Characteristic classes of foliations via SAYD-twisted cocycles

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Abstract

We find the first non trivial "SAYD-twisted" cyclic cocycle over the groupoid action algebra under the symmetry of the affine linear transformations of the Euclidian space. We apply the cocycle to construct a characteristic map by which we transfer the characteristic classes of transversely orientable foliations into the cyclic cohomology of the groupoid action algebra. In codimension 1, our result matches with the (only explicit) computation done by Connes-Moscovici. We carry out the explicit computation in codimension 2 to present the transverse fundamental class, the Godbillon-Vey class, and the other four residual classes as cyclic cocycles on the groupoid action algebra. For the general codimension we show that the introduced explicit cochain is always a Hochschild cocycle and its cyclic homology class is non-trivial.

Contents

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1 Introduction

Following Connes-Moscovici [2], let $\mathcal{A} := C_c^\infty(F^+) \rtimes \Gamma$. Here F^+ is the oriented frame bundle over $\mathbb{R}^n$, and Γ is a discrete subgroup of $\mathrm{Diff}^+(\mathbb{R}^n)$, the group of orientation preserving diffeomorphisms of $\mathbb{R}^n$.

For an arbitrary Γ, the cyclic cohomology of $\mathcal{A}$ is not known [1, Sect. III.2]. However, there is a map, even in the level of complexes,

$$\begin{array}{ccc} H_{\mathrm{GF}}(\mathfrak{a}_n, \mathbb{C}) & \xrightarrow{\Phi\circ\mathcal{V}} & HP(\mathcal{A}_\Gamma) \\ & {\searrow}^{\mathcal{V}_{\mathrm{vanEst}}} \quad {\nearrow}^{\Phi_{\mathrm{Connes}}} & \\ & H_\tau(F^+, \mathbb{C}). & \end{array} \tag{1.1}$$

from the Gelfand-Fuks cohomology of $\mathfrak{a}_n$, the Lie algebra of formal vector fields on $\mathbb{R}^n$, that factors through the twisted cohomology computed by the Bott bicomplex [1, Prop. III.2.11]. One notes that a basis for the Gelfand-Fuks classes are known, [4], but it is difficult to transfer them to the cyclic cohomology of $\mathcal{A}$. The reader is referred to [3] for a complete account of this computation in codimension 1.

The Hopf-cyclic cohomology, invented by Connes-Moscovici [2], made it possible to have a very explicit characteristic map

$$\chi_\tau : HP(\mathcal{H}, \mathbb{C}_\delta) \to HP(\mathcal{A}), \tag{1.2}$$

where $\mathcal{H} := \mathcal{H}_n$ is the Connes-Moscovici Hopf algebra of codimension n, $\mathbb{C}_\delta$ is the canonical one dimensional SAYD module over $\mathcal{H}$, and τ is the canonical trace on $\mathcal{A}$.

Although (1.2) has a simple presentation on the level of complexes, and $HP(\mathcal{H}, \mathbb{C}_\delta)$ is canonically isomorphic to $H_{\mathrm{GF}}(\mathfrak{a}_n, \mathbb{C})$, the isomorphism is not easy to present [2, 11]. Therefore, from the point of view of (1.2), the obstacle to transfer the characteristic classes of transversely orientable foliations to the cyclic cohomology of $\mathcal{A}$ is to find a basis of the representatives of the Hopf-cyclic classes of $\mathcal{H}$. There is an intensive ongoing study [9, 10, 11] on the Hopf-cyclic cohomology of the geometric bicrossed product Hopf algebras such as $\mathcal{H}$.

In the present paper we develop a new characteristic map, whose source is the Hopf-cyclic cohomology of $\mathcal{K} := U(g\ell_n)$, the enveloping algebra of the general linear Lie algebra $g\ell_n$. Since the Hopf algebra $\mathcal{K}$ is not as sophisticated as $\mathcal{H}$, one expects, by the conservation of work, a more sophisticated characteristic map and SAYD module than χ_τ and $\mathbb{C}_\delta$ respectively.

In fact, the first step of our mission was taken in [13], where the authors showed that the truncated Weil algebra is a Hopf-cyclic complex. As a result, the characteristic classes of transversely orientable foliations can be calculated from $HC(\mathcal{K}, V)$. Here $V := S(g\ell_n^*)_{[2n]}$, the algebra of n-truncated polynomials on $g\ell_n$, is a canonical and nontrivial SAYD module over $\mathcal{K}$.

The backbone of this new characteristic map is a SAYD twisted cyclic n-cocycle $\varphi \in C^n_{\mathcal{K}}(\mathcal{A}, V)$ by which we apply the cup product introduced in [8] by Khalkhali and the first named author. We use the explicit formula derived in [12] to compute the characteristic classes of foliations as cyclic cocycles in $HC(\mathcal{A})$ via

$$\chi_\varphi : HC^\bullet(\mathcal{K}, V) \to HC^{\bullet+n}(\mathcal{A}), \quad \chi_\varphi(x) = x \cup \varphi. \tag{1.3}$$

In order to test our method we first carry out the computation for codimension 1 and observe that our result matches with the classes obtained by Connes-Moscovici in [3]. The result of [11] shows that the amount of work in codimension 2 is not comparable with that of codimension 1. However, we completely determine the representatives of all classes in $HC(\mathcal{K}, V)$, in addition to an explicit formula for $\varphi \in HC^2_{\mathcal{K}}(\mathcal{A}, V)$. Then (1.3) yields our desired cyclic cocycles in $HC(\mathcal{A})$. For the general codimension n, we introduce a Hochschild cocycle in $C^n_{\mathcal{K}}(\mathcal{A}, V)$ and prove that its cyclic cohomology class is nontrivial.

Throughout the paper, all vector spaces and their tensor products are over $\mathbb{C}$ unless otherwise is specified. We use the Sweedler's notation for comultiplication and coaction. We denote the comultiplication of a coalgebra C by $\Delta : C \to C \otimes C$ and its action on $c \in C$ by $\Delta(c) = c_{(1)} \otimes c_{(2)}$. The image of $v \in V$ under a left coaction $\nabla : V \to C \otimes V$ is denoted by $\nabla(v) = v_{<-1>} \otimes v_{<0>}$, summation suppressed. By the coassociativity, we simply write $\Delta(c_{(1)}) \otimes c_{(2)} = c_{(1)} \otimes \Delta(c_{(2)}) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$. Unless stated otherwise, a Lie algebra $\mathfrak{g}$ is finite dimensional with a basis $\{X_i \,|\, 1 \leq i \leq n\}$ and a dual basis $\{\theta^i \,|\, 1 \leq i \leq n\}$. In particular, for $\mathfrak{g} = g\ell_n$ we use $\{Y_i^j \,|\, 1 \leq i,j \leq n\}$ for a basis and $\{\theta_j^i \,|\, 1 \leq i,j \leq n\}$ for a dual basis. We denote the Weil algebra of $\mathfrak{g}$ by $W(\mathfrak{g})$, and $W(\mathfrak{g})_{[2n]}$ stands for the n-truncated Weil algebra of $\mathfrak{g}$. We denote the Kronecker symbol by δ_j^i. We also adopt the Einstein summation convention on the repeating indices unless otherwise is stated. Finally, for the sake of simplicity we use

$$B_{\sigma(1)} \otimes \cdots \otimes B_{\sigma(q)} := \sum_{\sigma \in S_q} \operatorname{sgn}(\sigma) B_{\sigma(1)} \otimes \cdots \otimes B_{\sigma(q)}$$

for any set of objects $\{B_1, \ldots, B_q\}$. Here S_q is the group of all permutations on q objects and $\operatorname{sgn}(\sigma)$ stands for the signature of σ.

2 Preliminaries

In this section we recall the definition of Hopf-cyclic cohomology, the Connes-Moscovici characteristic map, and basics of the cyclic cohomology of Lie.

2.1 Hopf-cyclic cohomology with coefficients

Let H be a Hopf algebra, $\delta : H \to \mathbb{C}$ be a character, and σ be a group-like element $\sigma \in H$. The pair (δ, σ) is called a modular pair in involution (MPI for short) if

$$\delta(\sigma) = 1, \quad \text{and} \quad S_\delta^2 = \operatorname{Ad}_\sigma, \tag{2.1}$$

where $\operatorname{Ad}_\sigma(h) = \sigma h \sigma^{-1}$, for any $h \in H$, and S_δ is defined by

$$S_\delta(h) = \delta(h_{(1)}) S(h_{(2)}), \quad h \in H. \tag{2.2}$$

A vector space M is called a right-left stable-anti-Yetter-Drinfeld module (SAYD for short) over H if it is a right H-module, a left H-comodule, and for any $v \in V$ and $h \in H$

$$\nabla(m \cdot h) = S(h_{(3)}) m_{<-1>} h_{(1)} \otimes m_{<0>} \cdot h_{(2)}, \qquad m_{<0>} \cdot m_{<-1>} = m. \tag{2.3}$$

Using δ and σ one endows ${}^{\sigma}\mathbb{C}_{\delta} := \mathbb{C}$ with a right module and left comodule structures over H.

Let M be a right-left SAYD module over H and C an H-module coalgebra, *i.e.* , $\Delta(h(c)) = h_{(1)}(c_{(1)}) \otimes h_{(2)}(c_{(2)})$, and $\varepsilon(h(c)) = \varepsilon(h)\varepsilon(c)$, for $h \in H$ and $c \in C$.

We have the cocyclic module

$$C_H(C, M) := \bigoplus_{q \geq 0} C_H^q(C, M), \quad C_H^q(C, M) := M \otimes_H C^{\otimes q+1} \tag{2.4}$$

$$\begin{aligned}
&\mathfrak{d}_i : C_H^q(C, M) \to C_H^{q+1}(C, M), \quad 0 \leq i \leq q+1 && (2.5)\\
&\mathfrak{d}_i(m \otimes_H c^0 \otimes \cdots \otimes c^q) = m \otimes c^0 \otimes \cdots \otimes \Delta(c^i) \otimes \cdots \otimes c^q, \\
&\mathfrak{d}_{q+1}(m \otimes_H c^0 \otimes \cdots \otimes c^q) = \\
&m_{<0>} \otimes_H c^0{}_{(2)} \otimes c^1 \otimes \cdots \otimes c^q \otimes m_{<-1>}(c^0{}_{(1)}), \\
&\mathfrak{s}_j : C_H^q(C, M) \to C_H^{q-1}(C, M), \quad 0 \leq j \leq q-1 && (2.6)\\
&\mathfrak{s}_j(m \otimes_H c^0 \otimes \cdots \otimes h^q) = m \otimes_H c^0 \otimes \cdots \otimes \varepsilon(c^{j+1}) \otimes \cdots \otimes c^q, \\
&\mathfrak{t}_q : C_H^q(C, M) \to C_H^q(C, M), && (2.7)\\
&\mathfrak{t}_q(m \otimes_H c^0 \otimes \cdots \otimes c^q) = m_{<0>} \otimes_H c^1 \otimes \cdots \otimes c^q \otimes m_{<-1>}(c^0).
\end{aligned}$$

Using the above operators one defines the Hochschild coboundary b and the Connes boundary operator B,

$$b : C_H^q(C, M) \to C_H^{q+1}(C, M), \qquad b := \sum_{i=0}^{q+1} (-1)^i \mathfrak{d}_i, \tag{2.8}$$

$$B : C_H^q(C, M) \to C_H^{q-1}(C, M), \quad B := \left(\sum_{i=0}^{q} (-1)^{qi} \mathfrak{t}^i \right) \mathfrak{s}_{q-1} \mathfrak{t}. \tag{2.9}$$

We denote the cyclic cohomology of $C_H^{\bullet}(C, M)$ by $HC_H^{\bullet}(C, M)$.

For $C = H$, with the multiplication action, the map

$$\begin{aligned}
&\mathcal{J} : M \otimes_H H^{\otimes(n+1)} \to M \otimes H^{\otimes n}, \\
&\mathcal{J}(m \otimes_H h^0 \otimes \cdots \otimes h^n) = m h^0{}_{(1)} \otimes S(h_{(2)}) \cdot (h^1 \otimes \cdots \otimes h^n).
\end{aligned} \tag{2.10}$$

identifies the standard Hopf-cyclic complex (2.4) with

$$C(H, M) := \bigoplus_{q \geq 0} C^q(H, M), \quad C^q(H, M) := M \otimes H^{\otimes q}. \tag{2.11}$$

$$\mathfrak{d}_0(m\otimes h^1\otimes\cdots\otimes h^q)=m\otimes 1\otimes h^1\otimes\cdots\otimes h^q,$$
$$\mathfrak{d}_i(m\otimes h^1\otimes\cdots\otimes h^q)=m\otimes h^1\otimes\cdots\otimes h^i{}_{(1)}\otimes h^i{}_{(2)}\otimes\cdots\otimes h^q,$$
$$\mathfrak{d}_{q+1}(m\otimes h^1\otimes\cdots\otimes h^q)=m_{<0>}\otimes h^1\otimes\cdots\otimes h^q\otimes m_{<-1>},$$
$$\mathfrak{s}_j(m\otimes h^1\otimes\cdots\otimes h^q)=m\otimes h^1\otimes\cdots\otimes\varepsilon(h^{j+1})\otimes\cdots\otimes h^q,$$
$$\mathfrak{t}(m\otimes h^1\otimes\cdots\otimes h^q)=m_{<0>}h^1{}_{(1)}\otimes S(h^1{}_{(2)})\cdot(h^2\otimes\cdots\otimes h^q\otimes m_{<-1>}).$$

Let A be a H-module algebra, that is, a (left) H-module and

$$h(ab)=h_{(1)}(a)h_{(2)}(b),\quad h(1_A)=\varepsilon(h)1_A,\quad \forall h\in H, a\in A.$$

Then one endows $V\otimes A^{\otimes n+1}$ with the action of H as

$$(v\otimes a^0\otimes\cdots\otimes q^q)\cdot h=mh_{(1)}\otimes S(h_{(q+2)})a^0\otimes\cdots\otimes S(h_{(2)})a^q. \quad (2.12)$$

We set

$$C^n_H(A,V)=\mathrm{Hom}_H(V\otimes A^{\otimes n+1},\mathbb{C}) \quad (2.13)$$

as the space of H-linear maps. It is checked in [5] that for any $v\otimes\tilde{a}:=v\otimes a^0\otimes\cdots\otimes a^{n+2}\in V\otimes A^{\otimes n+2}$ the morphisms

$$(\partial_i\varphi)(v\otimes\tilde{a})=\varphi(v\otimes a^0\otimes\ldots\otimes a^ia^{i+1}\otimes\ldots\otimes a^{n+1}),\quad 0\le i\le n,$$
$$(\partial_{n+1}\varphi)(v\otimes\tilde{a})=\varphi(v_{<0>}\otimes(S^{-1}(v_{<-1>})a^{n+1})a^0\otimes a^1\otimes\ldots\otimes a^n),$$
$$(\sigma_i\varphi)(v\otimes\tilde{a})=\varphi(v\otimes a^0\otimes\ldots\otimes a^i\otimes 1\otimes\ldots\otimes a^{n-1}),\quad 0\le i\le n-1,$$
$$(\tau\varphi)(v\otimes\tilde{a})=\varphi(v_{<0>}\otimes(S^{-1}(v_{<-1>})a^n)\otimes a^0\otimes\ldots\otimes a^{n-1})$$

define a cocyclic module structure on $C^n_H(A,V)$, whose cyclic cohomology is denoted by $HC_H(A,V)$.

