbf{p}}} > 0.$$

Let $0 < \varrho \leq t < \Im$. Then

$$\frac{\partial}{\partial \Im}\left[\frac{\partial^{\mathbf{p}}}{\partial t^{\mathbf{p}}}\Big\{\mathfrak{K}(\Im; 0, \varrho)\Big\}\right] = \frac{(\mathbf{q}-\mathbf{p}-1)(\mathbf{p})!(\Im-\varrho)^{\mathbf{q}-\mathbf{p}-2}\varrho}{\Gamma(\mathbf{q})\Im^{\mathbf{q}-\mathbf{p}}} > 0.$$

□

The following are the results of our extremal point analysis.

Theorem 1.1. [14] *Let $N : B \to B$ be a compact and positive linear operator. Then N has an essentially unique eigenvector in P, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.*

Theorem 1.2. [18] *Let $N_{\Im}$, $\nu \leq \Im \leq \vartheta$ be a family of compact, linear operators on Banach space s.t. the mapping $\Im \mapsto N_{\Im}$ is continuous in the uniform operator topology. Then the mapping $\Im \mapsto \mathfrak{r}\big(N_{\Im}\big)$ is continuous.*

Theorem 1.3. [14, 15] *Assume $\mathfrak{r}(N) > 0$. Then $\mathfrak{r}(N)$ is an eigenvalue of N, and there is a corresponding eigenvalue in P.*

Theorem 1.4. [14, 15] *Suppose there exists $\Im > 0$, $\varpi \in B$, $-\varpi \notin P$, s.t. $\Im\varpi \preceq N\varpi$ w.r.t. P. Then N has an eigenvector in P which corresponding to an eigenvalue χ with $\chi \geq \Im$.*

2. Main Results

Let $\mathtt{B} = \Big\{\varpi \in \mathcal{C}[0, \Im] : \varpi = t^{\mathbf{q}-\mathbf{p}-1}z,\ z \in \mathcal{C}[0, \Im]\Big\}$ be the Banach space with the norm

$$\|\varpi\| = \sup_{t \in [0, \Im]} |z(t)| = |z|_0.$$

Define a cone $\mathtt{P} \subset \mathtt{B}$ by

$$\mathtt{P} = \Big\{\varpi \in \mathtt{B} : \varpi(t) \geq 0 \text{ for } t \in [0, \Im]\Big\}.$$

Next, for each $\vartheta > 0$, define the Banach space

$$\mathtt{B}_{\vartheta} = \Big\{\varpi \in \mathcal{C}[0, \vartheta] \ : \ \varpi = t^{\mathbf{q}-\mathbf{p}-1}z, z \in \mathcal{C}[0, \vartheta]\Big\}$$

with the norm

$$\|\varpi\|_{\vartheta} = \sup_{t \in [0, \vartheta]} \big|z(t)\big| = \big|z\big|_{*}.$$

Notice that for $\varpi \in \mathtt{B}_{\vartheta}$, we have

$$\big|\varpi(t)\big| = \big|t^{\mathbf{q}-\mathbf{p}-1}z(t)\big| \leq t^{\mathbf{q}-\mathbf{p}-1}\|\varpi\|_{\vartheta},\ t \in [0, \vartheta].$$

This also gives the inequality

$$\big|\varpi(t)\big| \leq \vartheta^{\mathbf{q}-\mathbf{p}-1}\|\varpi\|_\vartheta,\ t\in[0,\vartheta].$$

For each $\vartheta>0$, define the cone $\mathtt{P}_\vartheta\subset\mathtt{B}_\vartheta$ to be

$$\mathtt{P}_\vartheta=\Big\{\varpi\in\mathtt{B}_\vartheta:\varpi(t)\geq 0 \text{ for } t\in[0,\vartheta]\Big\}.$$

Lemma 2.1. *The cone $\mathtt{P}_\vartheta$ is solid in $\mathtt{B}_\vartheta$ and hence reproducing.*

Proof. Set $\Omega_\vartheta=\Big\{\varpi=t^{\mathbf{q}-\mathbf{p}-1}z\in\mathtt{B}_\vartheta:\varpi(t)>0 \text{ for } t\in(0,\vartheta],z(0)>0\Big\}$. We show that $\Omega_\vartheta\subset\mathtt{P}^\circ_\vartheta$. Let $\varpi\in\Omega_\vartheta$. Then there exists an $\zeta_1>0$ s.t. $z(0)-\zeta_1>0$ since $z(0)>0$. For $z\in\mathcal{C}[0,\vartheta]$, there exists a $\mathrm{m}_1\in(0,\vartheta)$ s.t. $z(t)>\zeta_1$, $t\in(0,\mathrm{m}_1)$. Therefore, $\varpi(t)=t^{\mathbf{q}-\mathbf{p}-1}z(t)>\zeta_1 t^{\mathbf{q}-\mathbf{p}-1}$ for $t\in(0,\mathrm{m}_1)$. Furthermore, $\varpi(t)>0$ on $[\mathrm{m}_1,\vartheta]$. Thus there exists an $\zeta_2>0$ s.t. $\varpi(t)>\zeta_2$, $\forall t\in[\mathrm{m}_1,\vartheta]$.

