TWMS J. App. and Eng. Math. V.14, N.2, 2024, pp. 719-729

2-ODD LABELING OF GRAPHS USING SOME GRAPH OPERATIONS

A. A. PIR[1], A. PARTHIBAN[2*], K. SINGH[1], §

Abstract. 2-odd labeling assigns distinct integers to the nodes of a graph $G(V, E)$ in such a manner, that the positive difference of adjacent nodes is either 2 or an odd integer, $2k \pm 1, k \in N$. So, G is a 2-odd graph if and only if it permits 2-odd labeling. It is interesting and challenging to study certain important modifications through various graph operations on a given graph. These operations mainly modify the structure of the underlying graph and so having some understanding of the complex operations which can be done over a graph or a set of graphs is inevitable. The motivation behind the development of this article is to apply the concept of 2-odd labeling on graphs generated by using various graph operations, besides recalling a few interesting applications of graph labeling and graph coloring.

Keywords: 2-odd labeling, union, intersection, duplication, extension.

AMS Subject Classification (2020): 05C12, 05C76, 05C78.

1. Introduction

A graph labeling is a function that assigns integers to the lines or nodes, or both of G under some conditions. The importance of graph labeling includes its numerous applications in many areas like circuit design, radar, communication network address, etc. For a detailed study, see [2, 4, 5, 20]. The graphs used in the present study are simple, finite, undirected, and connected. Let Z be a set of all integers. A $2-$odd labeling of $G(V(G), E(G))$ is a 1-1 function $g : V(G) \to Z$ such that the positive difference between every pair of adjacent nodes, x_1 and x_2, i.e., $|g(x_1) - g(x_2)|$ is either $2k \pm 1; k \in N$ or exactly 2. Thus G is a $2-$odd graph if and only if there exists $2-$odd labeling of G. For more results, see [1, 8, 12].

The word "operation" is derived from the Latin word called opus ("work"). Operations may involve mathematical objects like graphs other than numbers. Likewise, in graph

[1] Lovely Professional University, Department of Mathematics, Phagwara, 144 411, Punjab, India.
email: pirajazahmad789@gmail.com; ORCID: https://orcid.org/0000-0002-7155-8461.
email: kulwinder.11033@lpu.co.in; ORCID: https://orcid.org/0000-0002-2433-6962.
[2] Mathematics Division, School of Advanced Sciences, Vellore Institute of Technology, Vellore, 632 014,Tamil Nadu, India.
email: mathparthi@gmail.com; ORCID: https://orcid.org/0000-0002-8375-4072.
[*] Corresponding author.
§ Manuscript received: April 20, 2022; accepted: September 14, 2022.

theory, graph operations produce new graphs from the initial ones. They can be divided into mainly two categories: (a) Unary operations: They give a new graph from the initial one. Elementary or editing operations produce a new graph from the initial one by a simple local change, such as addition or deletion of a vertex or of an edge, merging and splitting of vertices, edge contraction, etc. Advanced operations create a new graph from one initial one by complex changes, for instance; transpose graph, complement graph, line graph, dual graph, etc. (b) Binary operations: create a new graph from two initial ones H_1 and H_2, etc. For a detailed study on the same, one can see [11]. A few relevant definitions are recalled here. By $N(v)$ and $N[v]$, we mean an open and closed neighborhood of $v \in G$, respectively.

Definition 1.1. [21] *Duplication of a line $e = xy$ in H by a node z is formed by adding a new node z to H such that $N(z) = x, y$.*

Definition 1.2. [21] *Duplication of a node v_k in H by a line $e = v'_k v''_k$ is constructed by inserting new nodes v'_k, v''_k such that $N(v'_k) = v_k, v''_k$ and $N(v''_k) = v_k, v'_k$.*

These graph operations mainly modify the structure of the underlying graph and so having some understanding of the complex operations which can be done over a graph or a set of graphs are inevitable. So in this paper, 2 - odd labeling of graphs generated by using various graph operations such as union, disjoint union, intersection, duplication, and extension are obtained.

2. Main Results

This section is dedicated to deriving 2-odd labeling of some new classes of graphs. First, a few established results that are relevant to the study undertaken are given below.

