TWMS J. App. and Eng. Math. V.14, N.2, 2024, pp. 605-617

SECURE POINT SET DOMINATION IN GRAPHS

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Abstract. In this paper, we introduce the notion of secure point-set domination in graphs. A point-set dominating D of graph G is called a secure point-set dominating set if for every vertex $u \in V - D$, there exists a vertex $v \in D \cap N(u)$ such that $(D - \{v\}) \cup \{u\}$ is also a point-set dominating set of G. The minimum cardinality of a secure point-set dominating set is called secure point-set domination number of graph G and will be denoted by $\gamma_{spsd}(G)$ (or simply γ_{spsd}). For any graph G of order n, $\gamma_{spsd}(G) \geq 1$ and equality holds if and only if $G \cong K_n$. Also, for any graph G of order n, $\gamma_{spsd}(G) \leq n - 1$ and equality holds if and only if $G \cong K_{1,n-1}$. Here we characterize graphs G with $\gamma_{spsd}(G) = 2$. We also establish a family $\mathcal{F}$ of 11 graphs such that being F-free is necessary as well as sufficient for a graph G to satisfy $\gamma_{spsd}(G) = n - 2$.

Keywords: Domination, Point-Set Domination, Secure Domination, Secure Point-Set Domination, Secure Point-Set Domination Number.

AMS Subject Classification: 05C69.

1. Introduction

By a graph $G = (V, E)$ we mean a finite, nontrivial, connected, undirected graph with neither loops nor multiple edges. For standard terminology in graphs, we refer to Chartrand and Lesniak [8].

Notion of Domination in graphs has a rich historical background and has attracted a lot of researchers in recent times due to its wide range of applications. The concept of domination was introduced formally for the first time by O.Ore [18] and then studied subsequently by Vizing [22, 23], C. Berge [7] and Nieminen [16, 17]. In 1977, Cockayne and Hedetniemi [11] gave a survey of the results and applications of the concept of domination in graphs which gave strong impetus and an independent status to the study of dominating sets in a graph.

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§ Manuscript received: May 21, 2022; accepted: March 17, 2023.
TWMS Journal of Applied and Engineering Mathematics, Vol.14, No.2

In 1993, Sampathkumar and Latha [20] defined a variant of notion of domination termed point-set domination. After the introduction of the concept in 1993, Acharya and Gupta worked extensively on point-set domination and wrote series of papers [1–6, 12].

Definition 1.1. [2, 20] *A subset D of the vertex set $V(G)$ in a graph G is a point-set dominating set (or, in short, psd-set) of G if for every set $S \subseteq V(G) - D$, there exists a vertex $v \in D$ such that the induced subgraph $\langle S \cup \{v\} \rangle$ is connected. The set of all point-set dominating sets of G is denoted by $\mathcal{D}_{ps}(G)$.*

During the third century AD, the Roman empire under Emperor Constantine the Great (274-337 A.D.) dominated most of Europe with their 50 legions. But in the fourth century AD, the number of legions reduced to 25 and Constantine faced the problem of deploying the limited legions available efficiently to secure the entire empire. Constantine devised a strategy by dividing 25 legion into four groups (three groups of 6 legions and one group of 7 legions) and deployed the troops using following rules:

1. A territory is securable if a group can reach it in one step from an adjacent territory.
2. At least two groups must occupy a territory before a group can leave it (that is, at least one group must stay behind).

American General Douglas MacArthur, during World War II, while conducting military operations in the Pacific theater, adopted a similar strategy of moving army from one island to a nearby one making sure that he leaves behind a large enough command post to keep the first island secure.

Ian Stewart in his article "Defend the Roman Empire!" [21] discussed the above strategy adopted by Constantine and contemplated whether there is a possibility to improve Constantine's deployment. Stewart presented the algebraic representation of Constantine's Problem through matrix and used the techniques developed by ReVelle and Rosing [19] to find the solution.

Motivated from the article by Ian Stewart, Henning and Hedetniemi [15] introduced a new graph theoretic strategy for defense based on the notion of domination, called Roman Domination, to potentially reduce the costs of maintaining troops while still defending the empire. Inspired from the definition of Roman Domination, Cockayne et. al. [10] introduced two more strategies Weak Roman Domination and Secure Domination to protect the graph $G = (V, E)$ where one or more guard vertex are chosen in such a way that if any vertex is in any kind of trouble then we can interchange the position of guards at these vertices and the entire graph still deemed to be protected.

Definition 1.2 (Secure Domination)**.** *[9] Let G be a graph and D be a dominating set of G. A vertex $u \in V(G) - D$ is said to be securely defended (or s-defended) by a vertex x in $D \cap N(u)$ if $(D - \{x\}) \cup \{u\}$ is also a dominating set of G. The set D is said to be a secure dominating set of G if every vertex $u \in V(G) - D$ is securely defended (or s-defended) by a vertex x in D.*

In this paper, we introduce another protection strategy, namely, Secure point-set domination which is motivated from Secure Domination and Point-set Domination. The notion of Secure point-set domination can be explained using the following situation. Suppose we want to secure a detention center by placing guards at some selected positions of the center in such a way that every area of the center is either occupied by a guard or is in the neighborhood of some guard. If under any situation one guard moves out to some other unoccupied area, then the new configuration of guards still protects the entire detention center.

Following definitions of the private neighbor of a vertex and of a set will be needed further in our study.

