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SOME PLANAR GRAPHS WITH TEN-SIDED FACES AND THEIR METRIC DIMENSION

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ABSTRACT. Let $\Gamma = (V, E)$ be a non-trivial planar connected graph with vertex set V and edge set E. A set of ordered vertices R from $V(\Gamma)$ is said to be a resolving set for Γ if each vertex of Γ is uniquely determined by its vector of distances to the vertices of R. The number of vertices in a smallest resolving set is called the metric dimension of Γ. In this article, we study the metric dimension for two families of planar graphs, each of which is shown to have an independent minimum resolving set with cardinality three.

Keywords: Metric dimension, independent set, metric basis, planar graph, resolving set, connected graph.

AMS Subject Classification: 05C10, 05C12, 05C90.

1. INTRODUCTION AND PRELIMINARIES

The graphs considered in this article are simple, undirected, and connected. For the graph $\Gamma = \Gamma(V, E)$, $E(\Gamma)$ and $V(\Gamma)$ represent its edge set and vertex set respectively. The minimum number of edges between the two vertices p and q in Γ, denoted by $d(p, q)$, is the distance between p and q. The totality of edges that are incident to a vertex of Γ is known as its degree (valency).

An ordered subset $R \subseteq V$ of distinct vertices is said to be a *resolving set* if every pair of different vertices of Γ are resolved by at least one vertex of R. In other words, for a subset of vertices, $R = \{x_1, x_2, x_3, ..., x_q\}$ of Γ, any vertex $\beta \in V$ can be represented uniquely in the form of a q-vector $\varphi(\beta|R) = (d(x_1, \beta), d(x_2, \beta), d(x_3, \beta), ..., d(x_q, \beta))$. Then, the set R is a resolving set for Γ, if $\varphi(p|R) = \varphi(q|R)$ implies that $p = q$ for all $p, q \in V$. Next, the resolving set R is said to be the *metric basis* for Γ, if the set R has the least possible cardinality in Γ, and this least cardinality is known as the *metric dimension* (*location number*) of Γ, represented by $dim_v(\Gamma)$. A subset R of distinguishable vertices in Γ is said

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to be an independent resolving set for Γ, if R is independent as well as resolving set.

For a subset $R = \{x_1, x_2, x_3, ..., x_q\}$ of distinct ordered vertices in Γ, the l^{th} component (distance coordinate) of $\varphi(x|R)$ is zero if and only if $x = x_l$. Therefore, in order to check that the set R is a resolving set in Γ, it is sufficient to prove that $\varphi(p|R) \neq \varphi(q|R)$ for each pair of distinct vertices $p, q \in V(\Gamma) \setminus R$.

The first paper consisting of the concepts of resolving set and that of minimum resolving set were introduced by Slater [18], in association with the problem of recognizing the location of a thief or an intruder in a given network. He used the terms *location number* and *locating set*, to describe the cardinality of a minimum resolving set and a resolving set of a given network, respectively. Harary and Melter [7] independently introduced the same concept, but used the terms resolving set and metric dimension, rather than locating set and location number as used by Slater, respectively.

Afterward, these concepts were studied in-depth, Melter and Tomescu [11] employed the concept of metric dimension in image processing and pattern recognition, Sebo and Tannier [13] discussed the notion of metric dimension in terms of combinatorial optimization, Cáceres et al. [4] employed these concepts on coin weighing problems and mastermind games, Khuller et al. [10] found an application of metric dimension in the navigation of robots, Beerloiva et al. [2] discussed these ideas to network discovery and verification, Chartrand et al. [5] studied applications to chemical science, Slater [18] discussed problems related to SONAR (Sound Navigation and Ranging), coastguard LORAN (Long-Range Navigation), facility location problems, etc.

The notion of metric dimension has recently been addressed for a lot of significant graph families. For instance, Javaid et al. [9] studied metric dimension for regular graphs, circulant graphs & Harary graphs, Sharma and Bhat [14, 15, 16] considered several graph families and obtained their metric dimension, Khuller et al. [10] investigated these notions for trees, Sharma and Bhat [17] also obtained the metric dimension of the line graph of the subdivision graph of the graph of convex polytope. For a comparative study of graph parameters and metric dimension of more algebraic flavor, see [1] by Cameron and Bailey. For a survey on some variations and metric dimension, see Saenpholphat and Zhang [12] and Chartrand and Zhang [6]. For more detail on metric dimension, readers are refer to [3, 13, 19].

In this manuscript, we construct few planar graphs with some specific properties, that is, they consist of n sided faces (where even $n \geq 10$) and $\frac{m(n+12)}{4}$ number of vertices. We represent these graphs as $\mathbb{A}_m^n$ (or $\mathbb{B}_m^n$), where $m = 2s$; $s \geq 3$ and $n = 2k$; $k \geq 5$ with $s, k \in \mathbb{N}$. In this article, we take $k = 5$ and so consider two families, represented by $\mathbb{A}_m^{10}$ and $\mathbb{B}_m^{10}$, of planar graphs (see Fig. 1 and 2, respectively), for which we determine their metric dimension. We also prove that the graphs $\mathbb{A}_m^{10}$ and $\mathbb{B}_m^{10}$ possesses an independent minimum resolving sets of cardinality three, that is, only three vertices are the minimum requirement for the unique identification of all vertices in the planar graphs $\mathbb{A}_m^{10}$ and $\mathbb{B}_m^{10}$.

This article is organized as follows. In Section 2, we investigate the metric dimension of the planar graph $\mathbb{A}_m^{10}$. In Section 3, we investigate the metric dimension of the planar graph $\mathbb{B}_m^{10}$. Finally, the conclusion and future work of this paper is presented in Section 4.

2. Location Number of Planar graph $\mathbb{A}_m^{10}$

In this section, we introduce a new family of planar graph, represented by $\mathbb{A}_m^{10}$, and for which we investigate its metric dimension.