Theorem 2.1. [20] *A graph is bipartite if and only if it contains no cycle of odd length.*

Theorem 2.2. [1, 8] *Every bipartite graph is 2 - odd.*

Theorem 2.3. [1, 8] *Every subgraph of a 2 - odd graph is 2-odd.*

Proposition 2.1. [8] *The complete graph $K_n, n \geq 5$ is not 2-odd.*

Theorem 2.4. [1] *The graph obtained by performing duplication of the vertex by an edge at all the vertices of a 2-odd graph admits 2-odd labeling if the Twin prime conjecture is true.*

Definition 2.1. [21] *Duplication of a line $e = xy$ in H by a line produces G by adding a line $e' = x'y'$ where $x'y'$ are newly inserted nodes to H such that $N(x') = N(x) \cup y' - y$ and $N(y') = N(y) \cup x' - x$.*

Theorem 2.5. *The graph obtained by duplicating edge by an edge at all the edges in any 2-odd graph admits 2-odd labeling.*

Proof. Let H be the given 2-odd graph with labeling h. Let $V(H) = \{v_1, v_2, \ldots v_n\}$ and G be obtained by duplicating edge by an edge at all the edges of H. So, $V(G) = \{V(H) \cup \{v'_i\} \cup \{v''_i\} : 1 \leq i \leq n\}$ and $|V(G)| = 3n$. Defining a function $f : V(G) \to Z$ gives the following three cases.
Case 1 : When $h(v_k)$ and $h(v_{k+1})$ are even
Without loss of generality, let $f(v'_k) = p_1$ and $f(v''_k) = p_2; 1 \leq k \leq n$, where p_1, p_2 are sufficiently large twin primes.
Case 2: When $h(v_k)$ is odd (even) and $h(v_{k+1})$ is even (odd)
Let $2k_s$ be the sufficiently large even number. Then $f(v'_k) = 2k_s$ and $f(v''_k) = 2k_s + 1$:

$k_s \in N$. A similar argument holds good when $h(v_k)$ is even and $h(v_{k+1})$ is odd. Thus G admits 2-odd labeling. Case 3: When $h(v_k)$ and $h(v_{k+1})$ are odd
Let $2k_r$ be the sufficiently large even number. Then $f(v_k') = 2k_r$ and $f(v_k'') = 2k_r + 2; k_r \in N$.

□

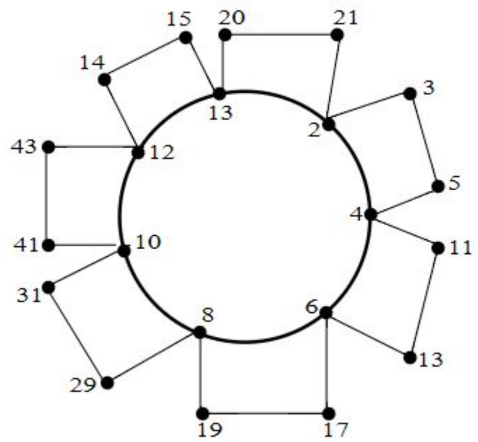

FIGURE 1. 2-odd labeling of the graph obtained by duplicating edge by an edge at all the edges in C_7

Definition 2.2. [21] *Duplication of a node v of H is formed by inserting a new node v' to H and introducing lines in such a way that $N(v') = N(v)$.*

Inspired by Theorem 2.4 and Theorem 2.5, the following conjecture is raised.

Conjecture 2.1. *The graph obtained by performing duplication of the vertex by a vertex at all the vertices of any 2-odd graph admits 2-odd labeling.*

Definition 2.3. [21] *An extension of a node x by a new node y in H produces G such that $N(y) = N[x]$.*

Theorem 2.6. *The graph obtained by performing an extension of the vertex by a vertex at all the vertices of any 2-regular graph admit 2-odd labeling.*

Proof. Let H be the 2-regular graph on n vertices, namely $v_1, v_2, \dots v_n$. Obtain G by performing extension of the vertex by a vertex at all the vertices of H by introducing the following new vertices $u_1, u_2, \dots u_n$, respectively to H as given in Definition 2.3 (See Fig. 7). Note that $V(G) = V_1 \cup V_2$ where $V_1 = \{v_i; 1 \leq i \leq n\}$ and $V_2 = \{u_i; 1 \leq i \leq n\}$ and also $|V(G)| = 2n$ and $|E(G)| = 4n$. Defining a function $g : V(G) \to Z$ gives rise to the following two cases.
Case 1 : When H is of order $n \equiv 0(mod \quad 2)$
Let $g(v_i) = 3i; 1 \leq i \leq n$ and without loss of generality, $g(u_1) = 1$. Then $g(u_i) = g(u_{i-1}) + 3; 2 \leq i \leq n$.
Case 2 : When H is of order $n \equiv 1(mod \quad 2)$