Definition 1.3. [13,14] *For any vertex u, v is called a private neighbor of u with respect to $S \in V$ if $N(v) \cap S = \{u\}$. Further private neighbor set of u with respect to S is $pn[u, S] = \{v : N(v) \cap S = \{u\}\}$.*

Definition 1.4. [13,14] *For any subset S of the vertex set V in a graph G and any vertex $u \in S$, a vertex v is called a* **private neighbor** *of u with respect to S if $N(v) \cap S = \{u\}$. If further $v \in V(G) - S$, then v is called an external private neighbor of u with respect to S. The set of all private neighbors of u with respect to S is denoted by $pn(u, S)$ and the set of all external private neighbors of u with respect to S is denoted by $epn(u, S)$.*

2. Secure Point-set Domination

In this section, we introduce a new protection strategy in graphs termed as secure point-set domination, where instead of a dominating set, we consider point-set dominating set.

Definition 2.1 (Secure Point Set Domination)**.** *Let G be a graph and D be a point-set dominating set (or simply, a psd-set) of G. A vertex $u \in V(G) - D$ is said to be securely defended (or s-defended) by a vertex x in $D \cap N(u)$ if $(D - \{x\}) \cup \{u\}$ is also a psd-set of G. The set D is said to be a secure point-set dominating set of G if every vertex $u \in V(G) - D$ is securely defended (or s-defended) by a vertex x in D. The minimum cardinality of a secure point-set dominating set of a graph G, denoted by $\gamma_{spsd}(G)$, is called the secure point set domination number of G.*

It is worthwhile to note that every graph possesses a secure point-set dominating set. Trivially, the vertex set $V(G)$ is a secure point-set dominating set of G. Hence the graph invariant secure point set domination number $\gamma_{spsd}(G)$ is well-defined. Also, since by definition, every secure point-set dominating set is non-empty, therefore $\gamma_{spsd}(G) \geq 1$. Further, for any graph G of order at least two, the set $V(G) - \{u\}$ is a secure point-set dominating set of G, for any vertex $u \in V(G)$. Thus for any graph G of order n (≥ 2),

$$1 \leq \gamma_{spsd}(G) \leq n - 1.$$

Moreover both the bounds are sharp. In fact, for a graph G, $\gamma_{spsd}(G) = 1$ if and only if $G \cong K_n$. Also, for any star $K_{1,n-1}$, the set of all pendant vertices forms a γ_{spsd}-set of $K_{1,n-1}$, hence $\gamma_{spsd}(K_{1,n-1}) = n - 1$.

Note that, by definition, every secure point-set dominating set is a point-set dominating set and hence $\gamma_{psd}(G) \leq \gamma_{spsd}(G)$. Further the difference between the two parameters can be arbitrarily large. In fact, for any star $K_{1,n-1}$, $\gamma_{spsd}(K_{1,n-1}) - \gamma_{psd}(K_{1,n-1}) = n - 2$. Also, every secure point-set dominating set is a secure dominating set and hence $\gamma_{sd}(G) \leq \gamma_{spsd}(G)$. Thus

$$\max\{\gamma_{sd}(G), \gamma_{psd}(G)\} \leq \gamma_{spsd}(G).$$

The following results relate the secure point set domination number of a graph to its subgraph and will be very useful in further study of secure point set domination in graphs.

Proposition 2.1. *Let G be a graph and H be any subgraph of G. If a subset D of $V(H)$ is a secure point set dominating set of H, then $D \cup (V(G) - V(H))$ is a secure point set dominating set for G.*

Proof. Let $D_1 = D \cup (V(G) - V(H))$. Then $V(G) - D_1 = V(H) - D$. For any independent subset $I \subseteq V(G) - D_1$, there exists a $x \in D \subseteq D_1$ such that $I \subseteq N(x)$. Also every vertex $u \in V(G) - D_1$ is s-defended by a vertex of D and therefore a vertex of D_1. Hence, D_1 is a spsd-set of G. □

Proposition 2.2. *For any graph G of order n and any subgraph H of G,*

$$\gamma_{spsd}(G) \leq n + \gamma_{spsd}(H) - |V(H)|.$$

Proof. Follows immediately from Proposition 2.1. □

Corollary 2.1. *For any graph G of order n and clique number $\omega(G)$,*

$$\gamma_{spsd}(G) \leq n + 1 - \omega(G).$$

Corollary 2.2. *If a graph G of order n has a subgraph isomorphic to path P_4, then*

$$\gamma_{spsd}(G) \leq n - 2.$$

Proof. Since $\gamma_{spsd}(P_4) = 2$ (see Figure 1), consequently, by Proposition 2.2,

$$\gamma_{spsd}(G) \leq n + 2 - 4 = n - 2.$$

□

FIGURE 1. The set $\{b, c\}$ is a γ_{spsd}-set of path P_4.

Corollary 2.3. *For any graph G of order n (≥ 2), $\gamma_{spsd}(G) = n - 1$ if and only if $G \cong K_{1,n-1}$.*

Proof. If $G \cong K_{1,n-1}$, then the set of all pendant vertices forms a γ_{spsd}-set of $K_{1,n-1}$, hence $\gamma_{spsd}(K_{1,n-1}) = n - 1$. Conversely, let $\gamma_{spsd}(G) = n - 1$. If $G \ncong K_{1,n-1}$, then either G has a subgraph H isomorphic to either K_3 or P_4. In either case, from Corollary 2.1 and Corollary 2.2, it follows that $\gamma_{spsd}(G) \leq n - 2$, a contradiction. Hence $G \cong K_{1,n-1}$. □

From above discussion, it is easy to observe that if $G \ncong K_n, K_{1,n-1}$, then

$$2 \leq \gamma_{spsd}(G) \leq n - 2. \tag{1}$$

Note that for cycle C_4, $\gamma_{spsd}(C_4) = 2$ and for cycle C_5, $\gamma_{spsd}(C_5) = 3 = 5 - 2$. Hence both the bounds in (1) are sharp. The main objective of this paper is to characterize graphs G for which $\gamma_{spsd}(G) = 2$ and graphs G for which $\gamma_{spsd}(G) = n - 2$. Since every secure point-set dominating set is a point-set dominating set, therefore to characterize graphs G with $\gamma_{spsd}(G) = 2$, we first look at psd-sets of cardinality at most 2.