One uses $HC_H(H,V)$ and $HC_H(A,V)$ to define a cup product

$$HC^p_H(A,V)\otimes HC^q_H(H,V)\to HC^{p+q}(A),$$

whose definition can be found in [12, 8].

As the simplest example, one notes that the cup product with the 0-cocycle $\tau\in C^0_H(A,\,{}^\sigma\mathbb{C}_\delta)$ defines the Connes-Moscovici characteristic map [2, 3],

$$\begin{aligned}&\chi_\tau: HC^\bullet(H,{}^\sigma\mathbb{C}_\delta)\to HC^\bullet(A)\\ &\chi_\tau(h^1\otimes\cdots\otimes h^n)(a^0\otimes\cdots\otimes a^n)=\tau(a^0h^1(a^1)\ldots h^n(a^n)).\end{aligned} \quad (2.14)$$

2.2 Lie algebra (co)homology

In this subsection we summarize our work in [13] on the cyclic cohomology of Lie algebras with coefficients in SAYD modules.

Let $\mathfrak{g}$ be a Lie algebra and V be a right $\mathfrak{g}$-module. Lets recall the Lie algebra homology complex

$$C(\mathfrak{g},V)=\bigoplus_{q\geq 0}C_q(\mathfrak{g},V),\quad C_q(\mathfrak{g},V):=\wedge^q\mathfrak{g}\otimes V, \tag{2.15}$$

with the Chevalley-Eilenberg boundary map

$$\cdots\xrightarrow{\partial_{\rm CE}}C_2(\mathfrak{g},V)\xrightarrow{\partial_{\rm CE}}C_1(\mathfrak{g},V)\xrightarrow{\partial_{\rm CE}}V\ , \tag{2.16}$$

$$\partial_{\rm CE}(X_0\wedge\cdots\wedge X_{q-1}\otimes v)=\sum_{i=0}^{q-1}(-1)^iX_0\wedge\cdots\wedge\widehat{X}_i\wedge\cdots\wedge X_{q-1}\otimes v\cdot X_i+$$
$$\sum_{0\leq i<j\leq q-1}(-1)^{i+j}[X_i,X_j]\wedge X_0\wedge\cdots\wedge\widehat{X}_i\wedge\cdots\wedge\widehat{X}_j\wedge\cdots\wedge X_{q-1}\otimes v.$$

The homology of the complex $(C(\mathfrak{g},V),\partial_{\rm CE})$ is called the Lie algebra homology of $\mathfrak{g}$ with coefficients in V, and it is denoted by $H_\bullet(\mathfrak{g},V)$. In a dual fashion, one defines the Lie algebra cohomology complex

$$W(\mathfrak{g},V)=\bigoplus_{q\geq 0}W^q(\mathfrak{g},V),\quad W^q(\mathfrak{g},V)={\rm Hom}(\wedge^q\mathfrak{g},V), \tag{2.17}$$

where ${\rm Hom}(\wedge^q\mathfrak{g},V)$ is the vector space of all alternating linear maps on $\mathfrak{g}^{\otimes q}$ with values in V. The Chevalley-Eilenberg coboundary

$$V\xrightarrow{d_{\rm CE}}W^1(\mathfrak{g},V)\xrightarrow{d_{\rm CE}}W^2(\mathfrak{g},V)\xrightarrow{d_{\rm CE}}\cdots\ , \tag{2.18}$$

is defined by

$$d_{\rm CE}(\alpha)(X_0,\ldots,X_q)=\sum_{0\leq i<j\leq q}(-1)^{i+j}\alpha([X_i,X_j],X_0\ldots\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,X_q)+$$
$$\sum_{i=0}^{q}(-1)^{i+1}\alpha(X_0,\ldots,\widehat{X}_i,\ldots X_q)\cdot X_i. \tag{2.19}$$

Alternatively, we may identify $W^q(\mathfrak{g},V)$ with $\wedge^q\mathfrak{g}^*\otimes V$ and the coboundary $d_{\rm CE}$ with

$$d_{\rm CE}(v)=-\theta^i\otimes v\cdot X_i,\quad d_{\rm CE}(\beta\otimes v)=d_{\rm dR}(\beta)\otimes v-\theta^i\wedge\beta\otimes v\cdot X_i,$$
$$d_{\rm dR}:\wedge^p\mathfrak{g}^*\to\wedge^{p+1}\mathfrak{g}^*,\quad d_{\rm dR}(\theta^i)=-\frac{1}{2}C^i_{jk}\theta^j\wedge\theta^k \tag{2.20}$$

The cohomology of the complex $(W(\mathfrak{g}, V), d_{\mathrm{CE}})$, the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in V, is denoted by $H^\bullet(\mathfrak{g}, V)$.

We are particularly interested in the SAYD modules over the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, and by [13], such SAYD modules are obtained from the SAYD modules over the Lie algebra $\mathfrak{g}$.

Definition 2.1 ([13])**.** *A vector space V is a left comodule over the Lie algebra $\mathfrak{g}$ if there is a linear map*

$$\nabla_{\mathfrak{g}} : V \to \mathfrak{g} \otimes V, \qquad \nabla_{\mathfrak{g}}(v) = v_{[-1]} \otimes v_{[0]} \tag{2.21}$$

such that

$$v_{[-2]} \wedge v_{[-1]} \otimes v_{[0]} = 0,$$

where

$$v_{[-2]} \otimes v_{[-1]} \otimes v_{[0]} = v_{[-1]} \otimes (v_{[0]})_{[-1]} \otimes (v_{[0]})_{[0]}.$$

It is clear that left $\mathfrak{g}$-comodules and right $S(\mathfrak{g}^*)$-modules are identical.

Definition 2.2 ([13])**.** *Let V be a right module and left comodule over a Lie algebra $\mathfrak{g}$. We call V a right-left anti-Yetter-Drinfeld module (AYD module) over $\mathfrak{g}$ if*

$$\nabla_{\mathfrak{g}}(v \cdot X) = v_{[-1]} \otimes v_{[0]} \cdot X + [v_{[-1]}, X] \otimes v_{[0]}. \tag{2.22}$$

Moreover, V is called stable if

$$v_{[0]} \cdot v_{[-1]} = 0. \tag{2.23}$$

Finally, V is said to be unimodular stable if $V_{-\delta}$ is stable,where $\delta := \mathrm{Tr} \circ \mathrm{ad} : \mathfrak{g} \to \mathbb{C}$, and $V_{-\delta}$ is the deformation of V via

$$v \triangleleft X := v \cdot X - \delta(X)v.$$

Example 2.3. The truncated polynomial algebra $V = S(\mathfrak{g}^*)_{[2n]}$, of a Lie algebra $\mathfrak{g}$, is a unimodular SAYD module over $\mathfrak{g}$ with the coadjoint action and the Koszul coaction defined by

$$\nabla_K : V \to \mathfrak{g} \otimes V, \quad \nabla_K(v) = \sum_{i=1}^{n} X_i \otimes v\theta^i. \tag{2.24}$$

Next, by using SAYD modules, we generalize the Lie algebra (co)homology complexes. Let us start with the Lie algebra homology by introducing the complex

$$C(\mathfrak{g}, V) = \bigoplus_{i \geq 0} \wedge^i \mathfrak{g} \otimes V, \qquad \partial = \partial_{\mathrm{CE}} + \partial_{\mathrm{K}} \tag{2.25}$$

with the Chevalley-Eilenberg boundary and the Koszul coboundary

$$\partial_{\mathrm{K}} : C_n(\mathfrak{g}, V) \to C_{n+1}(\mathfrak{g}, V), \qquad \partial_{\mathrm{K}}(e \otimes v) = v_{[-1]} \wedge e \otimes v_{[0]}. \tag{2.26}$$

Applying the Poincaré duality, see [13, Prop. 4.4], we obtain

$$W(\mathfrak{g}, V) = \bigoplus_{i \geq 0} \wedge^i \mathfrak{g}^* \otimes V \tag{2.27}$$

with $d = d_{\mathrm{CE}} + d_{\mathrm{K}}$, where $d_{\mathrm{CE}} : W^n(\mathfrak{g}, V) \to W^{n+1}(\mathfrak{g}, V)$ is the Chevalley-Eilenberg coboundary and

$$d_{\mathrm{K}} : W^n(\mathfrak{g}, V) \to W^{n-1}(\mathfrak{g}, V), \qquad d_{\mathrm{K}}(\alpha \otimes v) = \iota(v_{[-1]})(\alpha) \otimes v_{[0]}.$$

Here $\iota(X)$ denotes the contraction by X.

In particular, we recover the (truncated) Weil algebra [13]:

$$W(\mathfrak{g}, S(\mathfrak{g}^*)) = W(\mathfrak{g}), \qquad W(\mathfrak{g}, S(\mathfrak{g}^*)_{[2n]}) = W(\mathfrak{g})_{[2n]}. \tag{2.28}$$

3 SAYD-twisted cyclic cocycles

In this section we fix K to be a cocommutative Hopf subalgebra of a Hopf algebra H, A an H-module algebra, and V a SAYD module over K. We aim to develop a machinery to produce cyclic cocycles in $HC_K(A, V)$. In the first subsection we introduce equivariant Hopf-cyclic cohomology $HC_K(H, V, N)$, where N is a SAYD module over H. In the second subsection we construct a cup product

$$HC^p_K(H, V, N) \otimes HC^q_H(A, N) \to HC^{p+q}_K(A, V).$$

In the third and fourth subsections we apply the results of the first two subsections to produce a nontrivial SAYD-twisted cyclic cocycle over the groupoid action algebra under the symmetry of the general linear Lie algebra with coefficients in the truncated polynomials on this Lie algebra.

3.1 Equivariant Hopf-cyclic cohomology

For a SAYD module N over H and a module-comodule V over K we define the graded space

$$\begin{aligned} C_K(H,V,N) &= \bigoplus_{q\geq 0} C_K^q(H,V,N), \\ \mathcal{C}^q := C_K^q(H,V,N) &:= \mathrm{Hom}_K(V, N\otimes_H H^{\otimes q+1}). \end{aligned} \tag{3.1}$$

More precisely, $\phi\in\mathcal{C}^q$ if and only if for any $u\in K$ and any $v\in V$

$$\phi(v\cdot u)=\phi(v)\cdot u, \tag{3.2}$$

where the right action of K on $N\otimes_H H^{\otimes q+1}$ is the usual diagonal action, *i.e.*

$$(n\otimes_H h^0\otimes\cdots\otimes h^q)\cdot u = n\otimes_H h^0 u_{(1)}\otimes\cdots\otimes h^q u_{(q+1)}. \tag{3.3}$$

For $\phi\in C_K^q(H,V,N)$ and $v\in V$, we use the notation

$$\phi(v)=\phi(v)^{[-1]}\otimes_H \phi(v)^{[0]}\otimes\cdots\otimes\phi(v)^{[q]}. \tag{3.4}$$

Let us define the morphisms $d_i:\mathcal{C}^q\to\mathcal{C}^{q+1}$, $s_j:\mathcal{C}^q\to\mathcal{C}^{q-1}$, and $t_q:\mathcal{C}^q\to\mathcal{C}^q$ as

$$\begin{aligned} &d_i(\phi)(v)=\mathfrak{d}_i(\phi(v)), \quad 0\leq i\leq q \\ &d_{q+1}(\phi)(v)=\mathfrak{d}_{q+1}(\phi(v_{<0>}))\triangleleft S(v_{<-1>}), \\ &s_j(\phi)(v)=\mathfrak{s}_j(\phi(v)), \quad 0\leq j\leq q-1, \\ &t_q(\phi)(v)=\mathfrak{t}_q(\phi(v_{<0>}))\triangleleft S(v_{<-1>}), \end{aligned} \tag{3.5}$$

where the right action $\triangleleft$ of K on $N\otimes_H H^{\otimes q+1}$ is defined by

$$(n\otimes_H h^0\otimes\cdots\otimes h^q)\triangleleft u = n\otimes_H h^0\otimes\cdots\otimes h^{q-1}\otimes h^q u, \tag{3.6}$$

and the morphisms $(\mathfrak{d}_i,\mathfrak{s}_j,\mathfrak{t})$ are the usual morphisms of the cocyclic module $C_H(H,N)$ defined in (2.5), (2.6) and (2.7).