The planar graph $\mathbb{A}_m^{10}$ (where $m = 2s$; $s \geq 3$ & $s \in \mathbb{N}$) comprises of $\frac{11m}{2}$ and $8m$ number of vertices and edges respectively. It consists of m faces each are having four sides, m faces each are having five sides, $\frac{m}{2}$ faces each are having ten sides, and two faces each are having m-sides, and is shown in Fig. 1. The vertex set $V(\mathbb{A}_m^{10})$ and the edge set $E(\mathbb{A}_m^{10})$ for $\mathbb{A}_m^{10}$, respectively are given by $V(\mathbb{A}_m^{10}) = \{a_h, b_h, f_h, g_h | 1 \leq h \leq m\} \cup \{c_h, d_h, e_h | 1 \leq h \leq s\}$ and $E(\mathbb{A}_m^{10}) = \{a_h b_h, a_h a_{h+1}, f_h g_h, g_h g_{h+1} | 1 \leq h \leq m\} \cup \{c_h d_h, d_h e_h | 1 \leq h \leq s\} \cup \{b_{2h-1} c_h, b_{2h} c_h, f_{2h-1} e_h, f_{2h} e_{h+1} | 1 \leq h \leq s\} \cup \{b_h b_{h+1} | h \text{ is even } \& \ 1 \leq h \leq m\} \cup \{f_h f_{h+1} | h \text{ is odd } \& \ 1 \leq h \leq m\}$. Next, it is important to note that, $a_1 = a_{m+1}$, $b_1 = b_{m+1}$, $c_1 = c_{s+1}$, $d_1 = d_{s+1}$, $e_1 = e_{s+1}$, $f_1 = f_{m+1}$ and $g_1 = g_{m+1}$ (whenever necessary).

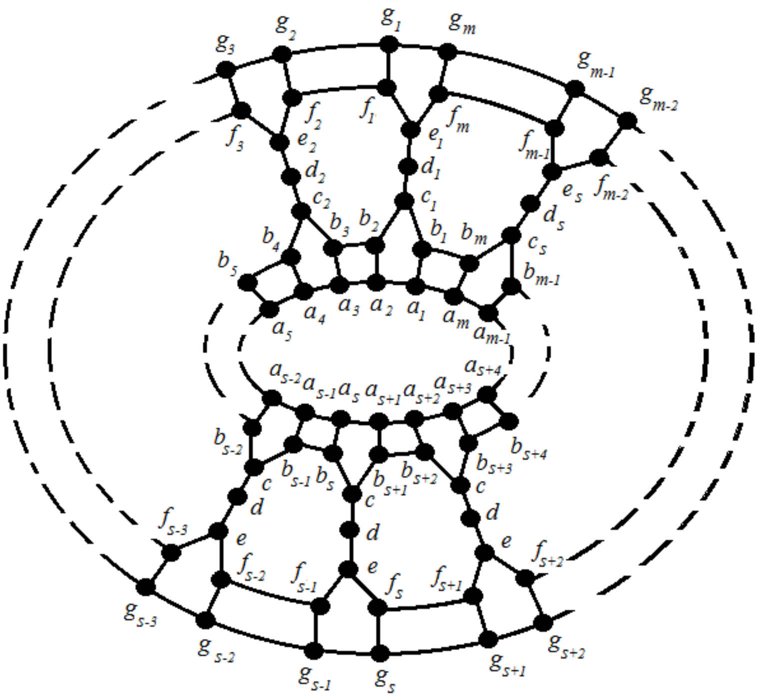

Figure 1. Planar graph $\mathbb{A}_m^{10}$

We refer the collection of vertices $\{a_l : 1 \leq l \leq m\}$ in $\mathbb{A}_m^{10}$, as the a-vertices, the collection of vertices $\{b_l : 1 \leq l \leq m\}$ in $\mathbb{A}_m^{10}$, as the b-vertices, the collection of vertices $\{c_l : 1 \leq l \leq s\}$ in $\mathbb{A}_m^{10}$, as the c-vertices, the collection of vertices $\{d_l : 1 \leq l \leq s\}$ in $\mathbb{A}_m^{10}$, as the d-vertices, the collection of vertices $\{e_l : 1 \leq l \leq s\}$ in $\mathbb{A}_m^{10}$, as the e-vertices, the collection of vertices $\{f_l : 1 \leq l \leq m\}$ in $\mathbb{A}_m^{10}$, as the f-vertices, and the collection of vertices $\{g_l : 1 \leq l \leq m\}$ in $\mathbb{A}_m^{10}$, as the g-vertices.

Next, for the graph $\mathbb{A}_m^{10}$, the vertex set $V(\mathbb{A}_m^{10})$ is as follows $\{a_h, b_h, f_h, g_h | 1 \leq h \leq m\} \cup \{c_h, d_h, e_h | 1 \leq h \leq s\}$. Then, we represent the set of metric codes for the planar graph $\mathbb{A}_m^{10}$ as follows: $R_1 = \{\varphi(a_h|R) | 1 \leq h \leq m\}$, $R_2 = \{\varphi(b_h|R) | 1 \leq h \leq m\}$, $R_3 =$

$\{\varphi(c_h|R)|1 \leq h \leq s\}$, $R_4 = \{\varphi(d_h|R)|1 \leq h \leq s\}$, $R_5 = \{\varphi(e_h|R)|1 \leq h \leq s\}$, $R_6 = \{\varphi(f_h|R)|1 \leq h \leq m\}$, and $R_7 = \{\varphi(g_h|R)|1 \leq h \leq m\}$. Next, we prove that the graph $\mathbb{A}_m^{10}$ consists of a minimum resolving set R with cardinality three i.e., $|R| = dim(\mathbb{A}_m^{10}) = 3$.

Theorem 2.1. *Let $\mathbb{A}_m^{10}$ be a planar graph as defined above. Then, $dim_v(\mathbb{A}_m^{10}) = 3 \; \forall \; m \geq 6$.*

Proof. The natural number m in $\mathbb{A}_m^{10}$ is always given by $m = 2s$; where $s \geq 3$ and $s \in \mathbb{N}$. Now, to investigate that the planar graph $\mathbb{A}_m^{10}$ consists of a resolving R with $|R| = dim_v(\mathbb{A}_m^{10}) = 3$, we consider the two cases based upon the natural number s i.e., when s is odd ($s \equiv 1 \; (mod\; 2)$) and when it is even ($s \equiv 0 \; (mod\; 2)$).