Observation 2.1. *For any graph G of order n and maximum degree $\Delta(G)$, $\gamma_{psd}(G) = 1$ if and only if $\Delta(G) = n - 1$.*

Theorem 2.1. *For any graph G with $\gamma_{psd}(G) \leq 2$, a dominating set $D = \{u, v\}$ is a psd-set of G if and only if $xy \in E(G)$ for each $x \in epn(v, D)$ and $y \in epn(u, D)$.*

Proof. Let $D = \{u, v\}$ be a psd-set of G. If either $epn(u, D) = \emptyset$ or $epn(v, D) = \emptyset$, then there is nothing to prove. Let both $epn(u, D)$ and $epn(v, D)$ be non-empty sets. For any $x \in epn(u, D)$ and $y \in epn(v, D)$, $N(x) \cap N(y) \cap D = \emptyset$. Since $x, y \in V(G) - D$, D is a psd-set of G and $N(x) \cap N(y) \cap D = \emptyset$, we have $xy \in E(G)$. Hence the necessity.

For the converse part, assume that the dominating set $D = \{u, v\}$ of G satisfies the hypothesis. We claim that D is a psd-set of G. Let I be any independent subset of $V(G) - D$. Since I is independent, by hypothesis, either $I \cap epn(u, D) = \emptyset$ or $I \cap epn(v, D) = \emptyset$. Without loss of generality, assume that, $I \cap epn(u, D) = \emptyset$. Since D is a dominating set of G and $I \cap epn(u, D) = \emptyset$, it follows that $I \subseteq N(v)$. Consequently, $\langle I \cup \{v\}\rangle$ is connected and hence D is a psd-set of G. □

Following result is an immediate consequence of Theorem 2.1.

Corollary 2.4. *For any graph G of order n and maximum degree $\Delta(G)$, $\gamma_{psd}(G) = 2$ if and only if $\Delta(G) \leq n - 2$ and there exists a dominating set $D = \{u, v\}$ of $V(G)$ such that $xy \in E(G)$ for each $x \in epn(v, D)$ and $y \in epn(u, D)$.*

In our next theorem, we completely characterize graphs whose secure point-set domination number is 2.

Theorem 2.2. *For any graph $G \ncong K_n$ of order n, $\gamma_{spsd}(G) = 2$ if and only if there exists a dominating set $D = \{u, v\}$ of G such that:*

- **a:** *$\langle epn(u, D) \cup epn(v, D)\rangle$ is a clique in G.*
- **b:** *$|(epn(u, D) \cup epn(v, D)) - N(x)| \leq 1$ for each $x \in N(u) \cap N(v)$. Further if D is independent, then $|(epn(u, D) \cup epn(v, D)) - N(x)| = 0$ for each $x \in N(u) \cap N(v)$.*
- **c:** *If $x, y \in N(u) \cap N(v)$ are such that $epn(u, D) - N(x) \neq \emptyset$ and $epn(v, D) - N(y) \neq \emptyset$, then x and y are adjacent vertices in G.*

Proof. Let G be a graph with $\gamma_{spsd}(G) = 2$ and $D = \{u, v\}$ be a γ_{spsd}-set of G. Since every secure psd-set is a psd-set, D is a psd-set of G and consequently, a dominating set of G. From Theorem 2.1, if $x \in epn(u, D)$ and $y \in epn(v, D)$, then $xy \in E(G)$. Therefore to prove (a), it is sufficient to show that both $\langle epn(u, D)\rangle$ and $\langle epn(v, D)\rangle$ are cliques in G. Suppose on the contrary, at least one of $\langle epn(u, D)\rangle$ and $\langle epn(v, D)\rangle$ is not a clique in G. Without loss of generality, assume that $\langle epn(u, D)\rangle$ is not a clique in G. Let $x, y \in epn(u, D)$ be a pair of non-adjacent vertices. Since $\{u, v\}$ is a secure psd-set of G and $x \in epn(u, D)$, therefore x is s-defended only by u. Consequently, $\{x, v\}$ is a psd-set of G. Since $xy \notin E(G)$ and $\{x, v\}$ is a psd-set of G, therefore y is adjacent to v, contradiction to the fact that $y \in epn(u, D)$. Hence our assumption is not correct and both $\langle epn(u, D)\rangle$ and $\langle epn(v, D)\rangle$ are cliques in G. Thus condition (a) is satisfied.

Next we shall show that $|(epn(u, D) \cup epn(v, D)) - N(x)| \leq 1$, for each $x \in N(u) \cap N(v)$. Suppose on the contrary, there exists $x \in N(u) \cap N(v)$ such that $|(epn(u, D) \cup epn(v, D)) - N(x)| \geq 2$. Since $x \in V(G) - \{u, v\}$ and $\{u, v\}$ is a secure psd-set of G, either $\{x, v\}$ or $\{x, u\}$ is a psd-set of G. Without loss of generality, assume that $\{x, v\}$ is a psd-set of G. Then $epn(u, D) \subseteq N(x)$. Since $|(epn(u, D) \cup epn(v, D)) - N(x)| \geq 2$, it follows that $|epn(v, D) - N(x)| \geq 2$. Let $w_1, w_2 \in epn(v, D) - N(x)$. Again since $\{u, v\}$ is a secure psd-set of G and $w_1 \in epn(v, D)$, $\{u, w_1\}$ is a psd-set of G. But then $\{x, w_2\}$ is an independent set in $V(G) - \{u, w_1\}$ such that $N(x) \cap N(w_2) \cap \{u, w_1\} = \emptyset$, a contradiction to the fact $\{u, w_1\}$ is a psd-set of G. Hence our assumption is not correct and

$|(epn(u, D) \cup epn(v, D)) - N(x)| \leq 1$, for each $x \in N(u) \cap N(v)$.