Theorem 3.1. *If V and N are SAYD modules over K and H respectively, then the morphisms d_i, s_j and t define a cocyclic module structure on $C_K(H,V,N)$.*

Proof. Let us prove that the morphisms d_i, s_j, and t are well-defined. To this end, it suffices to check that t, d_0, and s_n are well-defined as the other morphisms are made of these three. For d_0 and s_n the task is obvious as $\Delta : H \to H \otimes H$ and $\varepsilon : H \to \mathbb{C}$ are multiplicative respectively. As for t, we have

$$
\begin{aligned}
&t(\phi)(v \cdot y) = \tau(\phi((v \cdot y)_{<0>})) \triangleleft S((v \cdot y)_{<-1>}) \\
&= \tau(\phi(v_{<0>}) \cdot y_{(2)}) \triangleleft S(y_{(3)}) S(v_{<-1>}) y_{(1)} \\
&= \tau(\phi(v)^{[-1]} \otimes_H \phi(v)^{[0]} y_{(2)} \otimes \cdots \otimes \phi(v)^{[q]} y_{(q+2)}) \triangleleft S(y_{(q+3)}) S(v_{<-1>}) y_{(1)} \\
&= {\phi(v)^{[-1]}}_{<0>} \otimes_H \phi(v)^{[1]} y_{(3)} \otimes \ldots \\
&\ldots \otimes \phi(v)^{[q]} y_{(q+2)} \otimes {\phi(v)^{[-1]}}_{<-1>} \phi(v)^{[0]} y_{(2)} S(y_{(q+3)}) S(v_{<-1>}) y_{(1)} \\
&= {\phi(v)^{[-1]}}_{<0>} \otimes_H \phi(v)^{[1]} y_{(1)} \otimes \ldots \\
&\qquad\qquad \ldots \otimes \phi(v)^{[q]} y_{(q)} \otimes {\phi(v)^{[-1]}}_{<-1>} \phi(v)^{[0]} S(v_{<-1>}) y_{(q+1)} \\
&= t(\phi(v)) \cdot y.
\end{aligned} \tag{3.7}
$$

In the second and the sixth equalities we use the fact that K is cocommutative.

Let us next show that $C_K(H, V, N)$ is a cocyclic module, that is,

$$
d_j d_i = d_i d_{j-1}, \; i < j, \qquad s_j s_i = s_i s_{j+1}, \; i \leq j \tag{3.8}
$$

$$
s_j d_i = \begin{cases} d_i s_{j-1} & i < j \\ \mathrm{Id}_q & \text{if } i = j \text{ or } i = j+1 \\ d_{i-1} s_j & i > j+1; \end{cases} \tag{3.9}
$$

$$
t_{q+1} d_i = d_{i-1} t_q, \qquad 1 \leq i \leq q+1, \quad t_{q+1} d_0 = d_{q+1} \tag{3.10}
$$

$$
t_{q-1} s_i = s_{i-1} t_q, \qquad 1 \leq i \leq q-1, \quad t_q s_0 = s_{q-1} t_q^2 \tag{3.11}
$$

$$
t_q^{q+1} = \mathrm{Id}_q . \tag{3.12}
$$

The equalities (3.8), (3.9) and (3.11) follow directly from their counterparts for the operators $\mathfrak{d}_i, \mathfrak{s}_j$ and $\mathfrak{t}$.

We check (3.10) for $i = q+1$. Indeed

$$\begin{aligned}
& t_{q+1}(d_{q+1}(\phi))(v) = \mathfrak{t}_{q+1}(d_{q+1}(\phi)(v_{<0>})) \triangleleft S((v_{<-1>})) \\
& = \mathfrak{t}_{q+1}(\mathfrak{d}_{q+1}(\phi(v_{<0><0>})) \triangleleft S(v_{<0><-1>})) \triangleleft S(v_{<-1>}) \\
& = \mathfrak{t}_{q+1}(\mathfrak{d}_{q+1}(\phi(v_{<0>})) \triangleleft S(v_{<-1>})) \triangleleft S(v_{<-2>}) \\
& = \mathfrak{t}_{q+1}\Big(\phi(v_{<0>})^{[-1]}{}_{<0>} \otimes_H \phi(v_{<0>})^{[0]}{}_{(2)} \otimes \phi(v_{<0>})^{[1]} \otimes \cdots \otimes \\
& \phi(v_{<0>})^{[q]} \otimes \phi(v_{<0>})^{[-1]}{}_{<-1>} \phi(v_{<0>})^{[0]}{}_{(1)} S(v_{<-1>})\Big) \triangleleft S(v_{<-2>}) \\
& = \phi(v_{<0>})^{[-1]}{}_{<0>} \otimes_H \phi(v_{<0>})^{[1]} \otimes \cdots \otimes \phi(v_{<0>})^{[q]} \otimes \\
& \phi(v_{<0>})^{[-1]}{}_{<-2>} \phi(v_{<0>})^{[0]}{}_{(1)} S(v_{<-1>}) \otimes \\
& \phi(v_{<0>})^{[-1]}{}_{<-1>} \phi(v_{<0>})^{[0]}{}_{(2)} S(v_{<-2>}) \\
& = \phi(v_{<0>})^{[-1]}{}_{<0>} \otimes_H \phi(v_{<0>})^{[1]} \otimes \cdots \otimes \phi(v_{<0>})^{[q]} \otimes \\
& \Delta\Big(\phi(v_{<0>})^{[-1]}{}_{<-1>} \phi(v_{<0>})^{[0]} S(v_{<-1>})\Big) \\
& = \mathfrak{d}_q\Big(\Big[\phi(v_{<0>})^{[-1]}{}_{<0>} \otimes_H \phi(v_{<0>})^{[1]} \otimes \cdots \otimes \phi(v_{<0>})^{[q]} \otimes \\
& \phi(v_{<0>})^{[-1]}{}_{<-1>} \phi(v_{<0>})^{[0]}\Big] \triangleleft S(v_{<-1>})\Big) \\
& = \mathfrak{d}_q(\mathfrak{t}_q(\phi(v_{<0>})) \triangleleft S(v_{<-1>})) = d_q(t_q(\phi))(v).
\end{aligned} \tag{3.13}$$

A simple calculation then yields

$$\begin{aligned}
& t_q^{q+1}(\phi)(v) \\
& = \phi(v_{<0>})^{[-1]}{}_{<0>} \otimes_H \phi(v_{<0>})^{[-1]}{}_{<-q-1>} \phi(v_{<0>})^{[0]} S(v_{<-1>}) \otimes \ldots \\
& \ldots \otimes \phi(v_{<0>})^{[-1]}{}_{<-1>} \phi(v_{<0>})^{[q]} S(v_{<-q-1>}) \otimes \ldots \\
& = \phi(v_{<0>})^{[-1]}{}_{<0>} \phi(v_{<0>})^{[-1]}{}_{<-1>} \otimes_H \phi(v_{<0>})^{[0]} S(v_{<-1>}) \\
& \ldots \otimes \phi(v_{<0>})^{[q]} S(v_{<-q-1>}) \\
& = \phi(v_{<0>})^{[-1]} \otimes_H \phi(v_{<0>})^{[0]} S(v_{<-1>}) \otimes \cdots \otimes \phi(v_{<0>})^{[q]} S(v_{<-q-1>}) \\
& = \phi(v_{<0>}) \cdot S(v_{<-1>}) = \phi(v_{<0>} \cdot S(v_{<-1>})) = \phi(v),
\end{aligned}$$

where the last equality follows from [6, Lemma 4.9]. □

We denote the cyclic cohomology of $C_K(H, V, N)$ by $HC_K(H, V, N)$. One notes that by taking $K = \mathbb{C}$ and $V = \mathbb{C}$ the usual Hopf-cyclic cohomology $HC(H, N)$ is recovered.

3.2 Equivariant characteristic map

Let V and N be SAYD modules over K and H respectively. We define

$$\Psi : C^q_K(H,V,N)\otimes C^q_H(A,N)\longrightarrow C^q_K(A,V), \tag{3.14}$$
$$\Psi(\phi\otimes\psi)(v\otimes a_0\otimes\cdots\otimes a_q)$$
$$=\psi\big(\phi(v)^{[-1]}\otimes\phi(v)^{[0]}(a_0)\otimes\phi(v)^{[1]}(a_1)\otimes\cdots\otimes\phi(v)^{[q]}(a_q)\big). \tag{3.15}$$

One may check that Ψ is a map of cocyclic modules, where on the left hand side we consider the product of two cocyclic modules. This is enough to produce a generalization of the cup product in Hopf-cyclic cohomology [8, 12].

We define a bicocyclic module by tensoring (2.13) and (3.1). The bigraded module in the bidegree (p,q) is then defined by

$$\mathcal{C}^{p,q}:=\operatorname{Hom}_K(V,N\otimes_H H^{\otimes p+1})\otimes\operatorname{Hom}_H(A^{\otimes q+1},N) \tag{3.16}$$

with the horizontal structure $\overrightarrow{\partial}_i=\mathfrak{d}_i\otimes\operatorname{Id}$, $\overrightarrow{\sigma}_j=\mathfrak{s}\otimes\operatorname{Id}$ and $\overrightarrow{\tau}=\mathfrak{t}\otimes\operatorname{Id}$, and the vertical structure $\uparrow\!\partial_i=\operatorname{Id}\otimes\partial_i$, $\uparrow\!\sigma_j=\operatorname{Id}\otimes\sigma_j$ and $\uparrow\!\tau=\operatorname{Id}\otimes\tau$. Obviously $(\mathcal{C}^{\bullet,\bullet},\overrightarrow{\partial},\overrightarrow{\sigma},\overrightarrow{\tau},\uparrow\!\partial,\uparrow\!\sigma,\uparrow\!\tau)$ defines a bicocyclic module.

Now let us define the map

$$\Psi:\mathcal{D}^q\to\operatorname{Hom}_K(V\otimes A^{\otimes q+1},\mathbb{C}), \tag{3.17}$$
$$\Psi(\phi\otimes\psi)(v\otimes a_0\otimes\cdots\otimes a_q)$$
$$=\psi\big(\phi(v)^{[-1]}\otimes\phi(v)^{[0]}(a_0)\otimes\phi(v)^{[1]}(a_1)\otimes\cdots\otimes\phi(v)^{[q]}(a_q)\big),$$

where $\mathcal{D}^\bullet$ denotes the diagonal of the bicocyclic module $\mathcal{C}^{\bullet,\bullet}$. It is a cocyclic module whose qth component is $\mathcal{C}^{q,q}$ and its cocyclic structure maps are $\partial_i:=\overrightarrow{\partial}_i\circ\uparrow\!\partial_i$, $\sigma_j:=\overrightarrow{\sigma}_j\circ\uparrow\!\sigma_j$, and $\tau:=\overrightarrow{\tau}\circ\uparrow\!\tau$.

Proposition 3.2. *The map Ψ is a map of cocyclic modules.*

Proof. Let us first show that Ψ is well-defined. Indeed, using the fact that ϕ is K-linear, we see that

$$\Psi(\phi\otimes\psi)\big(vk_{(1)}\otimes S(k_{(q+2)})(a^0)\otimes\cdots\otimes S(k_{(2)})(a^q)\big)$$
$$=\psi(\phi(vk_{(1)})^{[-1]}\otimes\phi(vk_{(1)})^{[0]}S(k_{(q+2)})(a^0)\otimes\cdots\otimes\phi(vk_{(1)})^{[q]}S(k_{(2)})(a^q))$$
$$=\psi(\phi(v)^{[-1]}\otimes\phi(v)^{[0]}k_{(1)}S(k_{(2q+2)})(a^0)\otimes\cdots\otimes\phi(v)^{[q]}k_{(q+1)}S(k_{(q+2)})(a^q))$$
$$=\varepsilon(k)\Psi(\phi\otimes\psi)\big(v\otimes a^0\otimes\cdots\otimes a^q\big).$$

Next, we show that Ψ commutes with the cocyclic structures. To this end, it suffices to show the commutativity of Ψ with zeroth coface, the last codegeneracy and the cyclic operator. We check it only for the cyclic operators and leave the rest to the reader.

$$\begin{aligned}
&\tau\Big(\Psi(\phi\otimes\psi)\Big)\big(v\otimes a^0\otimes\cdots\otimes a^q\big)\\
&=\Psi(\phi\otimes\psi)\big(v_{<0>}\otimes S^{-1}(v_{<-1>})(a^q)\otimes a^0\otimes\cdots\otimes a^{q-1}\big)\\
&=\psi(\phi(v_{<0>})^{[-1]}\otimes\phi(v_{<0>})^{[0]}S^{-1}(v_{<-1>})(a^q)\otimes\\
&\qquad\qquad\qquad\qquad\phi(v_{<0>})^{[1]}(a^1)\otimes\cdots\otimes\phi(v_{<0>})^{[q]}(a^q).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&\Psi(\mathfrak{t}\phi\otimes\tau\psi)\big(v\otimes a^0\otimes\cdots\otimes a^q\big)\\
&=\tau\psi(\mathfrak{t}\phi(v)^{[-1]}\otimes\mathfrak{t}\phi(v)^{[0]}(a^0)\otimes\cdots\otimes\mathfrak{t}\phi(v)^{[q]}(a^q))\\
&=\psi\big(\phi(v_{<0>})^{[-1]}{}_{<0>}\otimes S^{-1}(\phi(v_{<0>})^{[-1]}{}_{<-1>})\phi(v_{<0>})^{[-1]}{}_{<-2>}\phi(v_{<0>})^{[0]}\\
&\qquad\qquad S(v_{<-1>})(a^q)\otimes\phi(v)^{[1]}(a^0)\otimes\cdots\otimes\phi(v)^{[q]}(a^{q-1})\big)\\
&=\psi(\phi(v_{<0>})^{[-1]}\otimes\phi(v_{<0>})^{[0]}S(v_{<-1>})(a^q)\otimes\\
&\qquad\qquad\qquad\qquad\phi(v_{<0>})^{[1]}(a^1)\otimes\cdots\otimes\phi(v_{<0>})^{[q]}(a^q).
\end{aligned}$$

The proof is as $S=S^{-1}$. ☐

Theorem 3.3. *Assume that K is a cocommutative Hopf subalgebra of a Hopf algebra H, $\mathcal{A}$ is a H-module algebra, and V and N are SAYD modules over K and H respectively. Then the map Ψ defines a cup product*

$$HC^p_K(H,V,N)\otimes HC^q_H(A,N)\to HC^{p+q}_K(A,V). \tag{3.18}$$

Proof. Let $[\phi]\in HC^p_K(H,V,N)$ and $[\psi]\in HC^q_H(A,N)$. Without loss of generality we assume that ϕ and ψ are respectively cyclic cocycles horizontally and vertically. This implies that $\phi\otimes\psi$ is a (b,B) cocycle of degree $p+q$ in total compex of $\mathcal{C}^{\bullet,\bullet}$. On the other hand, by the cyclic Eilenberg-Zilber theorem [7], the total complex of $\mathcal{C}^{\bullet,\bullet}$ is quasi-isomorphic with $\mathcal{D}^\bullet$ via the Alaxander-Witney map AW. So, $AW(\phi\otimes\psi)$ is a (b,B) cocycle in $\mathcal{D}^\bullet$. Since Ψ is cyclic, we conclude that $\Psi(AW(\phi\otimes\psi))$ defines a class in $HC^{p+q}_K(A,V)$. ☐

One notes that for $K=\mathbb{C}$ and $M=\mathbb{C}$, the trivial SAYD module over K, the above cup product becomes the one defined in [8, 12].