Without loss of generality, let $g(v_1) = 2, g(v_2) = 3, g(v_3) = 6$. Then $g(v_i) = g(v_{i-1})+2; 4 \leq i \leq n-1$, and $g(v_n) = g(v_{n-1}) + 1$. Again let, $g(u_1) = 0, g(u_2) = 4, g(u_3) = 5$. Then $g(u_i) = g(u_{i-1}) + 2; 4 \leq i \leq n-1$, and $g(u_n) = g(u_{n-1}) + 4$. Thus one can easily check that G admits 2-odd labeling. □

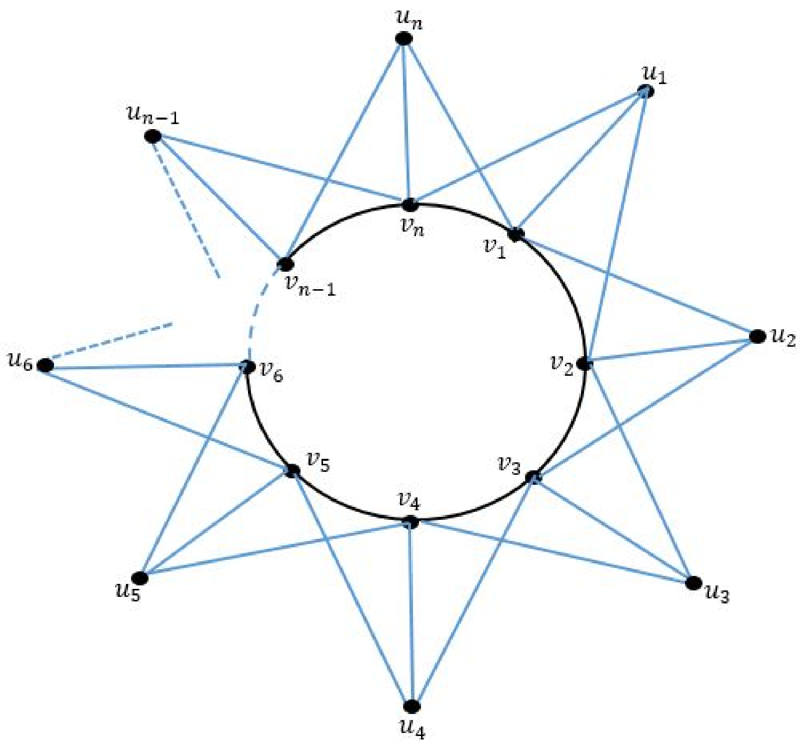

FIGURE 2. The graph obtained by performing an extension of vertex by a vertex at all the vertices of C_n

.

Motivated by Theorem 2.6, the following conjecture is formulated.

Conjecture 2.2. *The graph formed by performing an extension of vertex by a vertex at all the vertices of the given 2-odd graph admits 2- odd labeling.*

Definition 2.4. [13] *The disjoint union of graphs combines two or more graphs to form the largest graph.*

Theorem 2.7. *The disjoint union of any finite copies of 2-odd graph permits 2 - odd labeling.*

Proof. Let $H(n, m)$ be the given 2-odd graph with a 2-odd labeling h. Let $V(H) = \{u_1, u_2, ..., u_n\}$. Without loss of generality, let $h(u_k) = \max\limits_{1 \leq i \leq n} \{h(u_i)\} = r_n$. Construct the disconnected graph G as a disjoint union of H_i's, i.e., $G = kH$ as shown in Fig 4 with $V(G) = \{v_j^i : 1 \leq j \leq n; 1 \leq i \leq k\}$ and $E(G) = \{e_j^i : 1 \leq j \leq m; 1 \leq i \leq k\}$. Define a function $g : V(G) \to Z$ as follows: $g(v_j^1) = h(v_j)$ for $1 \leq j \leq n; g(v_j^2) = g(v_j^1) + r_{n_1}$, where $r_{n_1} = r_n + 1$. Next $g(v_j^3) = g(v_j^1) + r_{n_2}$, where $r_{n_2} = \max\limits_{1 \leq j \leq n} g(v_j^2) + 1$. Proceeding thus, we