Case(I) $s \equiv 0 \; (mod\; 2)$.
Then, $m = 2s$ such that $s \geq 4$ and $s \in \mathbb{E}_s$ (where $s = 2k$; $k \geq 2$ and $k \in \mathbb{N}$). Let $R = \{a_2, a_{s+1}, a_m\} \subset V(\mathbb{A}_m^{10})$. To complete the proof for this case, we have to show that the set R is a minimum resolving set for $\mathbb{A}_m^{10}$ (Note that, for $6 \leq m \leq 12$, one can verify easily that the set R is a resolving set for $\mathbb{A}_m^{10}$). For $m \geq 14$, we can give the metric codes for every element of $V(\mathbb{A}_m^{10})$ corresponding to the set R.

For the set of a-vertices $\{a_h : 1 \leq h \leq m\}$ in $\mathbb{A}_m^{10}$, metric codes are as follows:

$$\varphi(a_h|R) = \begin{cases} (1, s, 1), & h = 1; \\ (h-2, s-h+1, h), & 2 \leq h \leq s \\ (s-1, 0, s-1), & h = s+1 \\ (2s-h+2, h-s-1, 2s-h), & s+2 \leq h \leq 2s-1. \end{cases}$$

For the set of b-vertices $\{b_h : 1 \leq h \leq m\}$ in $\mathbb{A}_m^{10}$, metric codes are as follows: $\varphi(b_h|R) = \varphi(a_h|R) + (1,1,1)$ for all $1 \leq h \leq m$. Next, for the set of c-vertices $\{c_h : 1 \leq h \leq s\}$ in $\mathbb{A}_m^{10}$, metric codes are as follows:

$$\varphi(c_h|R) = \begin{cases} (2, s+1, 3), & h = 1; \\ (2h-1, s-2h+3, 2h+1), & 2 \leq h \leq k; \\ (s+1, 2, s), & h = k+1; \\ (2s-2h+4, 2h-s, 2s-2h+2), & k+2 \leq h \leq s. \end{cases}$$

For the set of d-vertices $\{d_h : 1 \leq h \leq s\}$ in $\mathbb{A}_m^{10}$, metric codes are as follows: $\varphi(d_h|R) = \varphi(c_h|R) + (1,1,1)$ for all $1 \leq h \leq s$. For the set of e-vertices $\{e_h : 1 \leq h \leq s\}$ in $\mathbb{A}_m^{10}$, metric codes are as follows: $\varphi(e_h|R) = \varphi(d_h|R) + (1,1,1)$ for all $1 \leq h \leq s$. Next, for the set of f-vertices $\{f_h : 1 \leq h \leq m\}$ in $\mathbb{A}_m^{10}$, metric codes are as follows:

$$\varphi(f_{even}|R) = \begin{cases} (6, s+2, 7), & h = 2; \\ (h+3, s-h+4, h+5), & 4 \leq h \leq s-2; \\ (s+3, 5, s+3), & h = s; \\ (2s-h+5, h-s+4, 2s-h+3), & s+2 \leq h \leq 2s-2; \\ (2s-h+5, h-s+4, 5), & h = 2s. \end{cases}$$

and

$$\varphi(f_{odd}|R) = \begin{cases} (5, s+3, 6), & h = 1; \\ (h+3, s-h+4, h+5), & 3 \leq h \leq s-3; \\ (s+2, 6, s+4), & h = s-1; \\ (2s-h+5, h-s+4, 2s-h+3), & s+1 \leq h \leq 2s-3; \\ (2s-h+5, h-s+4, 5), & h = 2s-1. \end{cases}$$

Finally, for the set of g-vertices $\{g_h : 1 \leq h \leq m\}$ in $\mathbb{A}_m^{10}$, metric codes are as follows: $\varphi(g_h|R) = \varphi(f_h|R) + (1,1,1)$ for all $1 \leq h \leq m$.

Now, from all the above metric codes, we find that no two codes are the same as the sets of metric codes holds $|R_1| = |R_2| = |R_6| = |R_7| = m$, $|R_3| = |R_4| = |R_5| = s$, and all are pairwise disjoint. Thus, from this we conclude that $dim_v(\mathbb{A}_m^{10}) \leq 3$. Next, in order to finish this case, we have to prove that $dim_v(\mathbb{A}_m^{10}) \geq 3$. To obtain this, we show that no resolving set R with $|R| = 2$ exist. Suppose $dim_v(\mathbb{A}_m^{10}) = 2$, on the contrary. Then, we must have to discuss the following possibilities.

Resolving sets	Contradictions
$R = \{a_1, a_j\}$; $2 \leq j \leq m$	$\varphi(b_1|R) = \varphi(a_m|R)$ for $2 \leq j \leq s$; and $\varphi(a_2|R) = \varphi(a_m|R)$ for $j = s+1$, a contradiction.
$R = \{b_1, b_j\}$; $2 \leq j \leq m$	$\varphi(f_m|R) = \varphi(f_1|R)$ for $j = 2$; $\varphi(g_m|R) = \varphi(f_{m-1}|R)$ for $j = 3$; $\varphi(a_3|R) = \varphi(b_3|R)$ for $4 \leq j \leq 5$; $\varphi(b_{m-1}|R) = \varphi(e_1|R)$ for $6 \leq j \leq s-1$; $\varphi(c_s|R) = \varphi(d_1|R)$ for $j = s$, and $\varphi(a_2|R) = \varphi(a_m|R)$ for $j = s+1$, a contradiction.
$R = \{c_1, c_j\}$; $2 \leq j \leq s$	$\varphi(c_s|R) = \varphi(f_m|R)$ for $2 \leq j \leq 3$; and $\varphi(a_2|R) = \varphi(b_3|R)$ for $4 \leq j \leq k+1$, a contradiction.
$R = \{d_1, d_j\}$; $2 \leq j \leq s$	$\varphi(b_4|R) = \varphi(f_3|R)$ for $2 \leq j \leq 3$; and $\varphi(a_2|R) = \varphi(b_3|R)$ for $4 \leq j \leq k+1$, a contradiction.
$R = \{e_1, e_j\}$; $2 \leq j \leq s$	$\varphi(d_1|R) = \varphi(f_m|R)$ for $2 \leq j \leq 3$; and $\varphi(a_2|R) = \varphi(b_3|R)$ for $4 \leq j \leq k+1$, a contradiction.
$R = \{f_1, f_j\}$; $2 \leq j \leq m$	$\varphi(d_1|R) = \varphi(f_m|R)$ for $2 \leq j \leq 3$; $\varphi(b_4|R) = \varphi(b_3|R)$ for $j = 4$; and $\varphi(f_2|R) = \varphi(g_1|R)$ for $5 \leq j \leq k+1$, a contradiction.
$R = \{g_1, g_j\}$; $2 \leq j \leq m$	$\varphi(f_1|R) = \varphi(g_m|R)$ for $2 \leq j \leq s$; and $\varphi(g_2|R) = \varphi(g_m|R)$ for $j = s+1$, a contradiction.
$R = \{a_1, b_j\}$; $1 \leq j \leq m$	$\varphi(a_2|R) = \varphi(a_m|R)$ for $j = 1, s+1$; $\varphi(f_1|R) = \varphi(f_m|R)$ for $2 \leq j \leq 3$; and $\varphi(b_1|R) = \varphi(a_m|R)$ for $4 \leq j \leq s$, a contradiction.
$R = \{a_1, c_j\}$; $1 \leq j \leq s$	$\varphi(f_1|R) = \varphi(f_m|R)$ for $j = 1$; $\varphi(b_2|R) = \varphi(a_3|R)$ for $j = 2$; $\varphi(b_1|R) = \varphi(a_m|R)$ for $3 \leq j \leq k$; and $\varphi(c_s|R) = \varphi(b_3|R)$ for $j = k+1$, a contradiction.
$R = \{a_1, d_j\}$; $1 \leq j \leq s$	$\varphi(f_1|R) = \varphi(f_m|R)$ for $j = 1$; $\varphi(b_2|R) = \varphi(a_3|R)$ for $j = 2$; $\varphi(b_1|R) = \varphi(a_m|R)$ for $3 \leq j \leq k$; and $\varphi(c_s|R) = \varphi(b_3|R)$ for $j = k+1$, a contradiction.
$R = \{a_1, e_j\}$; $1 \leq j \leq s$	$\varphi(f_1|R) = \varphi(f_m|R)$ for $j = 1$; $\varphi(b_2|R) = \varphi(a_3|R)$ for $j = 2$; $\varphi(b_1|R) = \varphi(a_m|R)$ for $3 \leq j \leq k$; and $\varphi(c_s|R) = \varphi(b_3|R)$ for $j = k+1$, a contradiction.