3.3 Equivariant charactrestic map for $\mathcal{H}_n$

In this subsection we apply the equivariant characteristic map of Subsection 3.2 to produce the desired cyclic cocycle on the groupoid action algebra $\mathcal{A} := C_c^\infty(F^+) \rtimes \Gamma$.

Let us first recall the Connes-Moscovici Hopf algebra $\mathcal{H} := \mathcal{H}_n$ from [2, 3]. To this end, let $\mathfrak{h}_n$ be the Lie algebra generated by

$$\{X_k,\, Y_i^j,\, \delta^i_{jk\,\ell_1\ldots\ell_r} \,|\, i,j,k,\ell_1\ldots\ell_r = 1,\ldots,n,\, r\in\mathbb{N}\} \tag{3.19}$$

with relations

$$\begin{aligned}
&[Y_i^j, Y_k^\ell] = \delta_k^j Y_i^\ell - \delta_i^\ell Y_k^j, \quad [Y_i^j, X_k] = \delta_k^j X_i, \quad [X_k, X_\ell] = 0,\\
&\delta^i_{jk\ell_1\ldots\ell_r} = [X_{\ell_r},\ldots[X_{\ell_1},\delta^i_{jk}]\ldots], \quad [\delta^i_{jk\ell_1\ldots\ell_r}, \delta^{i'}_{j'k'\ell'_1\ldots\ell'_r}] = 0,\\
&[Y_p^q, \delta^i_{j_1j_2\,j_3\ldots j_r}] = \sum_{s=1}^r \delta^q_{j_s}\, \delta^i_{j_1j_2\,j_3\ldots j_{s-1}pj_{s+1}\ldots j_r} - \delta^i_p\, \delta^q_{j_1j_2\,j_3\ldots j_r},\\
&\delta^i_{jk\ell_1\,\ldots\ell_r} = \delta^i_{jk\ell_{\pi(1)}\ldots\ell_{\pi(r)}}, \quad \forall\pi\in S_r.
\end{aligned} \tag{3.20}$$

As an algebra, $\mathcal{H}$ is $U(\mathfrak{h}_n)$ modulo the (Bianchi-type) identities

$$\delta^i_{j\ell k} - \delta^i_{jk\ell} = \delta^s_{jk}\,\delta^i_{s\ell} - \delta^s_{j\ell}\,\delta^i_{sk}. \tag{3.21}$$

The coalgebra structure of $\mathcal{H}$ is defined by a Leibniz rule that makes $\mathcal{A}$ an $\mathcal{H}$-module algebra.

In order to describe the action of $\mathcal{H}$ explicitly, let us first identify F^+ with $\mathbb{R}^n \rtimes \mathrm{GL}^+(n,\mathbb{R})$ and use the local coordinates $(x,y)\in F^+$. A typical element of the algebra $\mathcal{A}$ is a finite sum of $\sum_i f_i U^*_{\phi_i}$, where $U^*_{\phi_i}$ stands for $\phi_i^{-1}\in\Gamma$ and $f_i\in C_c^\infty(F^+)$. The elements of $\mathcal{H}$ then act as

$$\begin{aligned}
&X_k = y_k^\mu \frac{\partial}{\partial x^\mu}, \qquad X_k(fU^*_\phi) := X_k(f)U^*_\phi,\\
&Y_i^j = y_i^\mu \frac{\partial}{\partial y_\mu^j}, \qquad Y_i^j(fU^*_\phi) := Y_i^j(f)U^*_\phi,\\
&\delta^i_{jk\,\ell_1\ldots\ell_r}\,(f\,U^*_\phi) = \gamma^i_{jk\,\ell_1\ldots\ell_r}(\phi)\,f\,U^*_\phi,
\end{aligned} \tag{3.22}$$

where

$$\begin{aligned}
&\gamma^i_{jk\,\ell_1\ldots\ell_r}(\phi) = X_{\ell_r}\cdots X_{\ell_1}\big(\gamma^i_{jk}(\phi)\big),\\
&\gamma^i_{jk}(\phi)(x,y) = \big(y^{-1}\cdot\phi'(x)^{-1}\cdot\partial_\mu\phi'(x)\cdot y\big)^i_j\, y^\mu_k.
\end{aligned} \tag{3.23}$$

Therefore, for any $a, b \in \mathcal{A}$ we have the Leibniz rule

$$\begin{aligned} Y_i^j(ab) &= Y_i^j(a)\, b \,+\, a\, Y_i^j(b), \\ X_k(ab) &= X_k(a)\, b \,+\, a\, X_k(b) + \delta_{jk}^i(a)\, Y_i^j(b), \\ \delta_{jk}^i(ab) &= \delta_{jk}^i(a)\, b + a\, \delta_{jk}^i(b). \end{aligned} \tag{3.24}$$

Accordingly,

$$\Delta(Y_i^j) = Y_i^j \otimes 1 + 1 \otimes Y_i^j, \tag{3.25}$$

$$\Delta(\delta_{jk}^i) = \delta_{jk}^i \otimes 1 + 1 \otimes \delta_{jk}^i, \tag{3.26}$$

$$\Delta(X_k) = X_k \otimes 1 + 1 \otimes X_k + \delta_{jk}^i \otimes Y_i^j. \tag{3.27}$$

For simplicity, we will also employ the notation

$$\delta^a{}_{k\ell_1\ldots\ell_r} := \delta_{jk\ell_1\ldots\ell_r}^i, \;\; Y_a := Y_i^j \quad a = \begin{pmatrix} i \\ j \end{pmatrix}. \tag{3.28}$$

Let us also set $\mathfrak{g}_0 := g\ell_n$, $V = S(\mathfrak{g}_0^*)_{[2n]}$ with the canonical SAYD module structure over $\mathcal{K} := U(\mathfrak{g}_0)$ as recalled in (2.3), and $N = \mathbb{C}_\delta$ the SAYD module over $\mathcal{H}$ where $\delta : \mathcal{H} \to \mathbb{C}$ is the character defined on the generators by

$$\delta(Y_i^j) = \delta_i^j, \quad \delta(X_k) = \delta(\delta_{jk\ell_1\ldots\ell_r}^i) = 0, \quad 1 \leq i, j, k, l_t \leq n. \tag{3.29}$$

Applying the cup product (3.18) with the canonical trace $\tau \in C^0_{\mathcal{H}}(\mathcal{A}, N)$, [2], we get the characteristic map

$$\begin{aligned} &\chi_\tau^{\rm eq} : HC^q_{\mathcal{K}}(\mathcal{H}, V, N) \longrightarrow HC^q_{\mathcal{K}}(\mathcal{A}, V) \\ &\chi_\tau^{\rm eq}(\phi)(v \otimes a_0 \otimes \cdots \otimes a_q) = \tau\left(\phi(v)^{[0]}(a_0) \cdots \phi(v)^{[q]}(a_q)\right), \end{aligned} \tag{3.30}$$

as a mechanism to obtain "SAYD-twisted" cyclic cocycles. We conclude this section by the identification of $C^q_{\mathcal{K}}(\mathcal{K}, V, N)$ with $(V^* \otimes N \otimes \mathcal{H}^{\otimes q})^{\mathfrak{g}_0}$, where $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and $\mathfrak{g}_0$ acts on $V^* \otimes N \otimes \mathcal{H}^{\otimes q}$ via

$$\begin{aligned} &(\phi \otimes \mathbf{1} \otimes h^1 \otimes \cdots \otimes h^q)Z = -\sum_{i=1}^q \phi \otimes \mathbf{1} \otimes h^1 \otimes \cdots \otimes \mathrm{ad}_Z(h^i) \otimes \cdots \otimes h^q \\ &+ \phi \otimes \delta(Z) \otimes h^1 \otimes \cdots \otimes h^q + \phi \cdot Z \otimes \mathbf{1} \otimes h^1 \otimes \cdots \otimes h^q. \end{aligned} \tag{3.31}$$

Here, the action of $\mathfrak{g}_0$ on V^* is defined by $(\phi \cdot Z)(v) = -\phi(v \cdot Z)$. Accordingly, the aforementioned identification is given by the map

$$\begin{aligned} &\mathcal{I} : (V^* \otimes N \otimes \mathcal{H}^{\otimes q})^{\mathfrak{g}_0} \to C^q_{\mathcal{K}}(\mathcal{H}, V, N) \\ &\mathcal{I}(\phi \otimes \mathbf{1} \otimes h^1 \otimes \cdots \otimes h^q)(v) = \phi(v) \otimes_{\mathcal{H}} 1_{\mathcal{H}} \otimes h^1 \otimes \cdots \otimes h^q. \end{aligned} \tag{3.32}$$

Proposition 3.4. *The map $\mathcal{I}$, defined in* (3.32)*, is an isomorphism of vector spaces.*

Proof. Let us first check that $\mathcal{I}$ is well-defined. Indeed,

$$
\begin{aligned}
&\mathcal{I}(\phi\otimes \mathbf{1}\otimes h^1\otimes\cdots\otimes h^q)(v\cdot Z)=\phi(v\cdot Z)\otimes_{\mathcal{H}} 1_{\mathcal{H}}\otimes h^1\otimes\cdots\otimes h^q\\
&=-(\phi\cdot Z)(v)\otimes_{\mathcal{H}} 1_{\mathcal{H}}\otimes h^1\otimes\cdots\otimes h^q\\
&=-\sum_{i=1}^{q}\phi(v)\otimes_{\mathcal{H}} 1_{\mathcal{H}}\otimes h^1\otimes\cdots\otimes \mathrm{ad}_Z(h^i)\otimes\cdots\otimes h^q\\
&+\delta(Z)\phi(v)\otimes_{\mathcal{H}} 1_{\mathcal{H}}\otimes h^1\otimes\cdots\otimes h^q\\
&=-\sum_{i=1}^{q}\phi(v)\otimes_{\mathcal{H}} 1_{\mathcal{H}}\otimes h^1\otimes\cdots\otimes \mathrm{ad}_Z(h^i)\otimes\cdots\otimes h^q\\
&+\phi(v)\otimes_{\mathcal{H}} Z\otimes h^1\otimes\cdots\otimes h^q+\sum_{i=1}^{q}\phi(v)\otimes_{\mathcal{H}} 1_{\mathcal{H}}\otimes h^1\otimes\cdots\otimes Zh^i\otimes\cdots\otimes h^q\\
&=(\phi(v)\otimes 1_{\mathcal{H}}\otimes h^1\otimes\cdots\otimes h^q)\cdot Z=(\mathcal{I}(\phi\otimes\mathbf{1}\otimes h^1\otimes\cdots\otimes h^q)(v))\cdot Z.
\end{aligned}
$$

Next, we introduce an inverse map for $\mathcal{I}$. To this end we fix a basis for V, say $\{v_1,\ldots,v_m\}$, with a dual basis $\{\nu^1,\ldots,\nu^m\}$ for V^*. Then,

$$
\begin{aligned}
&\mathcal{I}^{-1}:C^q_{\mathcal{K}}(\mathcal{H},V,N)\to (V^*\otimes N\otimes \mathcal{H}^{\otimes q})^{\mathfrak{g}_0} \qquad (3.33)\\
&\mathcal{I}^{-1}(\phi)=\\
&\sum_{i=1}^{m}\nu^i\otimes \phi(v_i)^{[-1]}\delta(\phi(v_i)^{[0]}{}_{(1)})\otimes S(\phi(v_i)^{[0]}{}_{(2)})\cdot\Big(\phi(v_i)^{[1]})\otimes\cdots\otimes\phi(v_i)^{[q]}\Big)
\end{aligned}
$$

is inverse to $\mathcal{I}$, and is independent of the choice of bases. ☐

As a result, we can transfer the cocyclic structure of $C_{\mathcal{K}}(\mathcal{H},V,N)$ to

$$
\mathcal{E}(V^*,N,\mathcal{H})^{\mathrm{inv}}:=\bigoplus_{q\geq 0}\mathcal{E}^q,\qquad \mathcal{E}^q:=(V^*\otimes N\otimes\mathcal{H}^{\otimes q})^{\mathfrak{g}_0}. \qquad (3.34)
$$

This way we obtain the cocyclic structure with the cofaces

$$
\begin{aligned}
&\widetilde{\mathfrak{d}}_i:\mathcal{E}^q\longrightarrow\mathcal{E}^{q+1},\qquad 0\leq i\leq q+1,\\
&\widetilde{\mathfrak{d}}_0(\phi\otimes\mathbf{1}\otimes h^1\otimes\cdots\otimes h^q)=\phi\otimes\mathbf{1}\otimes 1\otimes h^1\otimes\cdots\otimes h^q,\\
&\widetilde{\mathfrak{d}}_i(\phi\otimes\mathbf{1}\otimes h^1\otimes\cdots\otimes h^q)=\phi\otimes\mathbf{1}\otimes h^1\otimes\cdots\otimes\Delta(h^i)\otimes\cdots\otimes h^q,\\
&\widetilde{\mathfrak{d}}_{q+1}(\phi\otimes\mathbf{1}\otimes h^1\otimes\cdots\otimes h^q)=\nu^i\phi(v_{i_{<0>}})\otimes\mathbf{1}\otimes h^1\otimes\cdots\otimes h^q\otimes S(v_{i_{<-1>}}),
\end{aligned}
\qquad (3.35)
$$

the codegeneracies

$$\widetilde{\mathfrak{s}}_j : \mathcal{E}^q \longrightarrow \mathcal{E}^{q-1}, \qquad 0 \le j \le q-1$$
$$\widetilde{\mathfrak{s}}_j(\phi \otimes \mathbf{1} \otimes h^1 \otimes \cdots \otimes h^q) = \phi \otimes \mathbf{1} \otimes h^1 \otimes \cdots \otimes \varepsilon(h^{j+1}) \otimes \cdots \otimes h^q, \tag{3.36}$$

and the cyclic operator

$$\widetilde{\mathfrak{t}}_q : \mathcal{E}^q \longrightarrow \mathcal{E}^q,$$
$$\widetilde{\mathfrak{t}}_q(\phi \otimes \mathbf{1} \otimes h^1 \otimes \cdots \otimes h^q) = \nu^i \phi(v_{i_{<0>}}) \otimes \mathbf{1} \otimes \widetilde{S}(h^1) \cdot (h^2 \otimes \cdots \otimes h^q \otimes S(v_{i_{<-1>}})). \tag{3.37}$$

3.4 A SAYD-twisted cyclic cocycle in codimension 1

In this subsection we keep the setting of Subsection 3.3 for $n = 1$. We construct an explicit equivariant cyclic 1-cocycle $\varphi \in C^1_{\mathcal{K}}(\mathcal{A}, V)$.