Now to prove the condition (b), we just need to show that if D is an independent set, then $|(epn(u, D) \cup epn(v, D)) - N(x)| = 0$ for each $x \in N(u) \cap N(v)$. Suppose, if possible, there exists $x \in N(u) \cap N(v)$ such that $|(epn(u, D) \cup epn(v, D)) - N(x)| = 1$. Let w be a vertex in G such that $(epn(u, D) \cup epn(v, D)) - N(x) = \{w\}$. Since $x \in V(G) - \{u, v\}$ and $\{u, v\}$ is a secure psd-set of G, either $\{x, v\}$ or $\{u, x\}$ is a psd-set of G. Without loss of generality we assume that $\{x, v\}$ is a psd-set of G. Then as $epn(u, D) \cap N(v) = \emptyset$, it follows that $epn(u, D) \subseteq N(x)$. Consequently, $w \in epn(v, D) - N(x)$. Since $\{w, u\}$ is an independent subset of $V(G) - \{x, v\}$ and $\{x, v\}$ is a psd-set of G, therefore either $\{w, u\} \subseteq N(x)$ or $\{w, u\} \subseteq N(v)$, which is not possible since $w \notin N(x)$ and $u \notin N(v)$. Hence our assumption is not correct and $|(epn(u, D) \cup epn(v, D)) - N(x)| = 0$ for each $x \in N(u) \cap N(v)$. Thus condition (b) is satisfied.

Finally to prove condition (c), we consider $x, y \in N(u) \cap N(v)$ such that $epn(u, D) - N(x) \neq \emptyset$ and $epn(v, D) - N(y) \neq \emptyset$. We need to show that $xy \in E(G)$. Let $w_1 \in epn(u, D) - N(x)$ and $w_2 \in epn(v, D) - N(y)$. Since x is not adjacent to $w_1 \in epn(u, D)$, x is s-defended only by v. Therefore $\{x, u\}$ is a psd-set of G. Since y and w_2 are non-adjacent vertices of $V(G) - \{x, u\}$, either $\{y, w_2\} \subseteq N(x)$ or $\{y, w_2\} \subseteq N(u)$. But as $w_2 \in epn(v, D)$, it follows that $\{y, w_2\} \subseteq N(x)$. In particular, $y \in N(x)$ i.e., y is adjacent to x. Hence condition (c) is also satisfied.

Conversely, suppose there exists a dominating set $D = \{u, v\}$ of G satisfying the hypothesis. We claim that D is a γ_{spsd}-set of G. Since $\langle epn(u, D) \cup epn(v, D) \rangle$ is a clique, from Theorem 2.1, D is a psd-set of G. Also $G \ncong K_n$, we shall show that D is a spsd-set of G. Let $x \in V(G) - D$ be any arbitrary vertex. We claim that either $\{x, u\}$ or $\{x, v\}$ is a psd-set of G. Suppose, if possible, neither $\{x, u\}$ nor $\{x, v\}$ is a psd-set of G.

Since $\{x, u\}$ is not a psd-set of G, we will show that either $u \in N(x)$ or $u \in N(v)$. Suppose on the contrary, $u \notin N(x) \cup N(v)$. Then $\{u, v\}$ is a dom-set, $x \in epn(v, D)$. Since $\{x, u\}$ is not a psd-set of G, there exists an independent set I of $V(G) - \{x, u\}$ such that neither $I \subseteq N(x)$ nor $I \subseteq N(u)$. If $v \in I$, then as I is independent, $I \cap N(v) = \emptyset$. Since $\{u, v\}$ is a dom-set, $I - \{v\} \subseteq epn(u, D)$. As $x \in epn(v, D)$ and $epn(u, D) \cup epn(v, D)$ is a clique, it follows that $I - \{v\} \subseteq N(x)$. Consequently, $I \subseteq N(x)$, a contradiction. Hence $v \notin I$. Let $w_1 \in I - N(x)$ and $w_2 \in I - N(u)$. Since $x \in epn(v, D)$, $w_1 \notin epn(u, D) \cup epn(v, D)$. Consequently, $w_1 \in N(u) \cap N(v)$. Again as $w_2 \notin N(u)$, $w_2 \in epn(v, D)$ and consequently $w_2, x \in epn(v, D) - N(w_1)$, contradiction to (b). Hence either either $u \in N(x)$ or $u \in N(v)$. Similarly as $\{x, v\}$ is not a psd-set of G, either $v \in N(x)$ or $v \in N(u)$.

Since $\{x, v\}$ is not a psd-set of G, there exists an independent set $S \subseteq V(G) - \{x, v\}$ such that neither $S \subseteq N(x)$ nor $S \subseteq N(v)$. Now we have two possibilities.

Case 1. D is independent.
Then $x \in N(u) \cap N(v)$. Then from (b), $epn(u, D) \cup epn(v, D) \subseteq N(x)$. Let z_1 $(\neq u) \in S - N(x)$. If $u \in S$, then $z_1 \notin N(u)$ and consequently, $z_1 \in epn(v, D) \subseteq N(x)$, a contradiction. Hence $u \notin S$. Let z_2 $(\neq u) \in S - N(v)$. Then $z_2 \in epn(u, D)$. Since $epn(u, D) \cup epn(v, D)$ is a clique, $z_1 \in N(u) \cap N(v)$. Then from (b),

$epn(u, D) \cup epn(v, D) \subseteq N(z_1)$ and consequently, $z_2 \in N(z_1)$, contradiction to the fact that S is independent.