Let $\zeta=\min\left\{\frac{\zeta_1}{2},\frac{\zeta_2}{2}\right\}$. Define $\mathtt{B}_\zeta(\varpi)=\Big\{\hat{\varpi}\in\mathcal{B}_\vartheta:\|\varpi-\hat{\varpi}\|_\vartheta<\zeta\Big\}$. Let $\hat{\varpi}\in\mathtt{B}_\zeta(\varpi)$, then $\hat{\varpi}=t^{\mathbf{q}-\mathbf{p}-1}\hat{z}$, where $\hat{z}\in\mathcal{C}[0,\vartheta]$. Now,

$$\big|\hat{\varpi}(t)-\varpi(t)\big|\leq t^{\mathbf{q}-\mathbf{p}-1}\|\hat{\varpi}-\varpi\|_\vartheta<t^{\mathbf{q}-\mathbf{p}-1}\zeta,\quad t\in[0,\vartheta].$$

So for $t\in(0,\mathrm{m}_1)$,

$$\hat{\varpi}(t)>\varpi(t)-t^{\mathbf{q}-\mathbf{p}-1}\zeta>t^{\mathbf{q}-\mathbf{p}-1}\zeta_1-\frac{1}{2}t^{\mathbf{q}-\mathbf{p}-1}\zeta_1=\frac{1}{2}t^{\mathbf{q}-\mathbf{p}-1}\zeta_1>0.$$

Consequently, we attain $\big|\hat{\varpi}(t)-\varpi(t)\big|\leq\|\hat{\varpi}-\varpi\|_\vartheta<\zeta$. For $t\in[\mathrm{m}_1,\vartheta]$,

$$\hat{\varpi}(t)>\varpi(t)-\zeta>\Big(\zeta_2-\frac{\zeta_2}{2}\Big)=\frac{\zeta_2}{2}>0.$$

Therefore $\hat{\varpi}\in\mathtt{P}_\vartheta$, and thus $\mathtt{B}_\zeta(\varpi)\subset\mathtt{P}_\vartheta$. Hence $\Omega_\vartheta\subset\mathtt{P}^\circ_\vartheta$. □

Next, let $\mathtt{N}_0\varpi(t)\equiv 0$, $t\in[0,\Im]$, and for each $\vartheta>0$, define $\mathtt{N}_\vartheta:\mathtt{B}\to\mathtt{B}$ by

$$\mathtt{N}_\vartheta\varpi(t)=\begin{cases}\int_0^\vartheta \aleph(\vartheta;t,\varrho)\mathtt{g}(\varrho)\varpi(\varrho)d\varrho, & 0\leq t\leq\vartheta,\\ \int_0^\vartheta t^{\mathbf{q}-\mathbf{p}-1}\mathfrak{K}(\vartheta;\vartheta,\varrho)\mathtt{g}(\varrho)\varpi(\varrho)d\varrho, & \vartheta\leq t\leq\Im.\end{cases}\tag{6}$$

We shall also refer to $\mathtt{N}_\vartheta:\mathtt{B}_\vartheta\to\mathtt{B}_\vartheta$, where $\mathtt{N}_\vartheta$ is represented by

$$\begin{aligned}\mathtt{N}_\vartheta\varpi(t)&=\int_0^\vartheta\aleph(\vartheta;t,\varrho)\mathtt{g}(\varrho)\varpi(\varrho)d\varrho\\&=t^{\mathbf{q}-\mathbf{p}-1}\int_0^\vartheta\mathfrak{K}(\vartheta;t,\varrho)\mathtt{g}(\varrho)\varpi(\varrho)d\varrho,\quad 0\leq t\leq\vartheta.\end{aligned}$$

By applying the methods used in [9], we establish a result for the possibility of the extremal point $\Im_0$ for FBVP($\Im$), (1) and (2).

Theorem 2.1. *Assume that $(\mathcal{H}_1)$-$(\mathcal{H}_4)$ hold. Then the FBVP(ϑ),* (1) *and* (2) *has a unique solution for $\vartheta\in(0,\wp)$. In particular, if $\vartheta\geq\wp$, then $\varpi\equiv 0$ is the only solution of FBVP(ϑ),* (1) *and* (2).

Proof. Let $\Upsilon=\max\limits_{t\in[0,\vartheta]}|\mathtt{g}(t)|$. We utilize the contraction mapping principle to prove the existence of a $\wp>0$, s.t. if $\vartheta\in(0,\wp)$, FBVP(ϑ), (1) and (2) has a unique solution. Let

$\varpi_1, \varpi_2 \in \mathtt{B}_\vartheta$ and consider

$$\Big(\mathtt{N}_\vartheta\varpi_2 - \mathtt{N}_\vartheta\varpi_1\Big)(t) = \begin{cases} t^{\mathbf{q}-\mathbf{p}-1}\Bigg(\displaystyle\int_0^\vartheta \frac{t^{\mathbf{p}}(\vartheta-\varrho)^{\mathbf{q}-\mathbf{p}-1}}{\Gamma(\mathbf{q})\vartheta^{\mathbf{q}-\mathbf{p}-1}}\mathtt{g}(\varrho)(\varpi_2-\varpi_1)(\varrho)d\varrho \\ \qquad -\displaystyle\int_0^t \frac{(t-\varrho)^{\mathbf{q}-1}}{\Gamma(\mathbf{q})t^{\mathbf{q}-\mathbf{p}-1}}\mathtt{g}(\varrho)(\varpi_2-\varpi_1)(\varrho)d\varrho\Bigg). \end{cases}$$

Set

$$z(t) = \begin{cases} \displaystyle\int_0^\vartheta \frac{t^{\mathbf{p}}(\vartheta-\varrho)^{\mathbf{q}-\mathbf{p}-1}}{\Gamma(\mathbf{q})\vartheta^{\mathbf{q}-\mathbf{p}-1}}\mathtt{g}(\varrho)(\varpi_2-\varpi_1)(\varrho)d\varrho \\ \qquad -\displaystyle\int_0^t \frac{(t-\varrho)^{\mathbf{q}-1}}{\Gamma(\mathbf{q})t^{\mathbf{q}-\mathbf{p}-1}}\mathtt{g}(\varrho)(\varpi_2-\varpi_1)(\varrho)d\varrho. \end{cases}$$