Let $\{R\}$ be the basis for $\mathfrak{g}_0^*$ as the dual basis of $\{Y := Y_1^1\}$ for $\mathfrak{g}_0$. Let also $\{1, R\}$ be the basis of V and $\{1^*, S\}$ as the dual basis for V^*.

For $1^* \otimes \mathbf{1} \otimes X$, $S \otimes \mathbf{1} \otimes \delta_1 \in \mathcal{E}^1$, we define

$$\begin{aligned} &\varphi_0, \varphi_1 : V \otimes \mathcal{A}^{\otimes 2} \to \mathbb{C}, \\ &\varphi_0 = \chi_\tau^{\rm eq}(1^* \otimes \mathbf{1} \otimes X), \quad \varphi_1 = \chi_\tau^{\rm eq}(S \otimes \mathbf{1} \otimes \delta_1). \end{aligned} \tag{3.38}$$

Lemma 3.5. *The 1-cochain $\varphi_0 - \varphi_1$ is a Hochschild 1-cocycle.*

Proof. Using (3.26) and (3.27) we have

$$\begin{aligned} &b(\varphi_0 - \varphi_1)((\alpha 1 + \beta\theta) \otimes a_0 \otimes a_1 \otimes a_2) = (\varphi_0 - \varphi_1)((\alpha 1 + \beta\theta) \otimes a_0 a_1 \otimes a_2) \\ &- (\varphi_0 - \varphi_1)((\alpha 1 + \beta\theta) \otimes a_0 \otimes a_1 a_2) + (\varphi_0 - \varphi_1)((\alpha 1 + \beta\theta) \otimes a_2 a_0 \otimes a_1) \\ &- (\varphi_0 - \varphi_1)(\alpha\theta \otimes Y(a_2) a_0 \otimes a_1) \\ &= \alpha\tau(a_0 a_1 X(a_2)) - \alpha\tau(a_0 X(a_1 a_2)) + \alpha\tau(a_2 a_0 X(a_1)) + \alpha\tau(Y(a_2) a_0 \delta_1(a_1)) \\ &- \beta\tau(a_0 a_1 \delta_1(a_2)) + \beta\tau(a_0 \delta_1(a_1 a_2)) - \beta\tau(a_2 a_0 \delta_1(a_1)) = 0. \end{aligned} \tag{3.39}$$

□

Proposition 3.6. *The 1-cocycle $\varphi_0 - \varphi_1$ is cyclic.*

Proof. By using the δ invariancy of τ, (3.26) and (3.27) we have

$$t(\varphi_0-\varphi_1)((\alpha 1+\beta\theta)\otimes a_0\otimes a_1)=(\varphi_0-\varphi_1)((\alpha 1+\beta\theta)\otimes a_1\otimes a_0)$$
$$-(\varphi_0-\varphi_1)(\alpha\theta\otimes Y(a_1)\otimes a_0)$$
$$=\alpha\tau(a_1X(a_0))+\alpha\tau(Y(a_1)\delta_1(a_0))-\beta\tau(a_1\delta_1(a_0))$$
$$=-\alpha\tau(a_0X(a_1))+\beta\tau(a_0\delta_1(a_1))=-(\varphi_0-\varphi_1)((\alpha 1+\beta\theta)\otimes a_0\otimes a_1).$$

□

3.5 A SAYD-twisted cyclic cocycle in codimension 2

As in the previous subsection, we keep the setting of Subsection 3.3 for $n=2$. We introduce an explicit cyclic 2-cocycle $\phi\in C^2_{\mathcal{K}}(\mathcal{A},V)$.

Let $\{R^i_j \mid 1\le i,j\le 2\}$ be the dual basis of $\mathfrak{g}_0$ with the pairing $\langle Y^j_i, R^k_l\rangle=\delta^j_k\delta^i_l$. We take

$$\left\{1, R^i_j, R^k_l R^p_q \;\middle|\; \begin{pmatrix} k\\ l\end{pmatrix}\le\begin{pmatrix} p\\ q\end{pmatrix}\right\}, \tag{3.40}$$

as a basis for V which is simplified by $\{1,R^a,R^{ab}\,|\,a\le b\}$. The dual basis for V^* is expressed by $\{1^*,S_a,S_{ab}\,|\,a\le b\}$.

We recall from [13] that the Koszul coaction (2.24) gives rise to a $\mathcal{K}$-coaction by the formula

$$\begin{aligned}
&\nabla_K: V\to\mathcal{K}\otimes V,\\
&\nabla_K(1)=1\otimes 1+Y_a\otimes R^a+\frac{1}{2!}Y_aY_b\otimes R^{ab},\\
&\nabla_K(R^a)=1\otimes R^a+Y_b\otimes R^{ab},\\
&\nabla_K(R^{ab})=1\otimes R^{ab}.
\end{aligned}\tag{3.41}$$

We decompose $V=V_0\oplus V_1\oplus V_2$, where $V_0=\mathbb{C}\langle 1\rangle$, $V_1=\mathbb{C}\langle R^a\rangle$, and $V_2=\mathbb{C}\langle R^{ab}\rangle$. Using this decomposition, any $\psi\in\mathrm{Hom}(V\otimes\mathcal{A}^{q+1},\mathbb{C})$ is decomposed uniquely as $\psi=\psi_0+\psi_1+\psi_2$ by $\psi_i=\psi|_{V_i\otimes\mathcal{A}^{\otimes q+1}}$.

We now consider the linear map $\psi : V \otimes \mathcal{A}^{\otimes 3} \to \mathbb{C}$ with components

$$\psi_0 := \chi_\tau^{\mathrm{eq}}\Big(\gamma_1 1^* \otimes \mathbf{1} \otimes X_{\sigma(1)} \otimes X_{\sigma(2)} + \gamma_2 1^* \otimes \mathbf{1} \otimes \delta^a{}_{\sigma(1)} \otimes X_{\sigma(2)} Y_a$$
$$+ \gamma_3 1^* \otimes \mathbf{1} \otimes \delta^a{}_{\sigma(1)} \delta^b{}_{\sigma(2)} Y_b \otimes Y_a + \gamma_4 1^* \otimes \mathbf{1} \otimes \delta^a{}_{\sigma(1)\sigma(2)} \otimes Y_a\Big),$$

$$\psi_1 := \chi_\tau^{\mathrm{eq}}\Big(\beta_1 S_a \otimes \mathbf{1} \otimes \delta^a{}_{\sigma(1)} \otimes X_{\sigma(2)} + \beta_2 S_a \otimes \mathbf{1} \otimes X_{\sigma(1)} \otimes \delta^a{}_{\sigma(2)}$$
$$+ \beta_3 S_a \otimes \mathbf{1} \otimes \delta^a{}_{\sigma(1)} \delta^b{}_{\sigma(2)} \otimes Y_b + \beta_4 S_a \otimes \mathbf{1} \otimes Y_b \otimes \delta^a{}_{\sigma(1)} \delta^b{}_{\sigma(2)}$$
$$+ \beta_5 S_a \otimes \mathbf{1} \otimes \delta^a{}_{\sigma(1)} Y_b \otimes \delta^b{}_{\sigma(2)} + \beta_6 S_a \otimes \mathbf{1} \otimes \delta^b{}_{\sigma(1)} Y_b \otimes \delta^a{}_{\sigma(2)}$$
$$+ \beta_7 S_a \otimes \mathbf{1} \otimes \delta^a{}_{\sigma(1)} \otimes \delta^b{}_{\sigma(2)} Y_b + \beta_8 S_a \otimes \mathbf{1} \otimes \delta^b{}_{\sigma(1)} \otimes \delta^a{}_{\sigma(2)} Y_b$$
$$+ \beta_9 S_a \otimes \mathbf{1} \otimes \Delta(\delta^a{}_{\sigma(1)\sigma(2)})\Big),$$

$$\psi_2 := \chi_\tau^{\mathrm{eq}}\Big(\alpha_1 S_{ab} \otimes \mathbf{1} \otimes \delta^a{}_{\sigma(1)} \otimes \delta^b{}_{\sigma(2)} + \alpha_2 S_{ab} \otimes \mathbf{1} \otimes \delta^b{}_{\sigma(1)} \otimes \delta^a{}_{\sigma(2)}\Big),$$

Our aim is to determine the coefficients $\alpha_i, \beta_j, \gamma_k$, such that ψ is a cyclic 2-cocycle. To do so we prove a series of technical lemmas.

Lemma 3.7. *For any $\alpha_i, \beta_j, \gamma_k$, $(b\psi)_2 = 0$.*

Proof. The result follows directly from the application of the Hochschild coboundary map and the fact that $\delta^a{}_k$ are derivations of $\mathcal{A}$. □

On the next move, we determine α_i, $1 \le i \le 2$, in such a way that $\psi \in C^2_{\mathcal{K}}(\mathcal{A}, V)$ is a cyclic cocycle on $V_2 \otimes \mathcal{A}^{\otimes n}$.

Lemma 3.8. *For $\alpha_1 = \alpha_2$, we have $(\tau\psi)_2 = \psi_2$.*

Proof. By definition of the cyclic operator, we have

$$\tau\psi(R^{ab} \otimes a_0 \otimes a_1 \otimes a_2) = \psi_2(R^{ab} \otimes a_2 \otimes a_0 \otimes a_1). \tag{3.42}$$

Hence, by the integration by parts property [3, (3.4)],

$$\begin{aligned} & t\psi(R^{ab} \otimes a_0 \otimes a_1 \otimes a_2) = \\ & \alpha_1 \tau\left(a_0(-\delta^a{}_{\sigma(1)})\delta^b{}_{\sigma(2)}(a_1)a_2\right) + \alpha_1 \tau\left(a_0 \delta^b{}_{\sigma(2)}(a_1)(-\delta^a{}_{\sigma(1)})(a_2)\right) \\ & + \alpha_2 \tau\left(a_0(-\delta^b{}_{\sigma(1)})\delta^a{}_{\sigma(2)}(a_1)a_2\right) + \alpha_2 \tau\left(a_0 \delta^a{}_{\sigma(2)}(a_1)(-\delta^b{}_{\sigma(1)})(a_2)\right) \\ & = \alpha_1 \tau\left(a_0 \delta^b{}_{\sigma(1)}(a_1)\delta^a{}_{\sigma(2)}(a_2)\right) + \alpha_2 \tau\left(a_0 \delta^a{}_{\sigma(1)}(a_1)\delta^b{}_{\sigma(2)}(a_2)\right). \end{aligned} \tag{3.43}$$

Therefore, $(t\psi)_2 = \psi_2$ if and only if $\alpha_1 = \alpha_2$. □

As a result we set

$$\alpha_1 = \alpha_2 = r. \tag{3.44}$$

On the next step, we find a constraint on β_j's such that ψ is a Hochschild cocycle over $V_1 \otimes \mathcal{A}^{\otimes q+1}$.