Resolving sets	Contradictions
$R = \{a_1, f_j\}$; $1 \le j \le m$	$\varphi(g_1\|R) = \varphi(f_2\|R)$ for $j = 1$; $\varphi(b_2\|R) = \varphi(a_3\|R)$ for $2 \le j \le 4$; $\varphi(b_1\|R) = \varphi(a_m\|R)$ for $5 \le j \le s-1$; $\varphi(a_2\|R) = \varphi(a_m\|R)$ for $j = s$; and $\varphi(b_m\|R) = \varphi(a_3\|R)$ for $j = s+1$, a contradiction.
$R = \{a_1, g_j\}$; $1 \le j \le m$	$\varphi(c_s\|R) = \varphi(b_3\|R)$ for $j = 1$; $\varphi(b_2\|R) = \varphi(a_3\|R)$ for $2 \le j \le 4$; $\varphi(b_1\|R) = \varphi(a_m\|R)$ for $5 \le j \le s-1$; $\varphi(a_2\|R) = \varphi(a_m\|R)$ for $j = s$; and $\varphi(b_m\|R) = \varphi(a_3\|R)$ for $j = s+1$, a contradiction.
$R = \{b_1, c_j\}$; $1 \le j \le s$	$\varphi(a_1\|R) = \varphi(b_m\|R)$ for $j = 1$; $\varphi(b_4\|R) = \varphi(d_2\|R)$ for $j = 2$; $\varphi(b_4\|R) = \varphi(a_5\|R)$ for $j = 3$; and $\varphi(g_1\|R) = \varphi(f_2\|R)$ for $4 \le j \le k+1$, a contradiction.
$R = \{b_1, d_j\}$; $1 \le j \le s$	$\varphi(a_1\|R) = \varphi(b_m\|R)$ for $j = 1$; $\varphi(b_4\|R) = \varphi(f_2\|R)$ for $j = 2$; $\varphi(b_4\|R) = \varphi(a_5\|R)$ for $j = 3$; and $\varphi(g_1\|R) = \varphi(f_2\|R)$ for $4 \le j \le k+1$, a contradiction.
$R = \{b_1, e_j\}$; $1 \le j \le s$	$\varphi(a_1\|R) = \varphi(b_m\|R)$ for $j = 1$; $\varphi(b_3\|R) = \varphi(e_1\|R)$ for $j = 2$; $\varphi(b_4\|R) = \varphi(a_5\|R)$ for $j = 3$, we have and $\varphi(g_1\|R) = \varphi(f_2\|R)$ for $4 \le j \le k+1$, a contradiction.
$R = \{b_1, f_j\}$; $1 \le j \le m$	$\varphi(f_2\|R) = \varphi(v_1\|R)$ for $j = 1$; $\varphi(c_2\|R) = \varphi(f_m\|R)$ for $j = 2$; $\varphi(f_2\|R) = \varphi(d_2\|R)$ for $3 \le j \le 4$; and $\varphi(g_1\|R) = \varphi(f_2\|R)$ for $5 \le j \le s+1$, a contradiction.
$R = \{b_1, g_j\}$; $1 \le j \le m$	$\varphi(b_m\|R) = \varphi(a_1\|R)$ for $j = 1$; and $\varphi(g_1\|R) = \varphi(f_2\|R)$ for $2 \le j \le s+1$, a contradiction.
$R = \{c_1, d_j\}$; $1 \le j \le s$	$\varphi(f_m\|R) = \varphi(f_1\|R)$ for $j = 1$; $\varphi(f_2\|R) = \varphi(b_4\|R)$ for $j = 2$; $\varphi(b_4\|R) = \varphi(a_5\|R)$ for $j = 3$; and $\varphi(g_1\|R) = \varphi(f_2\|R)$ for $4 \le j \le k+1$, a contradiction.
$R = \{c_1, e_j\}$; $1 \le j \le s$	$\varphi(f_m\|R) = \varphi(f_1\|R)$ for $j = 1$; $\varphi(f_2\|R) = \varphi(d_2\|R)$ for $j = 2$; $\varphi(b_4\|R) = \varphi(f_2\|R)$ for $j = 3$; and $\varphi(g_1\|R) = \varphi(f_2\|R)$ for $4 \le j \le k+1$, a contradiction.
$R = \{c_1, f_j\}$; $1 \le j \le m$	$\varphi(f_2\|R) = \varphi(g_1\|R)$ for $j = 1$; $\varphi(c_2\|R) = \varphi(f_m\|R)$ for $j = 2$; $\varphi(d_2\|R) = \varphi(f_2\|R)$ for $3 \le j \le 4$; and $\varphi(g_1\|R) = \varphi(f_2\|R)$ for $5 \le j \le s+1$, a contradiction.
$R = \{c_1, g_j\}$; $1 \le j \le m$	$\varphi(b_m\|R) = \varphi(a_1\|R)$ for $j = 1$; and $\varphi(g_1\|R) = \varphi(f_2\|R)$ for $2 \le j \le s+1$, a contradiction.
$R = \{d_1, e_j\}$; $1 \le j \le s$	$\varphi(f_m\|R) = \varphi(f_1\|R)$ for $j = 1$; $\varphi(g_2\|R) = \varphi(c_2\|R)$ for $j = 2$; $\varphi(f_m\|R) = \varphi(b_2\|R)$ for $j = 3$; and $\varphi(g_1\|R) = \varphi(f_2\|R)$ for $4 \le j \le k+1$, a contradiction.
$R = \{d_1, f_j\}$; $1 \le j \le m$	$\varphi(f_2\|R) = \varphi(g_1\|R)$ for $j = 1$; $\varphi(g_2\|R) = \varphi(e_2\|R)$ for $j = 2$; $\varphi(g_m\|R) = \varphi(b_3\|R)$ for $3 \le j \le 4$; and $\varphi(g_1\|R) = \varphi(f_2\|R)$ for $5 \le j \le s+1$, a contradiction.
$R = \{d_1, g_j\}$; $1 \le j \le m$	$\varphi(b_m\|R) = \varphi(a_1\|R)$ for $j = 1$; and $\varphi(g_1\|R) = \varphi(f_2\|R)$ for $2 \le j \le s+1$, a contradiction.
$R = \{e_1, f_j\}$; $1 \le j \le m$	$\varphi(f_m\|R) = \varphi(d_1\|R)$ for $1 \le j \le 4$; and $\varphi(g_1\|R) = \varphi(f_2\|R)$ for $5 \le j \le s+1$, a contradiction.
$R = \{e_1, g_j\}$; $1 \le j \le m$	$\varphi(b_m\|R) = \varphi(a_1\|R)$ for $j = 1$; and $\varphi(g_1\|R) = \varphi(f_2\|R)$ for $2 \le j \le s+1$, a contradiction.
$R = \{f_1, g_j\}$; $1 \le j \le m$	$\varphi(b_m\|R) = \varphi(a_1\|R)$ for $j = 1$; and $\varphi(g_1\|R) = \varphi(f_2\|R)$ for $2 \le j \le s+1$, a contradiction.