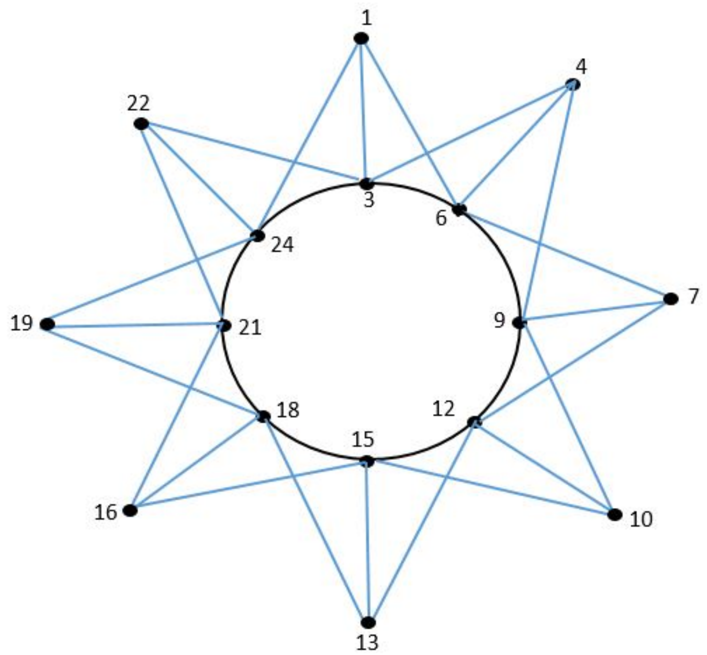

FIGURE 3. 2-odd labeling of the graph obtained by performing an extension of vertex by a vertex at all the vertices of C_8

.

have $g(v_j^k) = g(v_j^1) + r_{n_{k-1}}$, where $r_{n_{k-1}} = \max_{1\leq j\leq n} g(v_j^{k-1}) + 1$. A simple check shows that g is the desired 2-odd labeling of G. □

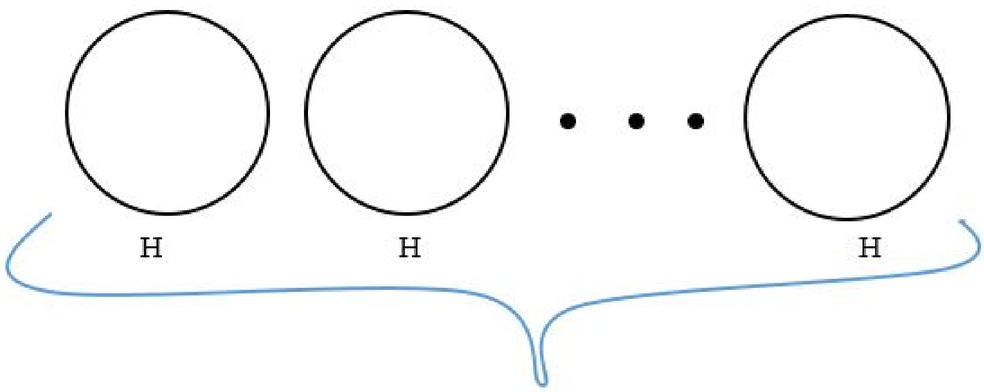

FIGURE 4. $G = kH$

.

Definition 2.5. [19] *The union of two graphs H_1 and H_2, is a graph such that $V(H_1 \cup H_2) = V(H_1) \cup\ V(H_2)$ and $E(H_1 \cup H_2) = E(H_1) \cup\ E(H_2)$.*

Theorem 2.8. *The graph obtained as a union of any finite copies of a 2-odd graph permits 2 - odd labeling.*