Lemma 3.9. *We have* $(b\psi)_1 = 0$ *if and only if*

$$\begin{aligned} \beta_1 - \beta_3 + \beta_7 + r &= 0 \\ \beta_3 + \beta_8 + r &= 0 \\ -\beta_2 - \beta_6 + \beta_8 &= 0 \\ \beta_4 - \beta_5 &= 0 \\ -\beta_4 - \beta_6 &= 0 \\ -\beta_5 + \beta_7 &= 0. \end{aligned} \tag{3.45}$$

Proof. Recalling the Koszul coaction (3.41) in the last coface,

$$\begin{aligned} &b\psi(R^a \otimes a_0 \otimes a_1 \otimes a_2 \otimes a_3) = \\ &\psi_1(R^a \otimes a_0a_1 \otimes a_2 \otimes a_3) - \psi_1(R^a \otimes a_0 \otimes a_1a_2 \otimes a_3) \\ &+ \psi_1(R^a \otimes a_0 \otimes a_1 \otimes a_2a_3) - \psi_1(R^a \otimes a_3a_0 \otimes a_1 \otimes a_2) \\ &+ \psi_2(R^{ab} \otimes Y_b(a_3)a_0 \otimes a_1 \otimes a_2). \end{aligned} \tag{3.46}$$

Therefore, as a result of the tracial property [3, Thm. 6] and the faithfullness [3, (3.12)] of the trace, we have $(b\psi)_1 = 0$ if and only if

$$\begin{aligned} &(\beta_1 - \beta_3 + \beta_7 + r)\, \mathbf{1} \otimes {\delta^a}_{\sigma(1)} \otimes {\delta^b}_{\sigma(2)} \otimes Y_b + (\beta_3 + \beta_8 + r)\mathbf{1} \otimes {\delta^b}_{\sigma(1)} \otimes {\delta^a}_{\sigma(2)} \otimes Y_b \\ &+ (-\beta_2 - \beta_6 + \beta_8)\mathbf{1} \otimes {\delta^b}_{\sigma(1)} \otimes Y_b \otimes {\delta^a}_{\sigma(2)} + (\beta_4 - \beta_5)\mathbf{1} \otimes Y_b \otimes {\delta^a}_{\sigma(1)} \otimes {\delta^b}_{\sigma(2)} \\ &+ (-\beta_4 - \beta_6)\mathbf{1} \otimes Y_b \otimes {\delta^b}_{\sigma(1)} \otimes {\delta^a}_{\sigma(2)} + (-\beta_5 + \beta_7)\mathbf{1} \otimes {\delta^a}_{\sigma(1)} \otimes Y_b \otimes {\delta^b}_{\sigma(2)} = 0. \end{aligned}$$

Accordingly we get the system (3.45). □

On the next step we determine β_j's in such a way that $\psi \in C^2_{\mathcal{K}}(\mathcal{A}, V)$ is a cyclic cocycle over $(V_1 \oplus V_2) \otimes \mathcal{A}^{\otimes n}$.

Lemma 3.10. *We have* $(\tau\psi)_1 = \psi_1$ *if and only if*

$$\begin{aligned} &\beta_1 = \beta_2 = -r, \quad \beta_3 = \beta_4 = \beta_5 = -\beta_6 = \beta_7 = s, \\ &\beta_8 = -r - s, \quad \beta_9 = \frac{1}{2}r + s. \end{aligned} \tag{3.47}$$

Proof. By the Koszul coaction, we have

$$t\psi(R^a\otimes a_0\otimes a_1\otimes a_2) = \psi_1(R^a\otimes a_2\otimes a_0\otimes a_1) - \psi_2(R^{ab}\otimes Y_b(a_2)\otimes a_0\otimes a_1).$$

Accordingly,

$$\begin{aligned}
&t\psi(R^a \otimes a_0 \otimes a_1 \otimes a_2) = \\
&\beta_1\tau\left({\delta^a}_{\sigma(1)}(a_0)X_{\sigma(2)}(a_1)a_2\right) + \beta_2\tau\left(X_{\sigma(1)}(a_0){\delta^a}_{\sigma(2)}(a_1)a_2\right) \\
&+\beta_3\tau\left({\delta^a}_{\sigma(1)}{\delta^b}_{\sigma(2)}(a_0)Y_b(a_1)a_2\right) + \beta_4\tau\left(Y_b(a_0){\delta^a}_{\sigma(1)}{\delta^b}_{\sigma(2)}(a_1)a_2\right) \\
&+\beta_5\tau\left({\delta^a}_{\sigma(1)}Y_b(a_0){\delta^b}_{\sigma(2)}(a_1)a_2\right) + \beta_6\tau\left({\delta^b}_{\sigma(1)}Y_b(a_0){\delta^a}_{\sigma(2)}(a_1)a_2\right) \\
&+\beta_7\tau\left({\delta^a}_{\sigma(1)}(a_0){\delta^b}_{\sigma(2)}Y_b(a_1)a_2\right) + \beta_8\tau\left({\delta^b}_{\sigma(1)}(a_0){\delta^a}_{\sigma(2)}Y_b(a_1)a_2\right) \\
&+\beta_9\tau\left(\Delta\left({\delta^a}_{\sigma(1)\sigma(2)}\right)(a_0\otimes a_1)a_2\right) - r\tau\left({\delta^a}_{\sigma(1)}(a_0){\delta^b}_{\sigma(2)}(a_1)Y_b(a_2)\right) \\
&-r\tau\left({\delta^b}_{\sigma(1)}(a_0){\delta^a}_{\sigma(2)}(a_1)Y_b(a_2)\right).
\end{aligned} \tag{3.48}$$

By the integration by parts property, $(t\psi)_1 = \psi_1$ if and only if

$$\begin{aligned}
\beta_1 - \beta_2 &= 0 \\
\beta_2 - \beta_3 + \beta_4 - \beta_5 + \beta_6 + \beta_7 - \beta_8 &= 0 \\
\beta_1 - 2\beta_2 + 2\beta_4 - 2\beta_5 - 2\beta_6 - 2\beta_9 &= 0 \\
\beta_2 + \beta_6 + \beta_7 + r &= 0 \\
\beta_2 - \beta_3 + \beta_5 + \beta_6 - \beta_8 &= 0 \\
\beta_5 + \beta_8 + r &= 0 \\
\beta_3 + \beta_5 + \beta_6 - \beta_7 &= 0 \\
\beta_3 - \beta_4 &= 0 \\
-\beta_2 + \beta_5 - \beta_6 - 2\beta_9 &= 0.
\end{aligned} \tag{3.49}$$

Solving the systems (3.45) and (3.49) we obtain (3.47). □

Finally we determine γ_k, $1 \le k \le 4$ such that $\psi \in C^2_{\mathcal{K}}(\mathcal{A}, V)$ is a Hochschild cocycle.

Lemma 3.11. *We have* $(b\psi)_0 = 0$ *if and only if*

$$\gamma_1 = \gamma_2 = r, \quad \gamma_3 = \gamma_4 = s. \tag{3.50}$$

Proof. By the Koszul coaction (3.41),

$$
\begin{aligned}
& b\psi(1\otimes a_0\otimes a_1\otimes a_2\otimes a_3) = \\
& \psi_0(1\otimes a_0a_1\otimes a_2\otimes a_3) - \psi_0(1\otimes a_0\otimes a_1a_2\otimes a_3) \\
& + \psi_0(1\otimes a_0\otimes a_1\otimes a_2a_3) - \psi_0(1\otimes a_3a_0\otimes a_1\otimes a_2) \\
& + \psi_1(R^a\otimes Y_a(a_3)a_0\otimes a_1\otimes a_2) - \frac{1}{2!}\psi_2(R^{ab}\otimes Y_bY_a(a_3)a_0\otimes a_1\otimes a_2).
\end{aligned}
$$

As a result, $(b\psi)_0 = 0$ if and only if

$$
\begin{aligned}
& \gamma_1\Big(-1\otimes {\delta^a}_{\sigma(1)}\otimes Y_a\otimes X_{\sigma(2)} + 1\otimes X_{\sigma(1)}\otimes {\delta^a}_{\sigma(2)}\otimes Y_a\Big) + \\
& \gamma_2\Big(1\otimes {\delta^a}_{\sigma(1)}\otimes X_{\sigma(2)}\otimes Y_a + 1\otimes {\delta^a}_{\sigma(1)}\otimes Y_a\otimes X_{\sigma(2)} \\
& \quad + 1\otimes {\delta^a}_{\sigma(1)}\otimes {\delta^b}_{\sigma(2)}Y_a\otimes Y_b + 1\otimes {\delta^a}_{\sigma(1)}\otimes {\delta^b}_{\sigma(2)}\otimes Y_bY_a\Big) + \\
& \gamma_3\Big(-1\otimes {\delta^a}_{\sigma(1)}{\delta^b}_{\sigma(2)}\otimes Y_b\otimes Y_a - 1\otimes Y_b\otimes {\delta^a}_{\sigma(1)}{\delta^b}_{\sigma(2)}\otimes Y_a \\
& \quad - 1\otimes {\delta^a}_{\sigma(1)}Y_b\otimes {\delta^b}_{\sigma(2)}\otimes Y_a - 1\otimes {\delta^a}_{\sigma(1)}\otimes {\delta^b}_{\sigma(2)}Y_b\otimes Y_a \\
& \quad - 1\otimes {\delta^b}_{\sigma(2)}Y_b\otimes {\delta^a}_{\sigma(1)}\otimes Y_a - 1\otimes {\delta^b}_{\sigma(2)}\otimes {\delta^a}_{\sigma(1)}Y_b\otimes Y_a\Big) \\
& - \gamma_4 1\otimes \Delta({\delta^a}_{\sigma(1)\sigma(2)})\otimes Y_a - r1\otimes {\delta^a}_{\sigma(1)}\otimes X_{\sigma(2)}\otimes Y_a \\
& + -r1\otimes X_{\sigma(1)}\otimes {\delta^a}_{\sigma(2)}\otimes Y_a + s1\otimes {\delta^a}_{\sigma(1)}{\delta^b}_{\sigma(2)}\otimes Y_b\otimes Y_a \\
& + s1\otimes Y_b\otimes {\delta^a}_{\sigma(1)}{\delta^b}_{\sigma(2)}\otimes Y_a + s1\otimes {\delta^a}_{\sigma(1)}Y_b\otimes {\delta^b}_{\sigma(2)}\otimes Y_a \\
& - s1\otimes {\delta^b}_{\sigma(1)}Y_b\otimes {\delta^a}_{\sigma(2)}\otimes Y_a + s1\otimes {\delta^a}_{\sigma(1)}\otimes {\delta^b}_{\sigma(2)}Y_b\otimes Y_a \\
& + (-r-s)1\otimes {\delta^b}_{\sigma(1)}\otimes {\delta^a}_{\sigma(2)}Y_b\otimes Y_a + (\frac{r}{2}+s)1\otimes \Delta\left({\delta^a}_{\sigma(1)\sigma(2)}\right)\otimes Y_a \\
& - r1\otimes {\delta^a}_{\sigma(1)}\otimes {\delta^b}_{\sigma(2)}\otimes Y_bY_a - \frac{r}{2}1\otimes\Delta\left({\delta^a}_{\sigma(1)\sigma(2)}\right)\otimes Y_a = 0.
\end{aligned}
\tag{3.51}
$$

Hence we obtain (3.50). □

Proposition 3.12. *The cochain* $\psi : V\otimes\mathcal{A}^{\otimes 3}\to\mathbb{C}$ *is a cyclic 2-cocycle if and only if* (3.44), (3.47), *and* (3.50) *are satisfied. The resulting cocycle is then a SAYD-twisted cyclic cocycle.*

Proof. We note that ψ is a Hochschild cocycle, *i.e.* $b\psi = (b\psi)_0 + (b\psi)_1 + (b\psi)_2 = 0$, if and only if $(b\psi)_t = 0$, $t = 0, 1, 2$. We see that $(b\psi)_2 = 0$ via Lemma 3.7, $(b\psi)_1 = 0$ via Lemma 3.9, $(b\psi)_0 = 0$ via Lemma 3.11.

On the other hand ψ is cyclic, *i.e.* $\tau\psi = \psi$, if and only if $(\tau\psi)_t = \psi_t$, $t = 0, 1, 2$. Indeed, for $t = 1$ Lemma 3.10, for $t = 2$ Lemma 3.8 yields the claims. As for $t = 0$ we have

$$t\psi(1 \otimes a_0 \otimes a_1 \otimes a_2) = \psi_0(1 \otimes a_2 \otimes a_0 \otimes a_1) - \psi_1(R^a \otimes Y_a(a_2) \otimes a_0 \otimes a_1) + \frac{1}{2!}\psi_2(R^{ab} \otimes Y_b Y_a(a_2) \otimes a_0 \otimes a_1).$$

Accordingly,

$$t\psi(1 \otimes a_0 \otimes a_1 \otimes a_2) = (3.52) + (3.53) + (3.54) + (3.55)$$

with

$$r\tau\Big(X_{\sigma(1)}(a_0)X_{\sigma(2)}(a_1)a_2\Big) + r\tau\Big({\delta^a}_{\sigma(1)}(a_0)X_{\sigma(2)}Y_a(a_1)a_2\Big) + s\tau\Big({\delta^a}_{\sigma(1)}\delta^b_{\sigma(2)}Y_b(a_0)Y_a(a_1)a_2\Big) + s\tau\Big({\delta^a}_{\sigma(1)\sigma(2)}(a_0)Y_a(a_1)a_2\Big), \tag{3.52}$$

$$r\tau\Big({\delta^a}_{\sigma(1)}(a_0)X_{\sigma(2)}(a_1)Y_a(a_2)\Big) + r\tau\Big(X_{\sigma(1)}(a_0){\delta^a}_{\sigma(2)}(a_1)Y_a(a_2)\Big) - s\tau\Big({\delta^a}_{\sigma(1)}\delta^b_{\sigma(2)}(a_0)Y_b(a_1)Y_a(a_2)\Big) - s\tau\Big(Y_b(a_0){\delta^a}_{\sigma(1)}\delta^b_{\sigma(2)}(a_1)Y_a(a_2)\Big) + (r+s)\tau\Big(\delta^b_{\sigma(1)}(a_0){\delta^a}_{\sigma(2)}Y_b(a_1)Y_a(a_2)\Big) - s\tau\Big({\delta^a}_{\sigma(1)}Y_b(a_0)\delta^b_{\sigma(2)}(a_1)Y_a(a_2)\Big) + s\tau\Big(\delta^b_{\sigma(1)}Y_b(a_0){\delta^a}_{\sigma(2)}(a_1)Y_a(a_2)) - s\tau\Big({\delta^a}_{\sigma(1)}(a_0)\delta^b_{\sigma(2)}Y_b(a_1)Y_a(a_2)\Big), \tag{3.53}$$

$$(-\frac{r}{2} - s)\tau\Big(\Delta\left({\delta^a}_{\sigma(1)\sigma(2)}\right)(a_0 \otimes a_1)Y_a(a_2)\Big), \tag{3.54}$$

and

$$\frac{r}{2!}\tau\Big({\delta^a}_{\sigma(1)}(a_0)\delta^b_{\sigma(2)}(a_1)Y_bY_a(a_2)\Big) + \frac{r}{2!}\tau\Big(\delta^b_{\sigma(1)}(a_0){\delta^a}_{\sigma(2)}(a_1)Y_bY_a(a_2)\Big). \tag{3.55}$$

Using once again the integration by parts property to put the above expressions into the standard form $\tau(a_0h^1(a_1)h^2(a_2))$, we obtain

$$t\psi(1 \otimes a_0 \otimes a_1 \otimes a_2) = \psi_0(1 \otimes a_0 \otimes a_1 \otimes a_2).$$

□

We can simplify this cocycle as follows.