Therefore, $\big\|\mathtt{N}_\vartheta\varpi_2 - \mathtt{N}_\vartheta\varpi_1\big\|_\vartheta = \big|z\big|_0$. For $t \in (0,\vartheta)$,

$$\begin{aligned} \big|z(t)\big| &= \begin{cases} \Bigg|\displaystyle\int_0^\vartheta \frac{t^{\mathbf{p}}(\vartheta-\varrho)^{\mathbf{q}-\mathbf{p}-1}}{\Gamma(\mathbf{q})\vartheta^{\mathbf{q}-\mathbf{p}-1}}\mathtt{g}(\varrho)(\varpi_2-\varpi_1)(\varrho)d\varrho \\ \qquad -\displaystyle\int_0^t \frac{(t-\varrho)^{\mathbf{q}-1}}{\Gamma(\mathbf{q})t^{\mathbf{q}-\mathbf{p}-1}}\mathtt{g}(\varrho)(\varpi_2-\varpi_1)(\varrho)d\varrho\Bigg| \end{cases} \\ &\le \begin{cases} \dfrac{\vartheta^{\mathbf{p}}\Upsilon}{\Gamma(\mathbf{q})\vartheta^{\mathbf{q}-\mathbf{p}-1}}\|\varpi_2-\varpi_1\|_\vartheta\displaystyle\int_0^\vartheta \Big[\varrho(\vartheta-\varrho)\Big]^{\mathbf{q}-\mathbf{p}-1}d\varrho \\ \qquad +\dfrac{\Upsilon t^{\mathbf{p}-\mathbf{q}+1}}{\Gamma(\mathbf{q})}\|\varpi_2-\varpi_1\|_\vartheta\displaystyle\int_0^t \varrho^{\mathbf{q}-\mathbf{p}-1}(t-\varrho)^{\mathbf{q}-1}d\varrho \end{cases} \\ &\le \left[\frac{\vartheta^{\mathbf{q}}\Upsilon\Gamma(\mathbf{q}-\mathbf{p})^2}{\Gamma(\mathbf{q})(2\mathbf{q}-2\mathbf{p})} + \frac{\Upsilon\vartheta^{\mathbf{q}}\Gamma(\mathbf{q})\Gamma(\mathbf{q}-\mathbf{p})}{\Gamma(\mathbf{q})(2\mathbf{q}-\mathbf{p})}\right]\|\varpi_2-\varpi_1\|_\vartheta. \end{aligned}$$

Choose $\wp > 0$ s.t. $\left[\dfrac{\Gamma(\mathbf{q}-\mathbf{p})^2}{(2\mathbf{q}-2\mathbf{p})} + \dfrac{\Gamma(\mathbf{q})\Gamma(\mathbf{q}-\mathbf{p})}{(2\mathbf{q}-\mathbf{p})}\right]\dfrac{\wp^{\mathbf{q}}\Upsilon}{\Gamma(\mathbf{q})} = 1$. As a result, if $0 < \vartheta < \wp$, $\mathtt{N}_\vartheta$ is a contraction map with a unique fixed point according to the contraction mapping principle. This fixed point is a solution to FBVP(ϑ), (1) and (2). However, $\varpi \equiv 0$ is a solution of FBVP(ϑ), (1) and (2), so FBVP(ϑ), (1) and (2) has only the trivial solution. □

Lemma 2.2. *Assume that $(\mathcal{H}_1)$-$(\mathcal{H}_4)$ hold. The linear operator $\mathtt{N}_\vartheta$ is positive w.r.t. $\mathtt{P}$ and $\mathtt{P}_\vartheta$ for each $\vartheta > 0$. Furthermore, $\mathtt{N}_\vartheta : \mathtt{P}_\vartheta \setminus \{0\} \to \mathtt{P}_\vartheta^\circ$.*

Proof. The sign properties of the Kernels $\aleph$ and $\mathfrak{K}$ yield a straightforward result of the positivity of $\mathtt{N}_\vartheta$ w.r.t. $\mathtt{P}$ and $\mathtt{P}_\vartheta$. We will clearly show that $\mathtt{N}_\vartheta : \mathtt{P}_\vartheta \setminus \{0\} \to \mathtt{P}_\vartheta^\circ$. From Lemma 2.1, we have $\Omega_\vartheta \subset \mathtt{P}_\vartheta^\circ$. Later, we prove that $\mathtt{N}_\vartheta : \mathtt{P}_\vartheta \setminus \{0\} \to \Omega_\vartheta$.

Let $\varpi \in \mathtt{P}_\vartheta \setminus \{0\}$, then there exists $[\mathrm{m}_1, \mathrm{m}_2] \subset [0,\vartheta]$ s.t. $\mathtt{g}(t) > 0$ and $\varpi(t) > 0$ for all $t \in [\mathrm{m}_1, \mathrm{m}_2]$. So

$$\mathtt{N}_\vartheta\varpi(t) = \int_0^\vartheta \aleph(\vartheta; t, \varrho)\mathtt{g}(\varrho)\varpi(\varrho)d\varrho \ge \int_{\mathrm{m}_1}^{\mathrm{m}_2} \aleph(\vartheta; t, \varrho)\mathtt{g}(\varrho)\varpi(\varrho)d\varrho > 0, \quad \forall\ t \in (0,\vartheta).$$

Note $z(t) = t^{\mathbf{q}-\mathbf{p}-1}\int_0^\vartheta \mathfrak{K}(\vartheta; t, \varrho)\mathtt{g}(\varrho)\varpi(\varrho)d\varrho$, we have

$$z(\vartheta) = \vartheta^{\mathbf{q}-\mathbf{p}-1}\int_0^\vartheta \mathfrak{K}(\vartheta; \vartheta, \varrho)\mathtt{g}(\varrho)\varpi(\varrho)d\varrho \ge \vartheta^{\mathbf{q}-\mathbf{p}-1}\int_{\mathrm{m}_1}^{\mathrm{m}_2} \mathfrak{K}(\vartheta; \vartheta, \varrho)\mathtt{g}(\varrho)\varpi(\varrho)d\varrho > 0.$$