Proof. Let H be the given 2-odd graph on n vertices. Take k copies of H, namely $H_i; 1 \leq i \leq k$ and obtain $G = \bigcup_{i=1}^{k} H_i$ with $|V(G)| = kn$. Let $h_1, h_2, ..., h_k$ be the 2-odd labeling of $H_1, H_2, ..., H_k$, respectively. Defining $g : V(G) \to Z$ gives rise to the following three cases.
Case 1: When h_i's are distinct
Clearly, the labels induced by h_i are unique for each H_i. Since each component, H_i admits 2-odd labeling, the graph is clearly a disjoint union of 2-odd graphs and the proof eventually follows from Theorem 2.7.
Case 2: When $h_i's$ are same
Clearly, $h_1(H_1) = h_2(H_2) = \cdots = h_k(H_k)$. The resultant graph is H itself considering the fact from definition 2.5 of the union of graphs. Further, $|V(G)| = n$ that is any one of the k copies of $H_i : 1 \leq i \leq k$, which is already a 2-odd graph.
Case 3: When some of the $h_i's$ are distinct and the others are not
Without loss of generality, assume that the first r copies of $H_i's$ have the 2-odd labeling with same labels and the remaining $k - r$ copies of $H_i's$ have 2-odd labeling with distinct labels. Now the resultant graph G has one of the r copies of $H_i's$ and all the $k - r$ copies of $H_i's$. Here $|V(G)| = n(k - r + 1)$. Establishing 2-odd labeling for the resultant graph G is done on similar lines that of Theorem 2.7.
Case 4: When some labels are repeated within the graphs $H_i's$
Here we get a new resultant graph G' obtained from G. Clearly, $|V(G')| < |V(G)|$ and $|E(G')| < |E(G)|$. Define $g' : V(G') \to Z$ as follows: without loss of generality, let $g'(V(G')) = g(V(G))$. Thus in all four cases, graph G admits 2-odd labeling. □

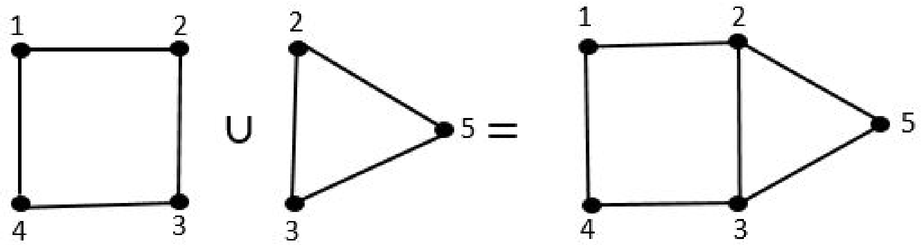

FIGURE 5. 2-odd labeling of the graph obtained by taking union of two 2-odd graphs

.

Definition 2.6. [14, 18] *A graph H is said to be an arbitrary super subdivision of $G(p, q)$, denoted by $ASS(G)$, if H is drawn from G by replacing every edge e_i of G by K_{2,m_i} (for some $m_i : 1 \leq i \leq q$) in such a way that the ends of each e_i are merged with the two vertices of the 2-vertices part of K_{2,m_i} after removing the edge e_i from G. Thus, $|V(H)| = p + \sum_{i=1}^{q} m_i$ and $|E(H)| = \sum_{i=1}^{q} 2m_i$.*

Lemma 2.1. *Arbitrary super subdivision of any graph is bipartite.*

Proof. Let H be the given graph and G be the graph obtained from H by taking an arbitrary super subdivision of H. There are two cases:
Case 1: When H is bipartite.
There is nothing to prove.
Case 2: When H is not bipartite.
By considering the fact from Definition 2.6 that the graph G obtained from H by taking an arbitrary super subdivision does not contain any odd cycle, thus the proof follows from the Theorem 2.1. □

Definition 2.7. [7, 17] *A graph H is said to be a subdivision of a graph $G(n,m)$, denoted by $S(G)$, if H is constructed by subdividing every edge of G exactly once. Thus, $|V(H)| = n+m$ and $|E(H)| = 2m$.*

Lemma 2.2. *The subdivision of any graph is bipartite.*

Definition 2.8. [6, 17] *A graph H is said to be a super subdivision of $G(p,q)$, denoted by $SS(G)$, if it is formed from G by replacing every edge e of G by a complete bipartite graph $K_{2,m}$. Thus, $|V(H)| = p+mq$ and $|E(H)| = 2mq$.*

Lemma 2.3. *The super subdivision of any graph is bipartite.*

Remark 2.1. *The proofs of Corollary 2.2 and Corollary 2.3 follow in similar lines of Lemma 2.1.*

Theorem 2.9. *The graph obtained by performing an arbitrary super subdivision of any graph admits 2-odd labeling.*

Proof. The proof clearly follows from Lemma 2.1 and Theorem 2.2. □

Conjecture 2.3. *The graph obtained by performing subdivision of any graph admits 2-odd labeling.*