Theorem 3.13. *The cochain* $\varphi = \varphi_0 + \varphi_1 + \varphi_2 \in C^2_{\mathcal{K}}(\mathcal{A}, V)$,

$$\varphi_2 = \chi^{\rm eq}_\tau \left(S_{ab} \otimes \mathbf{1} \otimes \delta^a{}_{\sigma(1)} \otimes \delta^b{}_{\sigma(2)} + S_{ab} \otimes \mathbf{1} \otimes \delta^b{}_{\sigma(1)} \otimes \delta^a{}_{\sigma(2)} \right) \tag{3.56}$$

$$\begin{aligned} \varphi_1 = \chi^{\rm eq}_\tau \Big(& - S_a \otimes \mathbf{1} \otimes \delta^a{}_{\sigma(1)} \otimes X_{\sigma(2)} - S_a \otimes \mathbf{1} \otimes X_{\sigma(1)} \otimes \delta^a{}_{\sigma(2)} \\ & - S_a \otimes \mathbf{1} \otimes \delta^b{}_{\sigma(1)} \otimes \delta^a{}_{\sigma(2)} Y_b + \frac{1}{2} S_a \otimes \mathbf{1} \otimes \Delta(\delta^a{}_{\sigma(1)\sigma(2)}) \Big) \end{aligned} \tag{3.57}$$

$$\varphi_0 = \chi^{\rm eq}_\tau \left(1^* \otimes \mathbf{1} \otimes X_{\sigma(1)} \otimes X_{\sigma(2)} + 1^* \otimes \mathbf{1} \otimes \delta^a{}_{\sigma(1)} \otimes X_{\sigma(2)} Y_a \right), \tag{3.58}$$

is cohomologous to ψ *which is defined in Proposition 3.12.*

Proof. As a result of Proposition 3.12 we can write $\psi = r\varphi + s\phi$ for a 2-cochain $\phi = \phi_0 + \phi_1 + \phi_2$ given by

$$\phi_2 = 0 \tag{3.59}$$

$$\begin{aligned} \phi_1 = \chi^{\rm eq}_\tau \Big(& S_a \otimes \mathbf{1} \otimes \delta^a{}_{\sigma(1)} \delta^b{}_{\sigma(2)} \otimes Y_b + S_a \otimes \mathbf{1} \otimes Y_b \otimes \delta^a{}_{\sigma(1)} \delta^b{}_{\sigma(2)} \\ & + S_a \otimes \mathbf{1} \otimes \delta^a{}_{\sigma(1)} Y_b \otimes \delta^b{}_{\sigma(2)} - S_a \otimes \mathbf{1} \otimes \delta^b{}_{\sigma(1)} Y_b \otimes \delta^a{}_{\sigma(2)} \\ & + S_a \otimes \mathbf{1} \otimes \delta^a{}_{\sigma(1)} \otimes \delta^b{}_{\sigma(2)} Y_b - S_a \otimes \mathbf{1} \otimes \delta^b{}_{\sigma(1)} \otimes \delta^a{}_{\sigma(2)} Y_b \\ & + S_a \otimes \mathbf{1} \otimes \Delta(\delta^a{}_{\sigma(1)\sigma(2)}) \Big) \end{aligned} \tag{3.60}$$

$$\phi_0 = \chi^{\rm eq}_\tau \left(1^* \otimes \mathbf{1} \otimes \delta^a{}_{\sigma(1)} \delta^b{}_{\sigma(2)} Y_b \otimes Y_a + 1^* \otimes \mathbf{1} \otimes \delta^a{}_{\sigma(1)\sigma(2)} \otimes Y_a \right). \tag{3.61}$$

We note that

$$\phi_1 = \chi^{\rm eq}_\tau \left(S_a \otimes \mathbf{1} \otimes \Delta \left(\delta^a{}_{\sigma(1)} \delta^b{}_{\sigma(2)} Y_b \right) + S_a \otimes \mathbf{1} \otimes \Delta(\delta^a{}_{\sigma(1)\sigma(2)}) \right). \tag{3.62}$$

It is then straightforward to check that the 1-cochain $\phi' = \phi'_0 + \phi'_1 + \phi'_2$ given by

$$\phi'_2 = 0 \tag{3.63}$$

$$\phi'_1 = \chi^{\rm eq}_\tau \left(S_a \otimes \mathbf{1} \otimes \delta^a{}_{\sigma(1)} \delta^b{}_{\sigma(2)} Y_b + S_a \otimes \mathbf{1} \otimes \delta^a{}_{\sigma(1)\sigma(2)} \right) \tag{3.64}$$

$$\phi'_0 = 0 \tag{3.65}$$

is an equivariant cyclic 1-cocycle, and that

$$b\phi' = \phi. \tag{3.66}$$

□

3.6 A SAYD-twisted cyclic cocycle in general codimension

Keeping the setting of Subsection 3.3, we prove in this subsection the existence of a nontrivial cyclic n-cocycle $\phi \in C^n_{\mathcal{K}}(\mathcal{A}, V)$.

We first construct for each $n \geq 0$ a (non-trivial) Hochschild n-cocycle $\varphi \in C^n_{\mathcal{K}}(\mathcal{H}, V)$, $\varphi = \chi_\tau^{\mathrm{eq}} \circ \mathcal{I}^{-1}(A_n)$, for some $A_n \in \mathcal{E}^n$.

Proposition 3.14. *If the n-cochain*

$$A_n = \sum_{k=0}^{n} (-1)^k S_{a_1 a_2 \dots a_k} \otimes \widetilde{h}^{a_1 a_2 \dots a_n}_{\sigma(1),\dots,\sigma(n)} \tag{3.67}$$

is a Hochschild n-cocycle in $\mathcal{H}$, then

$$A_{n+1} = \sum_{\ell=k-\mathrm{sgn}(i)}^{n} \; \sum_{j=0}^{1-\mathrm{sgn}(i)} \; \sum_{i=0}^{k} \sum_{k=0}^{n+1} (-1)^k S_{a_1 a_2 \dots a_k} \otimes \widetilde{h}^{a_1 \dots \widehat{a_i} \dots a_{\ell+\mathrm{sgn}(i)}}_{\sigma(1),\dots,\sigma(n+j)} \otimes F^{a_i}_{\sigma(n+1-j),\, a_{k+1} \dots a_{\ell+\mathrm{sgn}(i)}} \tag{3.68}$$

is a Hochschild $n+1$-cocycle in $\mathcal{H}_{n+1}$ for some

$$F^x_{y,\, z_1 \dots z_m} \in \mathcal{H}_{n+1}. \tag{3.69}$$

Proof. As a notational convention for (3.68), we mean

$$(-1)^k S_{a_1 a_2 \dots a_k} \otimes \widetilde{h}^{a_1 \dots a_\ell}_{\sigma(1),\dots,\sigma(n)} \otimes F_{\sigma(n+1),\, a_{k+1} \dots a_\ell} \tag{3.70}$$

when $i = j = 0$, and

$$(-1)^k S_{a_1 a_2 \dots a_k} \otimes \widetilde{h}^{a_1 \dots a_\ell}_{\sigma(1),\dots,\sigma(n+1)} \otimes F_{a_{k+1} \dots a_\ell} \tag{3.71}$$

when $i = 0$ and $j = 1$.

If (3.67) is a Hochschild n-cocycle, then

$$\widetilde{b}(A_n) = \sum_{k=0}^{n} (-1)^k S_{a_1 a_2 \dots a_k} \otimes b\left(\widetilde{h}^{a_1 \dots a_k}_{\sigma(1),\dots,\sigma(n)}\right) + (-1)^{n+1} \sum_{k=0}^{n} \sum_{t=0}^{k} \frac{(-1)^k}{t\,!} S_{a_1 \dots a_{k-t}} \otimes \widetilde{h}^{a_1 \dots a_k}_{\sigma(1),\dots,\sigma(n)} \otimes S(Y_{a_{k-t+1}} \dots Y_{a_k}) = 0. \tag{3.72}$$

As a result,

$$b\left(\widetilde{h}^{a_1\dots a_k}_{\sigma(1),\dots,\sigma(n)}\right) = (-1)^n\sum_{\ell=0}^{n-k}\frac{(-1)^\ell}{\ell\,!}\widetilde{h}^{a_1\dots a_{k+\ell}}_{\sigma(1),\dots,\sigma(n)}\otimes S(Y_{a_{k+1}}\dots Y_{a_{k+\ell}}) \tag{3.73}$$

for any $0\le k\le n$. Plugging this into $\widetilde{b}(A_{n+1})=0$, we obtain

$$\begin{aligned}&\sum_{\ell=k-\mathrm{sgn}(i)}^{n}\ \sum_{j=0}^{1-\mathrm{sgn}(i)}\ \sum_{i=0}^{k}\sum_{k=0}^{n+1}\\ &\qquad (-1)^k S_{a_1\dots a_k}\otimes\widetilde{h}^{a_1\dots\widehat{a_i}\dots a_{\ell+\mathrm{sgn}(i)}}_{\sigma(1),\dots,\sigma(n+j)}\otimes\Delta'\left(F^{a_i}_{\sigma(n+1-j),\,a_{k+1}\dots a_{\ell+\mathrm{sgn}(i)}}\right)\\ &=\sum_{\ell=k-\mathrm{sgn}(i)}^{n}\ \sum_{j=0}^{1-\mathrm{sgn}(i)}\ \sum_{i=0}^{k}\sum_{k=0}^{n+1}\sum_{t=0}^{k}\frac{(-1)^k}{t\,!}S_{a_1\dots a_{k-t}}\otimes\widetilde{h}^{a_1\dots\widehat{a_i}\dots a_{\ell+\mathrm{sgn}(i)}}_{\sigma(1),\dots,\sigma(n+j)}\otimes\\ &\Big(F^{a_i}_{\sigma(n+1-j),\,a_{k+1}\dots a_{\ell+\mathrm{sgn}(i)}}\otimes S(Y_{a_{k-t+1}}\dots Y_{a_k})\\ &\qquad\qquad +S(Y_{a_{k-t+1}}\dots Y_{a_k})\otimes F^{a_i}_{\sigma(n+1-j),\,a_{k+1}\dots a_{\ell+\mathrm{sgn}(i)}}\Big),\end{aligned} \tag{3.74}$$

where $\Delta'(u):=\Delta(u)-u\otimes 1-1\otimes u$. Equality (3.74) yields an expression for each $F^{\bullet}_{\bullet\dots\bullet}$ of a fixed number of indexes, in terms of the ones with smaller number of indexes. Using these equalities, we inductively find all $F^{\bullet}_{\bullet\dots\bullet}\in\mathcal{H}$. We in fact have, up to constant multiples,

$$F^{a_i}_{\sigma(n+1-j),\,a_{k+1}\dots a_{k+\ell}} = \begin{cases} X_{\sigma(n+1)}Y_{a_{k+1}}\dots Y_{a_{k+\ell}}, & i=0,\ j=0,\ \ell\neq 0,\\ X_{\sigma(n+1)}, & i=0,\ j=0,\ \ell=0\\ \delta^{a_i}_{\sigma(n+1)}, & i\neq 0,\ j=0,\ \ell=0,\\ \delta^{a_i}_{\sigma(n+1)}Y_{a_{k+1}}\dots Y_{a_{k+\ell}}, & i\neq 0,\ j=0,\ \ell\neq 0,\\ Y_{a_{k+1}}\dots Y_{a_{k+\ell}}, & i=0,\ j=1.\end{cases} \tag{3.75}$$

□

Proposition 3.15. *For any $n\ge 0$, $A_n\in\mathcal{E}^n$, that is, A_n is $\mathfrak{g}_0$-invariant.*

Proof. The claim follows from the observation

$$\begin{aligned}&\sum_{k=1}^{n}\delta^m_{\sigma(k)}\,B_{\sigma(1)}\otimes\dots\otimes B_{\sigma(k-1)}\otimes B_\ell\otimes B_{\sigma(k+1)}\otimes\dots\otimes B_{\sigma(n)}\\ &=\delta^m_\ell\,B_{\sigma(1)}\otimes\dots\otimes B_{\sigma(n)}\end{aligned} \tag{3.76}$$

in view of the fact that the action of $\mathfrak{g}_0$ on V^* is the adjoint action

$$S_{a_1 a_2 \dots a_r} \cdot Y_b = \sum_{k=1}^{r} C_{a_k b}^{c} \, S_{a_1 \dots a_{k-1} \, c \, a_{k+1} \dots a_r}. \tag{3.77}$$

□

Now recall the Connes periodicity exact sequence of the cyclic object $\mathcal{E} := \mathcal{E}(V^*, N, \mathcal{H})^{\mathrm{inv}}$,

$$\cdots \xrightarrow{S} HC^n(\mathcal{E}) \xrightarrow{I} HH^n(\mathcal{E}) \xrightarrow{B} HC^{n-1}(\mathcal{E}) \xrightarrow{S} HC^{n+1}(\mathcal{E}) \xrightarrow{I} \cdots \tag{3.78}$$

Having $[A_n] \in HH^n(\mathcal{E})$, its image $B([A_n]) \in HC^{n-1}(\mathcal{E})$ is a $b+B$-coboundary, therefore $[A_n] \in \ker(B) = \mathrm{Im}(I)$.