Notice from $(\mathcal{H}_4)$ and Lemma 1.3 that, we have

$$z^{(n-1)}(0) = \int_0^\vartheta \frac{\partial^{n-1}}{\partial t^{n-1}}\Big\{\mathfrak{K}(\vartheta; 0, \varrho)\Big\}\mathtt{g}(\varrho)\varpi(\varrho)d\varrho > 0.$$

Thus, $\mathtt{N}_\vartheta\varpi \in \Omega_\vartheta$ and $\mathtt{N}_\vartheta : \mathtt{P}_\vartheta \setminus \{0\} \to \mathtt{P}^\circ_\vartheta$. □

Lemma 2.3. *Assume that $(\mathcal{H}_1)$-$(\mathcal{H}_4)$ hold. The mapping $\vartheta \mapsto \mathfrak{r}(N_\vartheta)$ with N_ϑ defined on B for each $\vartheta \in (0, \Im]$ is continuous in the uniform topology.*

Proof. Define $f : (0, \Im] \to \{\mathtt{N}_\vartheta\}$ by $\mathtt{f}(\vartheta) = \mathtt{N}_\vartheta$. Assume $\varpi = t^{\mathbf{q}-\mathbf{p}-1}z \in \mathtt{B}$. Let $0 < \mathrm{m}_1 < \mathrm{m}_2 \leq \Im$. Then

$$\begin{aligned}\|\mathtt{f}(\mathrm{m}_2)-\mathtt{f}(\mathrm{m}_1)\| &= \|\mathtt{N}_{\mathrm{m}_2} - \mathtt{N}_{\mathrm{m}_1}\| = \sup_{\|\varpi\|=1} \|\mathtt{N}_{\mathrm{m}_2}\varpi - \mathtt{N}_{\mathrm{m}_1}\varpi\| \\ &= \sup_{\|\varpi\|=1}\left\{\sup_{t\in[0,\Im]}\left|\int_0^{\mathrm{m}_2} \mathfrak{K}(\mathrm{m}_2; t, \varrho)\mathrm{g}(\varrho)\varpi(\varrho)d\varrho - \int_0^{\mathrm{m}_1} \mathfrak{K}(\mathrm{m}_1; t, \varrho)\mathrm{g}(\varrho)\varpi(\varrho)d\varrho\right|\right\},\end{aligned}$$

where $\mathtt{N}_\vartheta\varpi(t) = t^{\mathbf{q}-\mathbf{p}-1}\mathfrak{K}_\vartheta\varpi(t)$. Notice if $\|\varpi\| = 1$, then $|\varpi(t)| \leq \vartheta^{\mathbf{q}-\mathbf{p}-1}$ for $t \in [0, \vartheta]$. Let $\Upsilon = \max_{t\in[0,\Im]} |\mathrm{g}(t)|$. Since $\mathfrak{K}(\vartheta; t, \varrho)$ is continuous w.r.t. ϑ, for $\zeta > 0$ there exists $\wp > 0$ s.t. $|\mathfrak{K}(\mathrm{m}_2; t, \varrho) - \mathfrak{K}(\mathrm{m}_1; t, \varrho)| < \dfrac{\zeta}{2\mathrm{m}_1^{\mathbf{q}-\mathbf{p}-1}\Upsilon}$ whenever $|\mathrm{m}_2 - \mathrm{m}_1| < \wp$. Now we shall discuss in three cases.

Case 1. Suppose $t \leq \mathrm{m}_1$. Let $\mathfrak{K}_1 = \sup_{t\in[0,\mathrm{m}_1],\varrho\in[\mathrm{m}_1,\mathrm{m}_2]} \left|\mathfrak{K}(\mathrm{m}_2; t, \varrho)\right|$. Choose $\wp = \dfrac{\zeta}{2\mathfrak{K}_1\Upsilon m_2^{\mathbf{q}-\mathbf{p}-1}}$. Then

$$\begin{aligned}&\left|\int_0^{\mathrm{m}_2} \mathfrak{K}(\mathrm{m}_2; t, \varrho)\mathrm{g}(\varrho)\varpi(\varrho)d\varrho - \int_0^{\mathrm{m}_1} \mathfrak{K}(\mathrm{m}_1; t, \varrho)\mathrm{g}(\varrho)\varpi(\varrho)d\varrho\right| \\ &\leq \left\{\begin{array}{l}\displaystyle\int_0^{\mathrm{m}_1} \left|\mathfrak{K}(\mathrm{m}_2; t, \varrho) - \mathfrak{K}(\mathrm{m}_1; t, \varrho)\right| \left|\mathrm{g}(\varrho)\right| \left|\varpi(\varrho)\right|d\varrho \\ \displaystyle\qquad + \int_{\mathrm{m}_1}^{\mathrm{m}_2} \left|\mathfrak{K}(\mathrm{m}_2; t, \varrho)\right| \left|\mathrm{g}(\varrho)\right| \left|\varpi(\varrho)\right|d\varrho\end{array}\right. \\ &\leq \frac{\zeta}{2\mathrm{m}_1^{\mathbf{q}-\mathbf{p}-1}\Upsilon}\Upsilon\mathrm{m}_1^{\mathbf{q}-\mathbf{p}-1} + \mathfrak{K}_1\Upsilon\mathrm{m}_2^{\mathbf{q}-\mathbf{p}-1}|\mathrm{m}_2 - \mathrm{m}_1| \\ &< \zeta.\end{aligned}$$