From the cases as discussed above, we obtain that for the planar graph $\mathbb{A}_m^{10}$, the set comprising with exactly two vertices can never be the resolving set, and so we find that $|R| \geq 3$ i.e., $dim_v(\mathbb{A}_m^{10}) = 3$ for this case.

Case(II) $s \equiv 1\ (mod\ 2)$.
Then, $m = 2s$ such that $s \geq 3$ and $s \in \mathbb{O}_s$ (where $s = 2k+1$; $k \geq 2$ and $k \in \mathbb{N}$). Let $R = \{a_2, a_{s+1}, a_m\} \subset V(\mathbb{A}_m^{10})$. To complete the proof for this case, we have to show that the set R is a minimum resolving set for $\mathbb{A}_m^{10}$ (Note that, for $7 \leq m \leq 11$, one can verify easily that the set R is a resolving set for $\mathbb{A}_m^{10}$). For $m \geq 13$, we can give the metric codes for every element of $V(\mathbb{A}_m^{10})$ corresponding to the set R.

For the set of a-vertices $\{a_h : 1 \leq h \leq m\}$ in $\mathbb{A}_m^{10}$, metric codes are as follows:

$$\varphi(a_h|R) = \begin{cases} (1, s, 1), & h = 1; \\ (h-2, s-h+1, h), & 2 \leq h \leq s \\ (s-1, 0, s-1), & h = s+1; \\ (2s-h+2, h-s-1, 2s-h), & s+2 \leq h \leq 2s. \end{cases}$$

For the set of b-vertices $\{b_h : 1 \leq h \leq m\}$ in $\mathbb{A}_m^{10}$, metric codes are as follows: $\varphi(b_h|R) = \varphi(a_h|R) + (1,1,1)$ for all $1 \leq h \leq m$. Next, for the set of c-vertices $\{c_h : 1 \leq h \leq s\}$ in $\mathbb{A}_m^{10}$, metric codes are as follows:

$$\varphi(c_h|R) = \begin{cases} (2, s+2, 3), & h = 1; \\ (2h-1, s-2h+3, 2h+1), & 2 \leq h \leq k; \\ (s, 2, s+1), & h = k+1; \\ (2s-2h+4, 2h-s, 2s-2h+2), & k+2 \leq h \leq s. \end{cases}$$

For the set of d-vertices $\{d_h : 1 \leq h \leq s\}$ in $\mathbb{A}_m^{10}$, metric codes are as follows: $\varphi(d_h|R) = \varphi(c_h|R) + (1,1,1)$ for all $1 \leq h \leq s$. For the set of e-vertices $\{e_h : 1 \leq h \leq s\}$ in $\mathbb{A}_m^{10}$, metric codes are as follows: $\varphi(e_h|R) = \varphi(d_h|R) + (1,1,1)$ for all $1 \leq h \leq s$. Next, for the set of f-vertices $\{f_h : 1 \leq h \leq m\}$ in $\mathbb{A}_m^{10}$, metric codes are as follows:

$$\varphi(f_{even}|R) = \begin{cases} (6, s+2, 7), & h = 2; \\ (h+3, s-h+4, h+5), & 4 \leq h \leq s-1; \\ (s+4, 6, s+2), & h = s+1; \\ (2s-h+5, h-s+4, 2s-h+3), & s+3 \leq h \leq 2s-2; \\ (2s-h+5, h-s+4, 6), & h = 2s. \end{cases}$$

and

$$\varphi(f_{odd}|R) = \begin{cases} (5, s+3, 6), & h = 1; \\ (h+3, s-h+4, h+5), & 3 \leq h \leq s-2; \\ (s+3, 5, s+3), & h = s; \\ (2s-h+5, h-s+4, 2s-h+3), & s+2 \leq h \leq 2s-3; \\ (2s-h+5, h-s+4, 5), & h = 2s-1. \end{cases}$$

Finally, for the set of g-vertices $\{g_h : 1 \leq h \leq m\}$ in $\mathbb{A}_m^{10}$, metric codes are as follows: $\varphi(g_h|R) = \varphi(f_h|R) + (1,1,1)$ for all $1 \leq h \leq m$.