Case 2. u and v are adjacent.
If $u \in S$, then $S - \{u\} \subseteq epn(v, D)$. Consequently, $S \subseteq N(v)$, a contradiction. Hence $u \notin S$. Let y_1 $(\neq u) \in S - N(x)$ and y_2 $(\neq u) \in S - N(v)$. Then $y_2 \in epn(u, D)$ and $y_1 \in N(u) \cap N(v)$. If $x \in epn(u, D) \cup epn(v, D)$, then $y_2, x \in epn(u, D) \cup epn(v, D) - N(y_1)$, contradiction to (b). Hence $x \in N(u) \cap N(v)$. Now as $\{x, u\}$ is not a psd-set of G, there exists an independent set $S_1 \subseteq V(G) - \{x, v\}$ such that neither $S \subseteq N(x)$ nor $S \subseteq N(v)$ and vertices a_1 $(\neq v) \in S_1 - N(x)$ and a_2 $(\neq v) \in S - N(u)$ such that $a_2 \in epn(v, D)$ and $a_1 \in N(u) \cap N(v)$. Then as $y_2 \in epn(u, D) - N(y_1)$ and $a_2 \in epn(v, D) - N(x)$, by (c), $y_1 \in N(x)$, contradiction.

Thus in both the possible cases, we arrive at a contradiction. Hence our assumption is not correct and either $\{x, u\}$ or $\{x, v\}$ is a psd-set of G. Consequently, $D = \{u, v\}$ is a spsd-set of G. □

Next we proceed to characterize graphs G of order n with secure point-set domination number $\gamma_{spsd}(G) = n - 2$. From Proposition 2.2, one is tempted to think that can there be some family set of graphs H with $\gamma_{spsd}(H) = V(H) - 3$ such that if a graph G is H-free, then $\gamma_{spsd}(G) = n - 2$. We have already seen how being P_4-free helps in characterization of graphs G with $\gamma_{spsd}(G) = n - 1$.

In the following theorem we present a family $\mathcal{F} = \{H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9, H_{10}, H_{11}\}$ (see Figure 2) of 11 forbidden subgraphs for a graph G of order n to have $\gamma_{spsd}(G) = n - 2$. In Figure 2, γ_{spsd}-set (vertices with unfilled circles) of each of the graph H_1, H_2,... H_{11} is exhibited. For each H_i in $\mathcal{F}$, $\gamma_{spsd}(H_i) = |V(H_i)| - 3$.

Theorem 2.3. *Let $G \ncong K_n, K_{1,n-1}$ be a graph of order n. Then $\gamma_{spsd}(G) = n - 2$ if and only if G does not have any subgraph isomorphic to any of the graphs $H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9, H_{10}, H_{11}$.*

Proof. Let G be a graph with $\gamma_{spsd}(G) = n-2$. Since for each $i = 1, 2, \ldots, 11$, $\gamma_{spsd}(H_i) = |V(H_i)|-3$, by Proposition 2.2, if G has a subgraph isomorphic to H_i for some $i = 1, \ldots, 11$, then

$$\gamma_{spsd}(G) \leq n + \gamma_{spsd}(H_i) - |V(H_i)| = n - 3,$$

a contradiction. Hence the necessity.

For the converse part, assume that G does not have any subgraph isomorphic to graphs H_1, H_2,... H_{11}. We will prove that $\gamma_{spsd}(G) = n - 2$. We will repeatedly make use of the following two observations:

- **A.** There does not exist any triplet x_1, x_2, x_3 of vertices in G such that $\langle\{x_1, x_2, x_3\}\rangle \cong K_3$ and $d(x_i) \geq 3$ for each i, for otherwise G will have a subgraph isomorphic to at least one of H_1, H_2, H_3, a contradiction.
- **B.** There does not exist a quartet x_1, x_2, x_3, x_4 of vertices in G such that $\langle\{x_1, x_2, x_3, x_4\}\rangle \cong C_4$ and $d(x_i) \geq 3$ for at least three i's, for otherwise G will have a subgraph isomorphic to at least one of H_3, H_4, H_5, a contradiction.

Suppose, if possible, $\gamma_{spsd}(G) \leq n - 3$. Let D be any γ_{spsd}-set of G. Then $|V(G) - D| \geq 3$. Let u, v, w be any three distinct vertices in $V(G) - D$. If

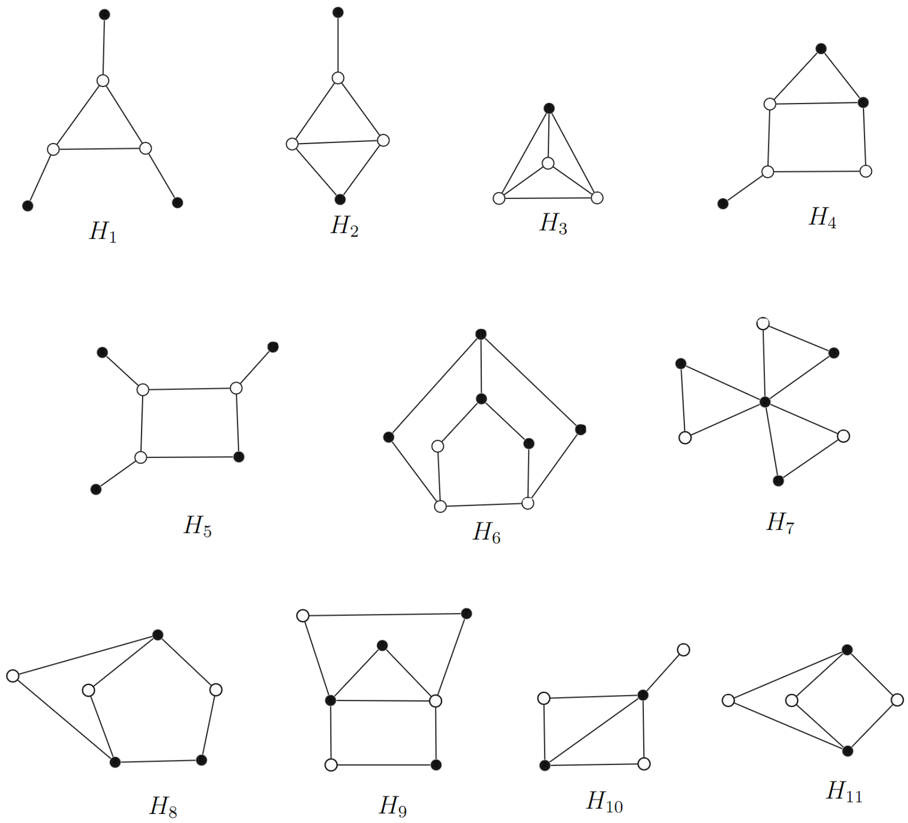

FIGURE 2. Forbidden subgraphs for $\gamma_{spsd}(G) = n - 2$.