Case 2. Suppose $\mathrm{m}_1 \leq t \leq \mathrm{m}_2$. Let $\mathfrak{K}_2 = \sup_{t,\varrho\in[\mathrm{m}_1,\mathrm{m}_2]} \left|\mathfrak{K}(\mathrm{m}_2; t, \varrho)\right|$. Choose $\wp = \dfrac{\zeta}{2\mathfrak{K}_2\Upsilon m_2^{\mathbf{q}-\mathbf{p}-1}}$. Then

$$\begin{aligned}&\left|\int_0^{\mathrm{m}_2} \mathfrak{K}(\mathrm{m}_2; t, \varrho)\mathrm{g}(\varrho)\varpi(\varrho)d\varrho - \int_0^{\mathrm{m}_1} \mathfrak{K}(\mathrm{m}_1; \mathrm{m}_1, \varrho)\mathrm{g}(\varrho)\varpi(\varrho)d\varrho\right| \\ &\leq \left\{\begin{array}{l}\displaystyle\int_0^{\mathrm{m}_1} \left|\mathfrak{K}(\mathrm{m}_2; t, \varrho) - \mathfrak{K}(\mathrm{m}_1; \mathrm{m}_1, \varrho)\right| \left|\mathrm{g}(\varrho)\right| \left|\varpi(\varrho)\right|d\varrho \\ \displaystyle\qquad + \int_{\mathrm{m}_1}^{\mathrm{m}_2} \left|\mathfrak{K}(\mathrm{m}_2; t, \varrho)\right| \left|\mathrm{g}(\varrho)\right| \left|\varpi(\varrho)\right|d\varrho\end{array}\right. \\ &\leq \frac{\zeta}{2\mathrm{m}_1^{\mathbf{q}-\mathbf{p}-1}\Upsilon}\Upsilon\mathrm{m}_1^{\mathbf{q}-\mathbf{p}-1} + \mathfrak{K}_2\Upsilon\mathrm{m}_2^{\mathbf{q}-\mathbf{p}-1}|\mathrm{m}_2 - \mathrm{m}_1| \\ &< \zeta.\end{aligned}$$

Case 3. Suppose $t > \mathrm{m}_2$. Let $\mathfrak{K}_3 = \sup_{\varrho\in[\mathrm{m}_1,\mathrm{m}_2]} \big|\mathfrak{K}(\mathrm{m}_2;\mathrm{m}_2,\varrho)\big|$. Choose $\wp = \dfrac{\zeta}{2\mathfrak{K}_3\Upsilon m_2^{\mathbf{q}-\mathbf{p}-1}}$. Then

$$\left|\int_0^{\mathrm{m}_2}\mathfrak{K}(\mathrm{m}_2;\mathrm{m}_2,\varrho)\mathtt{g}(\varrho)\varpi(\varrho)d\varrho - \int_0^{\mathrm{m}_1}\mathfrak{K}(\mathrm{m}_1;\mathrm{m}_1,\varrho)\mathtt{g}(\varrho)\varpi(\varrho)d\varrho\right|$$
$$\leq \begin{cases} \displaystyle\int_0^{\mathrm{m}_1}\big|\mathfrak{K}(\mathrm{m}_2;\mathrm{m}_2,\varrho)-\mathfrak{K}(\mathrm{m}_1;\mathrm{m}_1,\varrho)\big|\,\big|\mathtt{g}(\varrho)\big|\,\big|\varpi(\varrho)\big|d\varrho \\ \displaystyle\qquad + \int_{\mathrm{m}_1}^{\mathrm{m}_2}\big|\mathfrak{K}(\mathrm{m}_2;\mathrm{m}_2,\varrho)\big|\,\big|\mathtt{g}(\varrho)\big|\,\big|\varpi(\varrho)\big|d\varrho \end{cases}$$
$$\leq \frac{\zeta}{2\mathrm{m}_1^{\mathbf{q}-\mathbf{p}-1}\Upsilon}\Upsilon\mathrm{m}_1^{\mathbf{q}-\mathbf{p}-1} + \mathfrak{K}_3\Upsilon\mathrm{m}_2^{\mathbf{q}-\mathbf{p}-1}|\mathrm{m}_2-\mathrm{m}_1| < \zeta.$$

Thus f is continuous. Hence $\vartheta\mapsto\mathfrak{r}(\mathtt{N}_\vartheta)$ is continuous due to Theorem 1.2. □

Theorem 2.2. *Assume that $(\mathcal{H}_1)$-$(\mathcal{H}_4)$ hold. For $\vartheta\in(0,\Im]$, $\mathfrak{r}(N_\vartheta)$ is strictly increasing as a function of ϑ.*

Proof. Let $\chi>0$ and $\varpi\in\mathtt{P}_\vartheta\backslash\{0\}$. For $t\in[0,\vartheta]$, Theorem 1.1 claims that $\mathtt{N}_\vartheta\varpi(t)=\chi\varpi(t)$. Let $\varpi(t)=\varpi(\vartheta)$ for $t>\vartheta$. Then, for $t\in[0,\Im]$, $\mathtt{N}_\vartheta\varpi(t)=\chi\varpi(t)$ and $\mathfrak{r}(\mathtt{N}_\vartheta)\geq\chi>0$. Therefore $\mathfrak{r}(\mathtt{N}_\vartheta)>0$.