Conjecture 2.4. *The graph obtained by performing supersubdivision of any graph admits 2-odd labeling.*

Remark 2.2. *The proofs of Corollary 2.3 and Corollary 2.4 follow in the similar lines of Theorem 2.9.*

Definition 2.9. [15] *A graph G in which a node is distinguished from other nodes is called a rooted graph and the node is called the root of G. The graph G^k is formed by identifying the roots of k copies of a rooted graph G is called a one-point union of the k copies of G.*

Theorem 2.10. *One-point union of any finite number of 2-odd graphs permits 2-odd labeling if each 2-odd graph has distinct vertex labels except for one vertex.*

Proof. Let $G_i; 1 \leq i \leq k$ be the given 2-odd graphs and $n_1, n_2, n_3, \ldots n_k$ be the cardinalities of $G_i's$, respectively. Note that each G_i is a 2-odd graph with the condition that every G_i has distinct vertex labels except for exactly one vertex, say v_1. That is., if g_i is the 2-odd labeling of the corresponding $G_i : 1 \leq i \leq k$, respectively, then $g_1(G_1) \neq g_2(G_2) \neq g_3(G_3), ... \neq g_k(G_k)$ except that $g_1(v_1) = g_2(v_1) = g_3(v_1) = ... = g_k(v_1)$. Now without loss of generality, obtain G by fusing all the $G_i's : 1 \leq i \leq k$ at v_1 as a one-point union of G_i with $|V(G)| = \sum\limits_{i=1}^{k} n_i - k + 1$. Define $\beta : V(G) \to Z$ by letting $\beta(v_1) = g_1(v_1)$ and $\beta(G - v_1) = g_i(G_i - v_1) : 1 \leq i \leq k$. So one can evidently see that β induces 2-odd labeling of G. □

Inspired by Theorem 2.10, the following conjecture is proposed.

Conjecture 2.5. *One point union of any finite number of 2-odd graphs permits 2-odd labeling.*

Definition 2.10. [16] *Let $H_1, H_2, \dots H_k$ be $k \geq 2$ copies of a graph H. The graph $H(k)$ is formed by introducing an edge between H_i and $H_{i+1}; i = 1, 2, ..., k-1$. The graph $H(k)$ is called the path-union of k copies of H.*

Theorem 2.11. *If P_k is the path of length k and H is a 2-odd graph, then the graph obtained by guling H at every node of P_k admits 2-odd labeling.*

Proof. Let H be the given 2-odd graph on n vertices, namely $v_1, v_2, ..., v_n$. Since H is a 2-odd graph, it has 2-odd labeling say, $h : V(H) \to Z$ such that the induced edge labels are either 2 or $2r \pm 1, r \in N$. Also assume that $h(v_1) = 0$. Take k copies of H, namely $H_i; 1 \leq i \leq k$, and obtain G as a path union of $H_i's$ with $V(G) = v_i^j : 1 \leq i \leq n, 1 \leq j \leq k$. Let the vertices of the newly introduced path be merged with the vertices that are assigned 0 in each copy. Clearly $|V(G)| = nk$. Define $f : V(G) \to Z$ as follows: without loss of generality, let $f(H_1) = h(H_1)$. If k_1 is the largest label used in H_1, then let r_1 be sufficiently larger than k_1 so that $|f(v_1^1) - f(v_1^2)|$ is either 2 or odd and also $f(v_i^2) = h(v_i^2) + r_1 : 2 \leq i \leq n$. Similarly, if k_2 is the largest label used in H_2, then let r_2 be sufficiently larger than k_2 so that $|f(v_1^2) - f(v_1^3)|$ is either 2 or odd and also $f(v_i^3) = h(v_i^3) + r_2 : 2 \leq i \leq n$. By continuing this way, If k_t is the largest label used in H_{k-1}, then let r_t be sufficiently larger than k_t so that $|f(v_1^{k-1}) - f(v_1^k)|$ is either 2 or odd and also $f(v_i^k) = h(v_i^k) + r_t : 2 \leq i \leq n$. By the above process, one can see that the path union of finite copies of any 2-odd graph admits 2-odd labeling. □

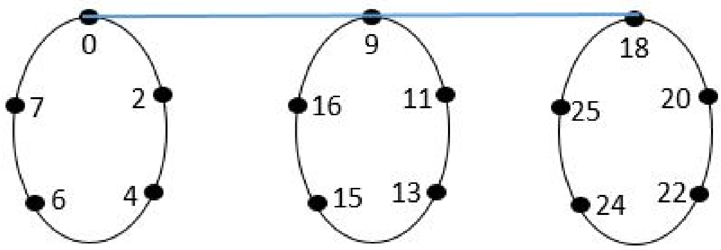

FIGURE 6. 2-odd labeling of the graph obtained by guling C_5 at every node of P_3

.