$\langle\{u, v, w\}\rangle \cong K_3$, then as D is a dominating set and $\{u, v, w\} \subseteq V(G) - D$, $d(u), d(v), d(w) \geq 3$, a contradiction to **A**. Hence $|E(\langle\{u, v, w\}\rangle)| \leq 2$, we have three possibilities:

Case 1. $|E(\langle\{u, v, w\}\rangle)| = 2$.
Without loss of generality, assume that $uv, wv \in E(G)$ and $uw \notin E(G)$. Since D is a psd-set and $u, w \in V(G) - D$, there exists a vertex $d_1 \in D$ such that $u, w \in N(d_1)$. We claim that $v \notin N(d_1)$. Suppose, if possible, $v \in N(d_1)$. Then $d(u) = 2 = d(w)$, for otherwise we get a subgraph isomorphic to H_2. Thus $N(u) = N(w) = \{v, d_1\}$. Therefore u is s-defended by d_1 only. But since w is not dominated by any vertex of the set $(D - \{d_1\}) \cup \{u\}$, $(D - \{d_1\}) \cup \{u\}$ is not a dominating set, a contradiction. Thus $v \notin N(d_1)$. Since D is a spsd-set, there exists some $d_2 \in D$ which defends v securely. Clearly, d_2 is not adjacent to d_1, for otherwise $\langle\{u, v, w, d_1, d_2\}\rangle \cong H_{11}$. If u and w are both adjacent to d_2, then $\langle\{u, v, w, d_2, d_1\}\rangle - wd_1$ is isomorphic to H_2, a contradiction. Therefore $d_2 \notin N(u) \cap N(w)$. Without loss of generality, assume that $d_2 \notin N(w)$. Since v is s-defended by d_2, $(D - \{d_2\}) \cup \{v\}$ is a psd-set. Consequently, for $w, d_2 \in V(G) - ((D - \{d_2\}) \cup \{v\})$, there exists $d_3 \in D$ such that $\{w, d_2\} \subseteq N(d_3)$. Also as $d_2 \notin N(d_1)$, either $u \in N(d_2)$ or there exists a vertex d_4(not necessarily distinct from d_3) in D such that $u, d_2 \in N(d_4)$. In both cases, $\langle\{u, v, w, d_1\}\rangle \cong C_4$ with at least three vertices of degree greater than 2, a contradiction to **B**.

Case 2. $|E(\langle\{u,v,w\}\rangle)| = 1$.
Without loss of generality, assume that $uv \in E(G)$ and $vw, wu \notin E(G)$. We have two possibilities:

Subcase 2.1. There exists $d \in D$ such that $u, v, w \in N(d)$. If w is s-defended by d, $(D - \{d\}) \cup \{w\}$ is a psd-set of G. Since $\langle\{u,v,d\}\rangle \cong K_3$ and $(D - \{d\}) \cup \{w\}$ is a dominating, $\langle\{u,v,d\}\rangle$ contradicts **A**. Hence w is not s-defended by d and there exists some d_1 in D such that w is s-defended by d_1.

Sub-Subcase 2.1.1 d_1 and d are adjacent vertices.
None of u and v can be adjacent to d_1, for otherwise, $\langle\{u,v,w,d,d_1\}\rangle$ will have a subgraph isomorphic to H_2. If u is s-defended by d, then $(D - \{d\}) \cup \{u\}$ is a dominating set and for non adjacent vertices v, w in set $V(G) - ((D - \{d\}) \cup \{u\})$, there exists some $x \in (D - \{d\}) \cup \{u\}$ such that $v, w \in N(x)$. Further since v is not adjacent to d_1, we conclude that $d_1 \neq x$. But then the set $\{d, v, x, w\}$ contradicts **B**. Hence u is not s-defended by d. Therefore there exists some $d_2(\neq d)$ in D such that u is s-defended by d_2. Next we claim that d_2 is not adjacent to d and d_1. Suppose, if possible, d_2 is adjacent to d. Then to avoid forbidden subgraphs, degree of v can not exceed 2. Therefore v is s-defended by d only. Then there exists some x_1 in $(D - \{d\}) \cup \{v\}$ for non adjacent vertices u, w of $V(G) - ((D - \{d\}) \cup \{v\})$ such that $u, w \in N(x_1)$. But in that case, the set $\{u, d, w, x_1\}$ contradicts **B**. This contradiction approves our claim that d_2 is not adjacent to d. Further if d_2 is adjacent to d_1, then the set $\{u, d, d_1, d_2\}$ contradicts **B**. Hence d_2 is not adjacent to any of vertex d and d_1. Now as u is s-defended by d_2 and v & d_2 are non adjacent vertices in $V(G) - ((D - \{d_2\}) \cup \{u\})$, there exists some y_1 in $(D - \{d_2\}) \cup \{u\}$ such that $v, d_2 \in N(y_1)$. But then $\{u, v, d\}$ contradicts **A**.