Now, from all the above metric codes, we find that no two codes are the same as the sets of metric codes holds $|R_1| = |R_2| = |R_6| = |R_7| = m$, $|R_3| = |R_4| = |R_5| = s$, and all are pairwise disjoint. Thus, from this we conclude that $dim_v(\mathbb{A}_m^{10}) \leq 3$. Next, in order to finish this case, we have to prove that $dim_v(\mathbb{A}_m^{10}) \geq 3$. To obtain this, we show that no resolving set R with $|R| = 2$ exist. Then, following the same pattern as we adopted in Case 1, we get contradictions similarly. This implies that $dim_v(\mathbb{A}_m^{10}) = 3$ for this case as well, which completes the proof of the theorem. □

In terms of independent resolving set, this result can also be presented as follows:

Theorem 2.2. *Let $\mathbb{A}_m^{10}$ be the planar graph as defined above. Then, for $\mathbb{A}_m^{10}$ their exist an independent resolving set with cardinality three, $\forall$ $m \geq 6$.*

Proof. Refer to Theorem 2.1, for proof. □

3. Location Number of planar graph $\mathbb{B}_m^{10}$

In this section, we introduce a new family of planar graph, represented by $\mathbb{B}_m^{10}$, and for which we investigate its metric dimension.

The planar graph $\mathbb{B}_m^{10}$ (where $m = 2s$; $s \geq 3$ & $s \in \mathbb{N}$) comprises of $\frac{11m}{2}$ and $\frac{17m}{2}$ number of vertices and edges respectively. It consists of $\frac{3m}{2}$ faces each are having four sides, m faces each are having five sides, $\frac{m}{2}$ faces each are having ten sides, and two faces each are having m-sides, and is shown in Fig. 1. In other words, the graph $\mathbb{B}_m^{10}$ can be obtained from $\mathbb{A}_m^{10}$ by introducing the set of $\frac{m}{2}$ new edges $\{f_h f_{h+1} | h \text{ is even } \& \ 2 \leq h \leq m\}$ in $\mathbb{A}_m^{10}$. The vertex set $V(\mathbb{B}_m^{10})$ and the edge set $E(\mathbb{B}_m^{10})$ for $\mathbb{B}_m^{10}$, respectively are given by $V(\mathbb{B}_m^{10}) = \{a_h, b_h, f_h, g_h | 1 \leq h \leq m\} \cup \{c_h, d_h, e_h | 1 \leq h \leq s\}$ and $E(\mathbb{B}_m^{10}) = \{a_h b_h, a_h a_{h+1}, f_h g_h, f_h f_{h+1}, g_h g_{h+1} | 1 \leq h \leq m\} \cup \{c_h d_h, d_h e_h | 1 \leq h \leq s\} \cup \{b_{2h-1} c_h, b_{2h} c_h, f_{2h-1} e_h, f_{2h} e_{h+1} | 1 \leq h \leq s\} \cup \{b_h b_{h+1} | h \text{ is even } \& \ 1 \leq h \leq m\}$. Next, it is important to note that, $a_1 = a_{m+1}$, $b_1 = b_{m+1}$, $c_1 = c_{s+1}$, $d_1 = d_{s+1}$, $e_1 = e_{s+1}$, $f_1 = f_{m+1}$ and $g_1 = g_{m+1}$ (whenever necessary).

We refer the collection of vertices $\{a_l : 1 \leq l \leq m\}$ in $\mathbb{B}_m^{10}$, as the a-vertices, the collection of vertices $\{b_l : 1 \leq l \leq m\}$ in $\mathbb{B}_m^{10}$, as the b-vertices, the collection of vertices $\{c_l : 1 \leq l \leq s\}$ in $\mathbb{B}_m^{10}$, as the c-vertices, the collection of vertices $\{d_l : 1 \leq l \leq s\}$ in $\mathbb{B}_m^{10}$, as the d-vertices, the collection of vertices $\{e_l : 1 \leq l \leq s\}$ in $\mathbb{B}_m^{10}$, as the e-vertices, the collection of vertices $\{f_l : 1 \leq l \leq m\}$ in $\mathbb{B}_m^{10}$, as the f-vertices, and the collection of vertices $\{g_l : 1 \leq l \leq m\}$ in $\mathbb{B}_m^{10}$, as the g-vertices.

Next, for the graph $\mathbb{B}_m^{10}$, the vertex set $V(\mathbb{B}_m^{10})$ is as follows $\{a_h, b_h, f_h, g_h | 1 \leq h \leq m\} \cup \{c_h, d_h, e_h | 1 \leq h \leq s\}$. Then, we represent the set of metric codes for the planar graph $\mathbb{B}_m^{10}$ as follows: $R_1 = \{\varphi(a_h|R) | 1 \leq h \leq m\}$, $R_2 = \{\varphi(b_h|R) | 1 \leq h \leq m\}$, $R_3 = \{\varphi(c_h|R) | 1 \leq h \leq s\}$, $R_4 = \{\varphi(d_h|R) | 1 \leq h \leq s\}$, $R_5 = \{\varphi(e_h|R) | 1 \leq h \leq s\}$, $R_6 = \{\varphi(f_h|R) | 1 \leq h \leq m\}$, and $R_7 = \{\varphi(g_h|R) | 1 \leq h \leq m\}$. Next, we prove that the graph $\mathbb{B}_m^{10}$ consists of a minimum resolving set R with cardinality three i.e., $|R| = dim(\mathbb{B}_m^{10}) = 3$.