Now, let $0<\vartheta_1<\vartheta_2\leq\Im$. Since $\mathfrak{r}(\mathtt{N}_{\vartheta_1})>0$, by Theorem 1.3, there exists $\varpi\in\mathtt{P}_{\vartheta_1}$ s.t. $\mathtt{N}_{\vartheta_1}\varpi=\mathfrak{r}(\mathtt{N}_{\vartheta_1})\varpi$. Let $\varpi_1=\mathtt{N}_{\vartheta_1}\varpi$ and $\varpi_2=\mathtt{N}_{\vartheta_2}\varpi$. Then for $t\in[0,\vartheta_1]$, we assert that $\varpi_2-\varpi_1\in\mathtt{P}^\circ_{\vartheta_1}$. In fact, by stating $(\varpi_2-\varpi_1)(t)=t^{\mathbf{q}-\mathbf{p}-1}z_{12}(t)$, we obtain

$$z_{12}(t)=\int_0^{\vartheta_1}\Big[\mathfrak{K}(\vartheta_2;t,\varrho)-\mathfrak{K}(\vartheta_1;t,\varrho)\Big]\mathtt{g}(\varrho)\varpi(\varrho)d\varrho+\int_{\vartheta_1}^{\vartheta_2}\mathfrak{K}(\vartheta_2;t,\varrho)\mathtt{g}(\varrho)\varpi(\vartheta_1)d\varrho.$$

Since $\varpi\in\mathtt{P}_{\vartheta_1}\setminus\{0\}$ and $(\mathcal{H}_2)$ for $[0,\vartheta_1]\subset[0,\Im]$, accordingly $z_{12}(t)>0$ as $\mathfrak{K}(\vartheta_2;t,\varrho)>\mathfrak{K}(\vartheta_1;t,\varrho)$. So, $\varpi_2(t)>\varpi_1(t)$ on $(0,\vartheta_1)$. Also from Lemma 1.3(iv) and $\varrho\in[0,\Im]$,

$$z_{12}^{(i)}(0)=\begin{cases}\displaystyle\int_0^{\vartheta_2}\frac{\partial^i}{\partial t^i}\Big\{\mathfrak{K}(\vartheta_2;0,\varrho)\Big\}\mathtt{g}(\varrho)\varpi(\varrho)d\varrho- \\ \displaystyle\qquad\int_0^{\vartheta_1}\frac{\partial^i}{\partial t^i}\Big\{\mathfrak{K}(\vartheta_1;0,\varrho)\Big\}\mathtt{g}(\varrho)\varpi(\varrho)d\varrho.\end{cases}$$

Thus $z_{12}^{(i)}(0)=0,\ i=\overline{0,n-2}$. By Lemma 1.3$(v)$ and $\varrho\in(0,\Im)$, one can get

$$z_{12}^{(n-1)}(0)=\begin{cases}\displaystyle\int_0^{\vartheta_1}\left[\frac{\partial^{n-1}}{\partial t^{n-1}}\Big\{\mathfrak{K}(\vartheta_2;0,\varrho)\Big\}-\frac{\partial^{n-1}}{\partial t^{n-1}}\Big\{\mathfrak{K}(\vartheta_1;0,\varrho)\Big\}\right]\mathtt{g}(\varrho)\varpi(\varrho)d\varrho \\ \displaystyle+\int_{\vartheta_1}^{\vartheta_2}\frac{\partial^{n-1}}{\partial t^{n-1}}\Big\{\mathfrak{K}(\vartheta_2;0,\varrho)\Big\}\mathtt{g}(\varrho)\varpi(\vartheta_1)d\varrho\end{cases}$$
$$>0.$$

Furthermore,

$$z_{12}(\vartheta_1)=\int_0^{\vartheta_1}\big[\mathfrak{K}(\vartheta_2;\vartheta_1,\varrho)-\mathfrak{K}(\vartheta_1;\vartheta_1,\varrho)\big]\mathtt{g}(\varrho)\varpi(\varrho)d\varrho+\int_{\vartheta_1}^{\vartheta_2}\mathfrak{K}(\vartheta_2;\vartheta_1,\varrho)\mathtt{g}(\varrho)\varpi(\vartheta_1)d\varrho>0,$$

due to Lemma 1.3(ii) and $\mathfrak{K}(\vartheta_2;\vartheta_1,\varrho)>0$ on $(\vartheta_1,\vartheta_2)$. As a result, the restriction of $\varpi_2-\varpi_1$ to $[0,\vartheta_1]$ pertains to $\Omega_{\vartheta_1}\subset\mathtt{P}^\circ_{\vartheta_1}$. So there exists $\wp>0$ s.t. $\varpi_2-\varpi_1\succeq\wp\varpi$ w.r.t. $\mathtt{P}_{\vartheta_1}$. Let $\varpi_1(t)=\varpi_1(\vartheta_1)$ for $t>\vartheta_1$. In consideration of $\varpi_2\in\mathtt{P}_{\vartheta_2}$, it concludes that $\varpi_2-\varpi_1\succeq\wp\varpi$ w.r.t. $\mathtt{P}_{\vartheta_2}$. Thus, $\varpi_2\succeq\varpi_1+\wp\varpi=\mathfrak{r}\big(\mathtt{N}_{\vartheta_1}\big)\varpi+\wp\varpi=\big[\mathfrak{r}(\mathtt{N}_{\vartheta_1})+\wp\big]\varpi$, i.e.,

$\mathtt{N}_{\vartheta_2}\varpi \succeq \big[\mathfrak{r}(\mathtt{N}_{\vartheta_1}) + \wp\big]\varpi$. As a result of Theorem 1.4, $\mathfrak{r}(\mathtt{N}_{\vartheta_2}) \geq \mathfrak{r}(\mathtt{N}_{\vartheta_1}) + \wp > \mathfrak{r}(\mathtt{N}_{\vartheta_1})$. Hence, $\mathfrak{r}(\mathtt{N}_{\vartheta})$ is strictly increasing for $\vartheta \in (0, \Im]$. □

Theorem 2.3. *The following are equivalent:*

($\mathcal{A}_1$) $\Im_0$ *is the FEP of the FBVP*($\Im$), (1) *and* (2).
($\mathcal{A}_2$) *There exists a nontrivial solution* ϖ *of the FBVP*($\Im_0$), (1) *and* (2) *s.t.* $\varpi \in P_{\Im_0}$.
($\mathcal{A}_3$) $\mathfrak{r}(N_{\Im_0}) = 1$.