Definition 2.11. [19] *Given two graphs H_1 and H_2, with a minimum one of node in common, then the intersection of H_1 and H_2 is a graph such that $V(H_1 \cap H_2) = V(H_1) \cap V(H_2)$ and $E(H_1 \cap H_2) = E(H_1) \cap E(H_2)$.*

Theorem 2.12. *The graph obtained as an intersection of a finite number of 2-odd graphs permits 2-odd labeling.*

Proof. Let H_i denote the given 2-odd graphs on n_i vertices with 2-odd labeling h_i, respectively for $1 \leq i \leq k$. Obtain $G = \bigcap_{i=1}^{k} H_i$. Defining $g : V(G) \to Z$ gives rise to the following two cases.
Case 1: When each H_i has distinct labeling h_i
Obviously, the resultant graph is null and there is nothing to prove.
Case 2: When H_i's have some common labels
The result followed by Theorem 2.3 as the resultant graph is clearly a subgraph of the given 2-odd graphs.
Thus g is clearly 2-odd labeling of G. □

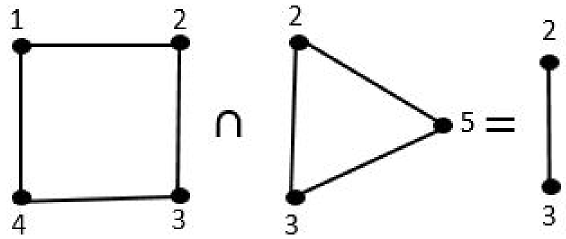

FIGURE 7. 2-odd labeling of the graph obtained by taking intersection of two 2-odd graphs

.

3. Applications of Graph Labeling and Graph Coloring

Graph labeling is one of the most interesting concepts in graph theory which has numerous applications in different fields. A graph labeling is a function that assigns integers to the lines or nodes, or both of G under some conditions. The importance of graph labeling includes its numerous applications in many areas like circuit design, radar, communication network addressing, fault- tolerant system design, and automatic channel allocation, etc. For a detailed study, see [4, 5, 20].

By a vertex $k-$ coloring of G we mean a mapping $c : V(G) \to \{1, 2, \ldots, k\}$ and c are called a proper $k-$ coloring if the adjacent vertices are colored differently. The chromatic number $\chi(G)$ of G is the smallest k such that G has a proper vertex $k-$ coloring. Graph coloring is especially used various in research areas of computer science such as data mining, image segmentation, clustering, image capturing, networking, etc. For instance, a data structure can be designed in the form of a tree which in turn utilized nodes and lines. In the same way, the most important concept of graph coloring is utilized in resource allocation, and scheduling. Moreover, paths; walks, and circuits in graph theory are used in tremendous applications like traveling salesman problem, database design concepts, and resource networking.

The application of graph coloring is also used in guarding an art gallery [3]. Final exam timetabling requires satisfactory assignment of timetable slots to a set of exams. Each exam is taken by a number of students, based on a set of constraints. Ozcan et. al. [10] proposed a mimetic algorithm (MA) for solving final exam timetabling at Yeditepe

University. Considering the task of minimizing the number of exam periods and removing the clashes, final exam timetabling reduces to the graph coloring problem [9].

4. Conclusion

In this present study, a few general results concerning 2-odd labeling of certain graphs using some graph operations such as union, intersection, disjoint union, one point union, path union, duplication, extension, and arbitrary subdivision are explored, besides, formulating some interesting conjectures. The study undertaken may serve as a path in finding the complete characterization of 2-odd labeling which is still an open problem. The concept of 2-odd labeling of graphs may find its applications in graph-based cryptography and network security.

Acknowledgement. The authors would like to extend their heartfelt gratitude to the anonymous reviewers for their valuable suggestions and comments to improve presentation of the paper.