Theorem 3.1. *Let $\mathbb{B}_m^{10}$ be the planar graph as defined above. Then, $dim(\mathbb{B}_m^{10}) = 3$ $\forall$ $m \geq 6$.*

Proof. The natural number m in $\mathbb{B}_m^{10}$ is always given by $m = 2s$; where $s \geq 3$ and $s \in \mathbb{N}$. Now, to investigate that the planar graph $\mathbb{B}_m^{10}$ consists of a resolving R with $|R| = dim_v(\mathbb{B}_m^{10}) = 3$, we consider the two cases based upon the natural number s i.e., when s is odd ($s \equiv 1$ $(mod\ 2)$) and when it is even ($s \equiv 0$ $(mod\ 2)$).

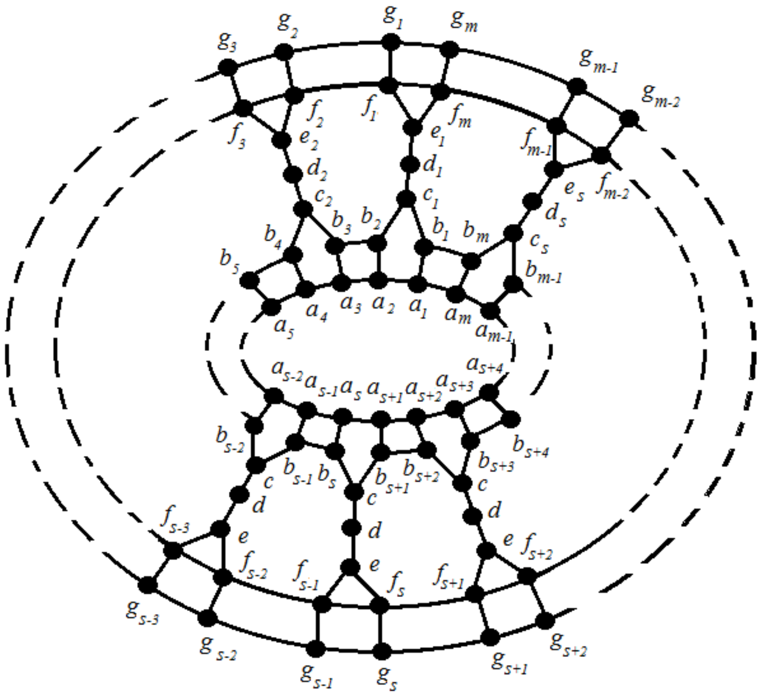

FIGURE 2. Plane graph $\mathbb{B}_m^{10}$

Case(I) $s \equiv 0 \ (mod\ 2)$.
Then, $m = 2s$ such that $s \geq 4$ and $s \in \mathbb{E}_s$ (where $s = 2k$; $k \geq 2$ and $k \in \mathbb{N}$). Let $R = \{a_2, a_{s+1}, a_m\} \subset V(\mathbb{B}_m^{10})$. To complete the proof for this case, we have to show that the set R is a minimum resolving set for $\mathbb{B}_m^{10}$ (Note that, for $6 \leq m \leq 12$, one can verify easily that the set R is a resolving set for $\mathbb{B}_m^{10}$). For $m \geq 14$, we can give the metric codes for every element of $V(\mathbb{B}_m^{10})$ corresponding to the set R.

For the set of a-vertices $\{a_h : 1 \leq h \leq m\}$ in $\mathbb{B}_m^{10}$, metric codes are as follows:

$$\varphi(a_h|R) = \begin{cases} (1, s, 1), & h = 1; \\ (h-2, s-h+1, h), & 2 \leq h \leq s \\ (s-1, 0, s-1), & h = s+1 \\ (2s-h+2, h-s-1, 2s-h), & s+2 \leq h \leq 2s-1. \end{cases}$$

For the set of b-vertices $\{b_h : 1 \leq h \leq m\}$ in $\mathbb{B}_m^{10}$, metric codes are as follows: $\varphi(b_h|R) = \varphi(a_h|R) + (1,1,1)$ for all $1 \leq h \leq m$. Next, for the set of c-vertices $\{c_h : 1 \leq h \leq s\}$ in $\mathbb{B}_m^{10}$, metric codes are as follows:

$$\varphi(c_h|R) = \begin{cases} (2, s+1, 3), & h = 1; \\ (2h-1, s-2h+3, 2h+1), & 2 \leq h \leq k; \\ (s+1, 2, s), & h = k+1; \\ (2s-2h+4, 2h-s, 2s-2h+2), & k+2 \leq h \leq s. \end{cases}$$

For the set of d-vertices $\{d_h : 1 \leq h \leq s\}$ in $\mathbb{B}_m^{10}$, metric codes are as follows: $\varphi(d_h|R) = \varphi(c_h|R) + (1,1,1)$ for all $1 \leq h \leq s$. For the set of e-vertices $\{e_h : 1 \leq h \leq s\}$ in $\mathbb{B}_m^{10}$, metric codes are as follows: $\varphi(e_h|R) = \varphi(d_h|R) + (1,1,1)$ for all $1 \leq h \leq s$. Next, for the set of f-vertices $\{f_h : 1 \leq h \leq m\}$ in $\mathbb{B}_m^{10}$, metric codes are as follows:

$$\varphi(f_{even}|R) = \begin{cases} (6, s+2, 7), & h = 2; \\ (h+3, s-h+4, h+5), & 4 \leq h \leq s-2; \\ (s+3, 5, s+3), & h = s; \\ (2s-h+5, h-s+4, 2s-h+3), & s+2 \leq h \leq 2s-2; \\ (2s-h+5, h-s+4, 5), & h = 2s. \end{cases}$$

and

$$\varphi(f_{odd}|R) = \begin{cases} (5, s+3, 6), & h = 1; \\ (h+3, s-h+4, h+5), & 3 \leq h \leq s-3; \\ (s+2, 6, s+4), & h = s-1; \\ (2s-h+5, h-s+4, 2s-h+3), & s+1 \leq h \leq 2s-3; \\ (2s-h+5, h-s+4, 5), & h = 2s-1. \end{cases}$$

Finally, for the set of g-vertices $\{g_h : 1 \leq h \leq m\}$ in $\mathbb{B}_m^{10}$, metric codes are as follows: $\varphi(g_h|R) = \varphi(f_h|R) + (1, 1, 1)$ for all $1 \leq h \leq m$.