Proof. ($\mathcal{A}_3$) $\Rightarrow$ ($\mathcal{A}_2$) is a direct result of Theorem 1.3.

Now, we prove ($\mathcal{A}_2$) $\Rightarrow$ ($\mathcal{A}_1$). Let $\varpi \in \mathtt{P}_{\Im_0} \setminus \{0\}$ satisfy FBVP($\Im_0$), (1) and (2) for $0 \leq t \leq \Im_0$. Extend $\varpi(t) = \varpi(\Im_0)$ for $t > \Im_0$. Clearly, we have $\mathfrak{r}(\mathtt{N}_{\Im_0}) \geq 1$ for $\mathtt{N}_{\Im_0}\varpi(t) = \varpi(t)$.

If $\mathfrak{r}(\mathtt{N}_{\Im_0}) = 1$, then by Theorem 2.2 that $\mathfrak{r}(\mathtt{N}_{\vartheta}) < \mathfrak{r}(\mathtt{N}_{\Im_0})$ for $\vartheta \in (0, \Im_0)$. Therefore $\mathfrak{r}(\mathtt{N}_{\vartheta}) < 1$. Thus the FBVP($\vartheta$), (1) and (2) has the only trivial solution. Hence $\Im_0$ is the FEP of FBVP($\Im$), (1) and (2).

If $\mathfrak{r}(\mathtt{N}_{\Im_0}) > 1$. Let $\mathfrak{v} \in \mathtt{P}_{\Im_0} \setminus \{0\}$ s.t. $\mathtt{N}_{\Im_0}\mathfrak{v} = \mathfrak{r}(\mathtt{N}_{\Im_0})\mathfrak{v}$. We see that restriction of $\mathfrak{v}$ to $[0, \Im_0]$ belongs to $\mathtt{P}^{\circ}_{\Im_0}$ due to Lemma 2.2. Thus, there exists $\wp > 0$ s.t. $\varpi \succeq \wp\mathfrak{v}$ w.r.t. $\mathtt{P}_{\Im_0}$, $0 \leq t \leq \Im_0$. Extend $\mathfrak{v}(t) = \mathfrak{v}(\Im_0)$ for $t > \Im_0$. Then $\varpi \succeq \wp\mathfrak{v}$ w.r.t. $\mathtt{P}$. Assume $\wp$ is maximal s.t. the inequality $\varpi \succeq \wp\mathfrak{v}$ holds. Then, $\varpi = \mathtt{N}_{\Im_0}\varpi \succeq \mathtt{N}_{\Im_0}(\wp\mathfrak{v}) = \wp\mathtt{N}_{\Im_0}\mathfrak{v} = \wp\mathfrak{r}(\mathtt{N}_{\Im_0})\mathfrak{v}$. Because $\mathfrak{r}(\mathtt{N}_{\Im_0}) > 1$, $\wp\mathfrak{r}(\mathtt{N}_{\Im_0}) > \wp$. However, this contradicts the premise that $\wp$ is the maximal value that can satisfy $\varpi \succeq \wp\mathfrak{v}$. So $\mathfrak{r}(\mathtt{N}_{\Im_0}) = 1$.

To prove ($\mathcal{A}_1$) $\Rightarrow$ ($\mathcal{A}_3$), notice that $\lim\limits_{\Im \to 0^+} \mathfrak{r}(\mathtt{N}_{\Im}) = 0$. Since ($\mathcal{A}_1$) implies $\mathfrak{r}(\mathtt{N}_{\Im_0}) \geq 1$ and if $\mathfrak{r}(\mathtt{N}_{\Im_0}) > 1$, then by the continuity of $\mathfrak{r}$ about $\Im$, there exists $\vartheta_0 \in (0, \Im_0)$ s.t. $\mathfrak{r}(\mathtt{N}_{\vartheta_0}) = 1$, contradicting ($\mathcal{A}_1$). Thus, ($\mathcal{A}_1$) $\Rightarrow$ ($\mathcal{A}_3$) follows from Theorem 1.2. □

3. Application to a Nonlinear FBVP

Consider a nonlinear FDEq of the form

$$\mathfrak{D}^{\mathbf{q}}_{0+}\varpi + \mathtt{f}(t, \varpi) = 0, \quad t \in (0, \Im) \tag{7}$$

with conditions (2), where $\mathtt{f}(t, \varpi) : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous, and $\mathtt{f}(t, 0) \equiv 0$, $\mathtt{f}(t, \varpi)$ is differentiable in ϖ. Assume $\dfrac{\partial}{\partial\varpi}\Big\{\mathtt{f}(t, 0)\Big\}$ is nonnegative continuous on $[0, \infty)$ and does not vanish identically on each compact subinterval of $[0, \infty)$. Then the variational equation along the zero solution of (7) is

$$\mathfrak{D}^{\mathbf{q}}_{0+}\varpi + \frac{\partial}{\partial\varpi}\Big\{\mathtt{f}(t, 0)\Big\}\varpi = 0, \quad t \in (0, \Im). \tag{8}$$

The following fixed point theorem [2, 24] will be used to obtain sufficient conditions for the existence of solutions of the FBVP($\Im$), (7) and (2).