Sub-Subcase 2.1.2 d_1 and d are not adjacent vertices.
If $d_1 \in N(u) \cap N(v)$, then the set $\{u, v, d\}$ will contradict **A**. Hence $d_1 \notin N(u) \cap N(v)$. Without loss of generality, assume that $d_1 \notin N(u)$. Then as w is s-defended by d_1, there exists some x_2 in $(D - \{d_1\}) \cup \{w\}$ such that x_2 is adjacent to both u and d_1. If $d_1 \in N(v)$, then again the set $\{u, v, d\}$ contradicts **A**. Hence v and d_1 are non adjacent vertices. Since $(D - \{d_1\}) \cup \{w\}$ is a psd-set, there exists y_2 in $(D - \{d_1\}) \cup \{w\}$ such that $v, d_1 \in N(y_2)$. But then again the set $\{u, v, d\}$ contradicts **A**.

Subcase 2.2. There does not exist any d in D such that $u, v, w \in N(d)$. Since $uw \notin E(G)$ and $vw \notin E(G)$, there exist two distinct vertices d_1 and d_2 in D such that $u, w \in N(d_1)$ and $v, w \in N(d_2)$. Next we claim that w can not be s-defended by any of d_1 or d_2. Suppose, if possible, w is s-defended by d_2. Then there exists a vertex x_3 in $(D - \{d_2\}) \cup \{w\}$ such that v is dominated by x_3. If d_1 is adjacent to d_2, $\langle\{u, v, d_1, d_2, x_3\}\rangle$ will have a subgraph isomorphic to H_4, a contradiction. Hence d_1 can not be adjacent to d_2. Now as u and d_2 are non adjacent vertices in $V(G) - ((D - \{d_2\}) \cup \{w\})$, there exists some x_4 (not necessarily distinct from x_3) in $(D - \{d_2\}) \cup \{w\}$ such that $u, d_2 \in N(x_4)$. But then the set $\{u, v, d_2, x_4\}$ contradicts **B**. Hence w is not s-defended by d_2. Due to symmetry of d_1 and d_2, we conclude that w can not be s-defended by both d_1 or d_2. Then there exists some d_3 in D such that w is s-defended by d_3. Again d_3 can not be adjacent to u and v, otherwise $\langle\{u, v, w, d_1, d_2, d_3\}\rangle$ will form a subgraph isomorphic to H_8. Also, d_3 is not adjacent to d_2, for otherwise $\{d_2, d_3, w\}$ will contradict **A**. But then for non adjacent vertices d_3 and v of $V(G) - ((D - \{d_3\}) \cup \{w\})$, there exists some d_4 in $(D - \{d_3\}) \cup \{w\}$ which is adjacent to both d_3 and v. Again d_3 is not adjacent to both d_1 and d_2, for otherwise $\{d, w, d_3\}$ will contradict **A**. Without loss of generality, assume that $d_3d_2 \notin E(G)$. Then v, d_3 are non adjacent vertices in $V(G) - ((D - \{d_3\}) \cup \{w\})$ and therefore there exists some d_4 in $(D - \{d_3\}) \cup \{w\}$ such that $\{v, d_3\} \subseteq N(d_4)$. If $d_1d_3 \in E(G)$, then $\{d_1, d_2, w\}$ contradicts **A**. Thus $d_1d_3 \notin E(G)$. But then there exists some d_5 in $(D - \{d_3\}) \cup \{w\}$ such that $u, d_3 \in N(d_5)$ and in that case $\langle\{u, d_1, w, d_2, v, d_4, d_3, d_5\}\rangle$ is a subgraph isomorphic to H_6, a contradiction.

Case 3. $|E(\langle\{u, v, w\}\rangle)| = 0$.
Since D is a psd-set of G, there exists a vertex d in D such that $u, v, w \in N(d)$. If there exists another vertex $x(\neq d)$ in D such that $u, v, w \in N(x)$, then G will have a subgraph isomorphic to H_{11}. Therefore, d in D is unique vertex such that $u, v, w \in N(d)$. Now we have four possibilities.

Subcase 3.1 All vertices of the set $\{u, v, w\}$ are s-defended by d.
As u is s-defended by d, then there exists a vertex d_1 in $(D - \{d\}) \cup \{u\}$ such that $v, w \in N(d_1)$. Similarly, as v, w are also s-defended by d, there exist a vertex $d_2 \in (D - \{d\}) \cup \{v\}$ and a vertex $d_3 \in (D - \{d\}) \cup \{w\}$ such that $u, w \in N(d_2)$ and $u, v \in N(d_3)$. But then $\{u, d, v, d_1\}$ contradicts **B**.

Subcase 3.2 Exactly two vertices of the set $\{u, v, w\}$ are s-defended by d.
Without loss of generality, let u and v be s-defended by d. Then w is s-defended by a vertex $d_1(\neq d)$. By uniqueness of d, $\{u, v\} \nsubseteq N(d_1)$. Due to symmetry, we assume that $v \notin N(d_1)$. Then v and d_1 are non adjacent vertices in $V(G) - ((D - \{d_1\}) \cup \{w\})$. Since $(D - \{d_1\}) \cup \{w\}$ is a psd-set, there exists a vertex $d_2 \in (D - \{d_1\}) \cup \{w\}$ such that $d_1, v \in N(d_2)$. If d_1 and u are adjacent, then we get a subgraph isomorphic to H_8, a contradiction. Thus $\{u, v, d_1\}$ is an independent set in $V(G) - ((D - \{d_1\}) \cup \{w\})$. Therefore there exists a vertex d_3 such that $\{u, v, d_1\} \subseteq N(d_3)$. But then $\langle\{u, v, w, d, d_1, d_3\}\rangle$ is isomorphic to H_8, a contradiction.