Now, from all the above metric codes, we find that no two codes are the same as the sets of metric codes holds $|R_1| = |R_2| = |R_6| = |R_7| = m$, $|R_3| = |R_4| = |R_5| = s$, and all are pairwise disjoint. Thus, from this we conclude that $dim_v(\mathbb{B}_m^{10}) \leq 3$. Next, in order to finish this case, we have to prove that $dim_v(\mathbb{B}_m^{10}) \geq 3$. To obtain this, we show that no resolving set R with $|R| = 2$ exist. Next, we consider the same possibilities as we adopted in Case 1 of Theorem 2.1, and obtained the contradictions similarly. From the cases as discussed above, we obtain that for the planar graph $\mathbb{B}_m^{10}$, the set comprising with exactly two vertices can never be the resolving set, and so we find that $|R| \geq 3$ i.e., $dim_v(\mathbb{B}_m^{10}) = 3$ for this case.

Case(II) $s \equiv 1 \ (mod\ 2)$.
Then, $m = 2s$ such that $s \geq 3$ and $s \in \mathbb{O}_s$ (where $s = 2k+1$; $k \geq 2$ and $k \in \mathbb{N}$). Let $R = \{a_2, a_{s+1}, a_m\} \subset V(\mathbb{B}_m^{10})$. To complete the proof for this case, we have to show that the set R is a minimum resolving set for $\mathbb{B}_m^{10}$ (Note that, for $7 \leq m \leq 11$, one can verify easily that the set R is a resolving set for $\mathbb{B}_m^{10}$). For $m \geq 13$, we can give the metric codes for every element of $V(\mathbb{B}_m^{10})$ corresponding to the set R.

For the set of a-vertices $\{a_h : 1 \leq h \leq m\}$ in $\mathbb{B}_m^{10}$, metric codes are as follows:

$$\varphi(a_h|R) = \begin{cases} (1, s, 1), & h = 1; \\ (h-2, s-h+1, h), & 2 \leq h \leq s \\ (s-1, 0, s-1), & h = s+1; \\ (2s-h+2, h-s-1, 2s-h), & s+2 \leq h \leq 2s. \end{cases}$$

For the set of b-vertices $\{b_h : 1 \leq h \leq m\}$ in $\mathbb{B}_m^{10}$, metric codes are as follows: $\varphi(b_h|R) = \varphi(a_h|R) + (1, 1, 1)$ for all $1 \leq h \leq m$. Next, for the set of c-vertices $\{c_h : 1 \leq h \leq s\}$ in

$\mathbb{B}_m^{10}$, metric codes are as follows:

$$\varphi(c_h|R) = \begin{cases} (2, s+2, 3), & h = 1; \\ (2h-1, s-2h+3, 2h+1), & 2 \leq h \leq k; \\ (s, 2, s+1), & h = k+1; \\ (2s-2h+4, 2h-s, 2s-2h+2), & k+2 \leq h \leq s. \end{cases}$$

For the set of d-vertices $\{d_h : 1 \leq h \leq s\}$ in $\mathbb{B}_m^{10}$, metric codes are as follows: $\varphi(d_h|R) = \varphi(c_h|R) + (1,1,1)$ for all $1 \leq h \leq s$. For the set of e-vertices $\{e_h : 1 \leq h \leq s\}$ in $\mathbb{B}_m^{10}$, metric codes are as follows: $\varphi(e_h|R) = \varphi(d_h|R) + (1,1,1)$ for all $1 \leq h \leq s$. Next, for the set of f-vertices $\{f_h : 1 \leq h \leq m\}$ in $\mathbb{B}_m^{10}$, metric codes are as follows:

$$\varphi(f_{even}|R) = \begin{cases} (6, s+2, 7), & h = 2; \\ (h+3, s-h+4, h+5), & 4 \leq h \leq s-1; \\ (s+4, 6, s+2), & h = s+1; \\ (2s-h+5, h-s+4, 2s-h+3), & s+3 \leq h \leq 2s-2; \\ (2s-h+5, h-s+4, 6), & h = 2s. \end{cases}$$

and

$$\varphi(f_{odd}|R) = \begin{cases} (5, s+3, 6), & h = 1; \\ (h+3, s-h+4, h+5), & 3 \leq h \leq s-2; \\ (s+3, 5, s+3), & h = s; \\ (2s-h+5, h-s+4, 2s-h+3), & s+2 \leq h \leq 2s-3; \\ (2s-h+5, h-s+4, 5), & h = 2s-1. \end{cases}$$

Finally, for the set of g-vertices $\{g_h : 1 \leq h \leq m\}$ in $\mathbb{B}_m^{10}$, metric codes are as follows: $\varphi(g_h|R) = \varphi(f_h|R) + (1,1,1)$ for all $1 \leq h \leq m$.

Now, from all the above metric codes, we find that no two codes are the same as the sets of metric codes holds $|R_1| = |R_2| = |R_6| = |R_7| = m$, $|R_3| = |R_4| = |R_5| = s$, and all are pairwise disjoint. Thus, from this we conclude that $dim_v(\mathbb{B}_m^{10}) \leq 3$. Next, in order to finish this case, we have to prove that $dim_v(\mathbb{B}_m^{10}) \geq 3$. To obtain this, we show that no resolving set R with $|R| = 2$ exist. Then, following the same pattern as we adopted in Case 1, we get contradictions similarly. This implies that $dim_v(\mathbb{B}_m^{10}) = 3$ for this case as well, which completes the proof of the theorem. □

In terms of independent resolving set, this result can also be presented as follows:

Theorem 3.2. *Let $\mathbb{B}_m^{10}$ be the planar graph as defined above. Then, for $\mathbb{B}_m^{10}$ their exist an independent resolving set with cardinality three, $\forall$ $m \geq 6$.*

Proof. Refer to Theorem 3.1, for proof. □

4. Conclusion

In this manuscript, we have studied the metric dimension of two planar graphs $\mathbb{A}_m^{10}$ and $\mathbb{B}_m^{10}$. For these two families, we proved that $dim_v(\mathbb{A}_m^{10}) = 3 = dim_v(\mathbb{B}_m^{10})$ (a partial answer to the problem posed in [8]) . We also observed that the minimum resolving sets R are independent for both of the graphs $\mathbb{A}_m^{10}$ and $\mathbb{B}_m^{10}$. In future, we will try to obtain other variants of metric dimension such as edge metric dimension, mixed metric dimension [20], etc. for these two graph $\mathbb{A}_m^{10}$ and $\mathbb{B}_m^{10}$.

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