Theorem 3.1. *Let* B *be a Banach space and let* $P \subset B$ *be a reproducing cone. Let* $M : B \to B$ *be a completely continuous nonlinear operator s.t.* $M : P \to P$ *and* $M(0) = 0$. *Assume* M *is Fréchet differentiable at* $\varpi = 0$ *whose Fréchet derivative* $N = M'(0)$ *has the property:*

($\widetilde{\mathcal{H}_1}$) *There exist* $w \in P$ *and* $\mu > 1$ *s.t.* $Nw = \mu w$, *and* $N\varpi = \varpi$ *implies* $\varpi \notin P$. *Furthermore, there exists* $\rho > 0$ *s.t., if* $\varpi = \left(\dfrac{1}{\chi}\right)M\varpi$, $\varpi \in P$ *and* $\|\varpi\| = \rho$, *then* $\chi \leq 1$.

Then the equation $\varpi = M\varpi$ *has a solution* $\varpi \in P \setminus \{0\}$.

We will now prove the following result using this theorem and the main conclusions of Section 3.

Theorem 3.2. *Suppose that* $\Im_0$ *is the FEP of FBVP*$(\Im)$, (8) *and* (2). *For each* $\vartheta > \Im_0$ *assume the property:*

$(\widetilde{\mathcal{H}_2})$ *There exists* $\rho(\vartheta) > 0$ *s.t. if* $\varpi(t)$ *is a nontrivial solution of the FDEq*

$$\mathfrak{D}_{0+}^{q}\varpi + \left(\frac{1}{\chi}\right) f(t, \varpi) = 0, \quad t \in (0, \Im), \tag{9}$$

with conditions (2), *and if* $\varpi \in P$ *with* $\|\varpi\| = \rho(\vartheta)$, *then* $\chi \leq 1$.

Then the FBVP(ϑ), (7) and (2) *has a nontrivial solution* $\varpi \in P, \forall\ \vartheta \geq \Im_0$.

Proof. For each $\vartheta > \Im_0$, let $\mathtt{N}_\vartheta : \mathtt{B} \to \mathtt{B}$ be defined by (6), where $\mathtt{g}(t) \equiv \dfrac{\partial}{\partial \varpi}\Big\{\mathtt{f}(t, 0)\Big\}$. Define the nonlinear operator $\mathtt{M}_\vartheta : \mathtt{B} \to \mathtt{B}$ by

$$\mathtt{M}_\vartheta \varpi(t) = \begin{cases} \displaystyle\int_0^\vartheta \aleph(\vartheta; t, \varrho)\mathtt{f}(\varrho, \varpi(\varrho))d\varrho, & 0 \leq t \leq \vartheta, \\ \displaystyle\int_0^\vartheta t^{\mathtt{q}-\mathtt{p}-1}\mathfrak{K}(\vartheta; \vartheta, \varrho)\mathtt{f}(\varrho, \varpi(\varrho))d\varrho, & \vartheta \leq t \leq \Im. \end{cases}$$

Then $\mathtt{M}_\vartheta$ is Fréchet differentiable at $\varpi = 0$. Since

$$\left|\int_0^\vartheta \aleph(\vartheta; t, \varrho)\Big[f\big(\varrho, \varpi(\varrho)\big) - \mathtt{g}(\varrho)\varpi(\varrho)\Big]d\varrho\right| = \left|\int_0^\vartheta \aleph(\vartheta; t, \varrho)\Big[f_\varpi\big(\varrho, \tilde{\varpi}(\varrho)\big) - \mathtt{g}(\varrho)\Big]\varpi(\varrho)d\varrho\right|$$

$$\leq \mathtt{Q}\vartheta\|\varpi\| \int_0^\vartheta \Big|f_\varpi\big(\varrho, \tilde{\varpi}(\varrho)\big) - \mathtt{g}(\varrho)\Big|d\varrho,$$

where $0 \leq \tilde{\varpi}(t) \leq \varpi(t)$ for $t \in [0, \vartheta]$ and $\mathtt{Q} = \sup\limits_{t, \varrho \in [0, \Im]} |\aleph(\vartheta; t, \varrho)|$. Moreover, $\mathtt{M}'_\vartheta(0) = \mathtt{N}_\vartheta$.

By Theorems 2.2 and 2.3, it follows that $\mathfrak{r}(\mathtt{N}_{\Im_0}) = 1$ and $\mathfrak{r}(\mathtt{N}_\vartheta) > 1$ if $\vartheta > \Im_0$. Moreover, since $\Im_0$ is the FEP of the FBVP$(\Im)$, (8), (2), it also follows from Theorem 2.3 that if $\mathtt{N}_\vartheta \varpi = \varpi$ and ϖ is nontrivial for $\vartheta > \Im_0$, then $\varpi \notin \mathtt{P}$. So, for $\vartheta > \Im_0$, we can apply $(\widetilde{\mathcal{H}_2})$ to check the condition $(\widetilde{\mathcal{H}_1})$ in Theorem 3.1. Thus we obtain the existence of a $\varpi \in \mathtt{P} \setminus \{0\}$ s.t. $\varpi = \mathtt{N}_\vartheta \varpi$. □

4. Conclusion

We have derived sufficient conditions for characterization of extremal points for a $(n, \mathbf{p})$-type Riemann–Liouville FBVP by employing the Krein–Rutman theorem. Further, these findings were implemented to a nonlinear FBVP using a fixed-point theorem.

Acknowledgment. The author expresses his profound gratitude to the editor and anonymous referees for their insightful comments and suggestions which led to the improvement of this article.

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Boddu Muralee Bala Krushna received his M.Sc., M.Phil. and Ph.D. degrees from Andhra University, Visakhapatnam, India. He is working as an Assistant Professor of Mathematics at MVGR College of Engineering, Vizianagaram. His major research interest includes ODE, FDE, and BVPs.)