References

[1] Abirami, P., Parthiban, A. and Srinivasan, N., (2020), On 2 - odd labeling of graphs, European Journal of Molecular and Clinical Medicine, 07, pp. 3914 – 3918.
[2] Acharya, B.D., Arumugam, S., and Rosa, A., (2008), Labeling of discrete structures and Applications, Narosa publishing house, New Delhi.
[3] Besjana Tosuni, (2015) Graph coloring problems in modern computer science, European Journal of Interdisciplinary Studies, 1 (2), pp. 87-95.
[4] Bloom, G.S., and Glomb, S.W., (1977), Application of numbered undirected graphs, Proc. IEEE, 65, pp. 562-570.
[5] Gross, J., and Yellen, J., (1999), Graph theory and its applications, CRC Press, London.
[6] Kathiresan, K. M., and Amutha, S., (2004), Arbitrary supersubdivision of stars are graceful, Indian J. Pure and Appl. Math., 35, pp. 81 – 84.
[7] Kathiresan, K. M., (1992), Subdivision of ladders are graceful, Indian Journal of Pure Appl Math., 23(1), pp. 21 – 23.
[8] Laison, J.D., Starr, C., and Walker, A., (2013), Finite prime distance graphs and 2-odd graphs, Discrete Mathematics, 313, pp. 2281-2291.
[9] Leighton, F.T., (1979), A graph coloring algorithm for large scheduling problems, Journal of Research of the National Bureau Standard, 84, pp. 79-100.
[10] Ozcan, E., and Ersoy, E., (2005), Final exam scheduler - FES, Proc. of IEEE Congress on Evolutionary Computation, 2, pp. 1356-1363.
[11] Meenakshi Paramasivan, (2017), Operations on graphs, arrays and automata, Ph.D. Thesis submitted to University of Trier, pp. 1-201.
[12] Parthiban, A., and Gnanamalar David, N., On prime distance labeling of graphs, (2017), in: Arumugam S., Bagga J., Beineke L., Panda B.(eds) Theoretical Computer Science and Discrete Mathematics.ICTCSDM 2016, Lecture Notes in Computer Science, 10398.Springer, Cham., 31, pp. 238-241.
[13] Rosen Kenneth, H., (1999), Handbook of discrete and combinatorial mathematics, Discrete Mathematics and Its Applications, CRC Press, pp. 515.
[14] Sethuraman, G., and Selvaraju, P., (2001), Gracefulness of super subdivision of graphs, Indian Journal of Pure and Appl. Math., 32(7), pp. 1059 – 1064.
[15] Sze-Chin Shee, and Yong-Song Ho., (1993), The cordiality of the one-point union of n copies of a graph, Discrete Mathematics, 117, pp. 225-243.
[16] Sze-Chin Shee, Yong-Song Ho, (1996), The cordiality of the path-union of n copies of a graph, Discrete Mathematics, 151, pp. 221-229.
[17] Ujwala Deshmukh, and Bhatavadekar, S. A., (2018), Gracefulness of some arbitrary super subdivision of cycle, Journal of Informatics and Mathematical Sciences, 10(4), pp. 599-611.
[18] Vaidya, S. K., Vihol, P. L., Dani, and Bantva, D. D., (2010), L(2,1)-Labeling in the context of some graph operations, Journal of Mathematics Research, 2(3), pp. 109-119.
[19] Vasudev, C., Graph theory with applications, New Age International Publishers.

[20] West, D.B., (2000), Introduction to Graph Theory, 2nd ed. Englewood Cliffs, NJ: Prentice-Hall.
[21] Yegnanarayanan, V., and Parthiban, A., (2012), On graph theory application-I, Applied Research and Development Institute Journal, 3(4), pp. 28-42.

Ajaz Ahmad Pir is a Ph.D. research scholar in the Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Punjab, India. He is doing his research in the area of graph labeling.

Dr. A. Parthiban is presently working as an Assistant Professor in Mathematics, Vellore Institute of Technology, Vellore, Tamil Nadu, India. He served as an Assistant Professor at LPU, Punjab (2018-2022). He has obtained his Ph.D from University of Madras, Chennai. His research area includes graph labeling, graph coloring, equitable power domination & algebraic graph theory.

Dr. Kulwinder Singh is presently working as an associate professor in the Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Punjab, India. His area of interest includes micropolar elasticity, thermoelasticity, fractional order thermoelasticity, wave propagation in composite structures and graph theory etc.