Subcase 3.3 Exactly one vertex of the set $\{u, v, w\}$ is s-defended by d, say u.
Then there exists a vertex d_1 in $D - \{d\} \cup \{u\}$ which is adjacent to both v and w. If both v and w are not s-defended by d_1, then $\{d, v, d_1, w\}$ will contradict **B**. Hence at least one of v or w is s-defended by d_1. Without loss of generality, assume that v is s-defended by d_1. If d_1 is adjacent to d, then G will have a subgraph isomorphic to H_{10}. Hence d_1 is not adjacent to d. Since d_1 is not adjacent to u, there exists d_2 in $(D - \{d_1\}) \cup \{v\}$ such that u and d_1 are adjacent to d_2. But then G will have a subgraph isomorphic to H_8, a contradiction.

Subcase 3.4 No vertex of the set $\{u, v, w\}$ is s-defended by d.
Let u be s-defended by $d_1 \in D$. We claim that d_1 is adjacent to d. Suppose, if possible, d_1 is not adjacent to d. If both v and w are not adjacent to d_1, then there exists a vertex $d_2 \in (D - \{d_1\}) \cup \{u\}$ such that $v, w, d_1 \in N(d_2)$. But then G will have a subgraph isomorphic to H_8, a contradiction. Let v be adjacent to d_1. By Uniqueness of d, w is not adjacent to d_1. Then there exists some $d_2 \in (D - \{d_1\}) \cup \{u\}$ such that $w, d_1 \in N(d_2)$, but then again G will have a subgraph isomorphic to H_8, a contradiction. Hence d_1 is adjacent to d. If any one of v or w is adjacent to d_1, then G will have a subgraph isomorphic to H_{10}. Similarly v and w can not be s-defended by a common vertex. Thus v and w are s-defended by two distinct vertices $d_2, d_3(\neq d_1)$, both adjacent to d. But in

that case G will have a subgraph isomorphic to H_7, a contradiction.

Hence in all the possible cases, we arrive at a contradiction. Thus our assumption is not correct and $\gamma_{spsd}(G) = n - 2$. □

It is important to note that all the graphs $H_1, \ldots H_{11}$ in figure 2, contains at least one cycle of length either 3, 4 or 5. Thus we conclude that if G does not have a subgraph isomorphic to C_3, C_4 or C_5, then $\gamma_{spsd}(G) = n - 2$.

Corollary 2.5. *If a graph G has a subgraph $H \cong C_3$ or C_4 such that H has at least three vertices of degree greater than 2. Then $\gamma_{spsd}(G) < n - 2$.*

Proof. By hypothesis, H has a subgraph isomorphic to H_1, H_2, H_3, H_4, H_5 in figure 2. Therefore by the above Theorem, $\gamma_{spsd}(G) < n - 2$. □

From the above Corollary we have the following results.

Corollary 2.6. *For a tree T of order n*

$$\gamma_{spsd}(T) = \begin{cases} 1 & \textit{if } T \cong K_1 \\ n-1 & \textit{if } T \cong K_{1,n-1} \\ n-2 & \textit{if otherwise} \end{cases}$$

Proof. Since T has no subgraph isomorphic to $H_1, \ldots, H_{11}$, by Theorem 2.3, $\gamma_{spsd}(T) = n-2$ whenever $T \ncong K_1, K_{1,n-1}$. Further, if $T \cong K_{1,n-1}$, then from Corollary 2.3, $\gamma_{spsd}(T) = n - 1$. □

Corollary 2.7. *If $G \cong P_n \times P_m (n + m > 5)$, then $\gamma_{spsd}(G) < nm - 2$.*

Proof. Since $G \cong P_n \times P_m (n + m > 5)$, G has a subgraph isomorphic to H_5. Also, as $G \ncong K_{nm}, K_{1,nm-1}$, from Theorem 2.3, it follows that $\gamma_{spsd}(G) < nm - 2$. □

Corollary 2.8. *Let G be a cylinder graph $C_n \times P_m$. Then $\gamma_{spsd}(G) < nm - 2$.*

3. Concluding Remarks

We observed that for any graph G of order n, $1 \leq \gamma_{spsd}(G) \leq n-1$. Moreover $\gamma_{spsd}(G) = 1$ if and only if $G \cong K_n$. Also, $\gamma_{spsd}(G) = n - 1$ if and only if $G \cong K_{1,n-1}$. Hence if $G \ncong K_n, K_{1,n-1}$, then $2 \leq \gamma_{spsd}(G) \leq n - 2$. We could obtain a forbidden subgraph criteria involving 11 forbidden subgraphs to characterize graphs G with $\gamma_{spsd}(G) = n - 2$. It would be interesting to explore whether there could be a forbidden subgraph criteria to characterize graphs G with $\gamma_{spsd}(G) = n - 3$.

Problem 1. *Is it possible to have forbidden subgraph criteria characterizing graphs G with $\gamma_{spsd}(G) = n - 3$.*

Also, one may look for a more general problem of characterizing graphs G with $\gamma_{spsd}(G) = n - k$, k varying from 3 to $n - 3$.

Problem 2. *For any k $(3 \leq k \leq n - 3)$, characterize graphs G with $\gamma_{spsd}(G) = n - k$.*

The Petersen Graph G has a subgraph isomorphic to H_6 (see Figure 2) (subgraph obtained by removing unfilled circles in Figure 3) and therfore $\gamma_{spsd}(G) < n - 2$.

It would be interesting to explore which of these 11 forbidden subgraphs $H_1, \ldots, H_{11}$ are present in generalized petersen graph $G(n, k)$ and how does it helps in establishing the secure point-set domination number of generalized petersen graph $G(n, k)$.

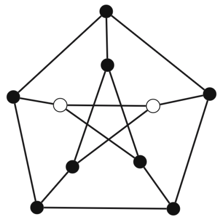

Figure 3. Petersen Graph

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