On the other hand, since we work over $\mathbb{C}$, a field that contains $\mathbb{Q}$, $HC^n(\mathcal{E}) \cong H^n_\lambda(\mathcal{E})$, where $H^*_\lambda(\mathcal{E})$ is the cohomology of the quotient $\mathcal{E}/\sim$ of $\mathcal{E}$ by the action of the cyclic group, [14, Lemma 9.6.10], [**?**, Sect. 2.4.2].

This indicates the existence of a class

$$[A'_n] \in HC^n(\mathcal{E}) \tag{3.79}$$

represented by a cyclic n-cocycle such that $I([A'_n]) = [A_n]$.

Let us illustrate. For $n = 0$

$$A_0 = 1^*, \tag{3.80}$$

which is a Hochschild 0-cocycle. It is also cyclic, therefore $A'_0 = A_0$.

For $n = 1$ we find

$$A_1 = 1^* \otimes \widetilde{h}_{\sigma(1)} - S_a \otimes \widetilde{h}^a_{\sigma(1)}, \qquad \sigma \in S_1, \ \widetilde{h}_{\sigma(1)} = X, \ \widetilde{h}^a_{\sigma(1)} = \delta_1, \tag{3.81}$$

which is a Hochschild 1-cocycle by

$$\begin{aligned} \widetilde{b}(A_1) &= 1^* \otimes b(\widetilde{h}_0) - S_a \otimes b(\widetilde{h}^a_1) - 1^* \otimes \widetilde{h}^a_1 \otimes S(Y_a) \\ &= -1^* \otimes \delta_1 \otimes Y - 0 + 1^* \otimes \delta_1 \otimes Y = 0. \end{aligned} \tag{3.82}$$

Since A_1 is cyclic, we have $A'_1 = A_1$. Note also that (3.81) is mapped, by $\chi^{\mathrm{eq}}_\tau \circ \mathcal{I}^{-1}$, to the 1-cocycle of Lemma 3.5.

For $n = 2$ we find

$$A_2 = 1^* \otimes \widetilde{h}_{\sigma(1)} \otimes X_{\sigma(2)} + 1^* \otimes \widetilde{h}^a_{\sigma(1)} \otimes X_{\sigma(2)} Y_a + 1^* \otimes \widetilde{h}_{\sigma(1)\sigma(2)} \otimes \frac{1}{2\,!} S(Y_a)$$
$$- S_a \otimes \widetilde{h}_{\sigma(1)} \otimes \delta^a_{\sigma(2)} - S_a \otimes \widetilde{h}^a_{\sigma(1)} \otimes X_{\sigma(2)} - S_a \otimes \widetilde{h}^b_{\sigma(1)} \otimes \delta^a_{\sigma(2)} Y_b$$
$$+ S_{ab} \otimes \widetilde{h}^a_{\sigma(1)} \otimes \delta^b_{\sigma(2)} + S_{ab} \otimes \widetilde{h}^b_{\sigma(1)} \otimes \delta^a_{\sigma(2)}, \tag{3.83}$$

where $\sigma \in S_2$, $\widetilde{h}_{\sigma(1)} = X_{\sigma(1)}$, and $\widetilde{h}^a_{\sigma(1)} = \delta^a_{\sigma(1)}$ (and hence $\widetilde{h}_{\sigma(1)\sigma(2)} = \delta^a_{\sigma(1)\sigma(2)}$). Then, $\widetilde{b}(A_2) = 0$ by

$$b(\widetilde{h}_{\sigma(1)}) + \widetilde{h}^a_{\sigma(1)} \otimes Y_a = 0, \qquad \text{and} \qquad b(\widetilde{h}^a_{\sigma(1)}) = 0 \tag{3.84}$$

which follows from (3.82).

Note that the Hochschild 2-cocycle (3.83) is not cyclic, yet, adding a Hochschild 2-coboundary

$$\widetilde{b}(-\frac{1}{2\,!} S_a \otimes \delta^a_{\sigma(1)\sigma(2)}) \tag{3.85}$$

we obtain the cyclic 2-cocycle

$$A_2' = A_2 + \frac{1}{2}\, S_a \otimes \Delta(\delta^a_{\sigma(1)\sigma(2)}) + \frac{1}{2}\, 1^* \otimes \delta^a_{\sigma(1)\sigma(2)} \otimes Y_a, \tag{3.86}$$

which is mapped, via $\chi_\tau^{\rm eq} \circ \mathcal{I}^{-1}$, to the 2-cochain of Theorem 3.13.

4 The characteristic map with coefficients

In this section we construct a new characteristic map from the truncated Weil complex of the Lie algebra $\mathfrak{g}_0$ to the cyclic complex of the algebra $\mathcal{A}$, and we illustrate it completely in codimensions $n = 1$ and $n = 2$. We observe that the resulting cocycles in codimension 1 match with those in [2, 3] by Connes-Moscovici .

Such a characteristic map is obtained by composing a series of maps

$$H(W(\mathfrak{g}_0, V)) \xrightarrow{\mathfrak{D}_P} H(C(\mathfrak{g}_0, V)) \xrightarrow{\cong} HC(\mathcal{K}, V) \xrightarrow{\chi_\varphi} HC(\mathcal{A}). \tag{4.1}$$

As it is shown in [13] the truncated Weil algebra is identical with $W(\mathfrak{g}_0, V)$. The Poincaré isomorphism $\mathfrak{D}_{\mathrm{P}}$ is defined in [13, Prop. 4.4]. The middle quasi-isomorphism is defined in [13, Thm. 6.2]. Finally the map χ_φ is given by the cup product, in the sense of [8, 12], with the SAYD-twisted cyclic cocycle φ defined in Proposition 3.6 for codimension 1, in Theorem 3.13 for codimension 2, and in (3.79) for the arbitrary codimension.

Let us recall the above mentioned cup product from [12]. Let C be a H-module coalgebra and A be an H-module algebra that are equipped with a mapping

$$C \otimes A \to A, \quad c \otimes a \mapsto c(a) \tag{4.2}$$

satisfying the conditions

$$(h \cdot c)(a) = h \cdot (c(a)), \quad c(ab) = c_{(1)}(a)c_{(2)}(b), \quad c(1) = \varepsilon(c)1. \tag{4.3}$$

Let also V be a SAYD module over a Hopf algebra H. One defines

$$\cup : C^p_H(C, V) \otimes C^q_H(A, V) \to C^{p+q}(A) \tag{4.4}$$

for any $\varphi \in C^q_H(A, V)$ and any $x = v \otimes_H c^0 \otimes \cdots \otimes c^p \in C^p_H(C, V)$,

$$\begin{aligned} &(x \cup \varphi)(a_0 \otimes \cdots \otimes a_{p+q}) := \\ &\sum_{\sigma \in \mathrm{Sh}(p,q)} (-1)^\sigma \partial_{\overline{\sigma}(p)} \dots \partial_{\overline{\sigma}(1)} \varphi(\langle \partial_{\overline{\sigma}(p+q)} \dots \partial_{\overline{\sigma}(p+1)} x, a_0 \otimes \cdots \otimes a_{p+q} \rangle), \end{aligned} \tag{4.5}$$

where $\langle x, a_0 \otimes \cdots \otimes a_n \rangle := v \otimes_H c^0(a^0) \otimes \cdots \otimes c^n(a^n)$, $\mathrm{Sh}(p,q)$ is the set of all (p,q)-shuffle permutations, and $\overline{\sigma}(n) = \sigma(n) - 1$.

We set $C = H := \mathcal{K}, \quad V = S(\mathfrak{g}_0^*)_{[2n]}, \quad A = \mathcal{A}$. Then H acts on C via multiplication, on A as (3.22) , and on V via the coadjoint action. The construction yields for any $\varphi \in C^n_{\mathcal{K}}(\mathcal{A}, V)$ a characteristic map

$$\chi_\varphi : C^\bullet_{\mathcal{K}}(\mathcal{K}, V) \to C^{\bullet+n}(\mathcal{A}). \tag{4.6}$$

4.1 The characteristic map in codimension 1

In this subsection we use the SAYD-twisted cyclic cocycle of Proposition 3.6 to illustrate (4.1) in codimension $n = 1$.

In order to verify that the new characteristic map is geometrically meaningful, we compare its image with the transverse fundamental class $TF := \chi_\tau(\mathrm{TF}) = \chi_\tau(X \otimes Y - Y \otimes X - \delta_1 Y \otimes Y) \in HC^2(\mathcal{A})$,

$$\begin{aligned} &TF(a_0 \otimes a_1 \otimes a_2) = \\ &= \tau(a_0 X(a_1) Y(a_2)) - \tau(a_0 Y(a_1) X(a_2)) - \tau(a_0 \delta_1 Y(a_1) Y(a_2)) \end{aligned} \tag{4.7}$$

and the Godbillon-Vey class $GV := \chi_\tau(\mathrm{GV}) = \chi_\tau(\delta_1) \in HC^1(\mathcal{A})$,

$$GV(a_0 \otimes a_1) = \tau(a_0 \delta_1(a_1)). \tag{4.8}$$

The next step is to find the representative cocycles of $H(W(\mathfrak{g}_0, V))$. Let $\{Y\}$ and $\{\theta\}$ be a dual pair of bases for $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$.

By the Vey basis [4], the cohomology of $W(\mathfrak{g}_0)_{[2]}$ is spanned by

$$\mathrm{TF} := 1 \in S(\mathfrak{g}_0^*)_{[2]}, \quad \mathrm{GV} := \theta \otimes R \in \mathfrak{g}_0^* \otimes S(\mathfrak{g}_0^*)_{[2]} \tag{4.9}$$

Applying the Poincaré duality [13, Prop. 4.4], we obtain

$$\mathfrak{D}_{\mathrm{P}}(1) = Y \otimes 1 \in \mathfrak{g}_0 \otimes V, \quad \mathfrak{D}_{\mathrm{P}}(\theta \otimes R) = R \in V. \tag{4.10}$$

Proposition 4.1. *The Hopf-cyclic cohomology $HC(\mathcal{K}, V)$ is generated by the classes*

$$[R] \in HC^0(\mathcal{K}, V), \tag{4.11}$$

$$[1 \otimes Y + \frac{1}{2} R \otimes Y^2] \in HC^1(\mathcal{K}, V). \tag{4.12}$$

Proof. It is straightforward to check that $R \in C^0(\mathcal{K}, V)$ and $1 \otimes Y + \frac{1}{2} R \otimes Y^2 \in C^1(\mathcal{K}, V)$ are cyclic cocycles. The claim, then follows from the observation

$$\mu(R) = R, \qquad \mu(1 \otimes Y + \frac{1}{2} R \otimes Y^2) = 1 \otimes Y, \tag{4.13}$$

for the quasi-isomorphism

$$\mu : C^\bullet(U(\mathfrak{g}), V) \longrightarrow C_\bullet(\mathfrak{g}, V), \tag{4.14}$$

which, for any Lie algebra $\mathfrak{g}$, is the left inverse of the anti-symmetrization map, see [3, 13]. □

Next we compute

$$\begin{aligned} &\chi_\varphi(\theta)(a_0 \otimes a_1) = (\varphi \cup (\theta \otimes 1))(a_0 \otimes a_1) \\ &= \varphi(\langle \partial_0(\theta \otimes 1), a_0 \otimes a_1 \rangle) = \varphi(\theta \otimes a_0 \otimes a_1) = -\tau(a_0 \delta_1(a_1)), \end{aligned}$$

and in the same way,

$$\chi_\varphi(1\otimes Y+\frac{1}{2}\theta\otimes Y^2)(a_0\otimes a_1\otimes a_2)=$$
$$\sum_{\sigma\in Sh(1,1)}(-1)^\sigma\partial_{\overline{\sigma}(1)}\varphi(\langle\partial_{\overline{\sigma}(2)}(1\otimes Y+\frac{1}{2}\theta\otimes Y^2),a_0\otimes a_1\otimes a_2\rangle)=$$
$$-\partial_0\varphi(\langle\partial_1(1\otimes Y+\frac{1}{2}\theta\otimes Y^2),a_0\otimes a_1\otimes a_2\rangle)+$$
$$\partial_1\varphi(\langle\partial_0(1\otimes Y+\frac{1}{2}\theta\otimes Y^2),a_0\otimes a_1\otimes a_2\rangle)$$

$$\begin{aligned}
&=\varphi(-1\otimes a_0a_1\otimes Y(a_2)-1\otimes a_0Y(a_1)\otimes a_2-\frac{1}{2}\theta\otimes a_0Y^2(a_1)\otimes a_2\\
&-\frac{1}{2}\theta\otimes a_0a_1\otimes Y^2(a_2)-\theta\otimes a_0Y(a_1)\otimes Y(a_2))+\varphi(1\otimes a_0\otimes a_1Y(a_2)\\
&+\frac{1}{2}\theta\otimes a_0\otimes a_1Y^2(a_2))\\
&=-\tau(a_0Y(a_1)X(a_2))+\tau(a_0X(a_1)Y(a_2))+\frac{1}{2}\tau(a_0Y^2(a_1)\delta_1(a_2))\\
&+\frac{1}{2}\tau(a_0\delta_1(a_1)Y^2(a_2))+\tau(a_0Y(a_1)\delta_1Y(a_2)).
\end{aligned}$$

As a result,

$$\chi_\varphi(1\otimes Y+\frac{1}{2}\theta\otimes Y^2)=\chi_\tau(\mathrm{TF}+\frac{1}{2}b(\delta_1Y^2)),\tag{4.15}$$

that is, we obtain the transverse fundamental class up to a coboundary. Similarly we obtain the Godbillon-Vey class

$$\chi_\varphi(\theta)=-\chi_\tau(\mathrm{GV}).\tag{4.16}$$

4.2 The characteristic map in codimension 2

In this subsection we exercise the machinery we developed in Subsection 3.3 for $n=2$. We note that there is no such computations in the literature that we know of.

Let us fix the following notation

$$c_1=\mathrm{Tr}=R_1^1+R_2^2\in S(\mathfrak{g}_0^*),\qquad c_2=R_2^1R_1^2\in S(\mathfrak{g}_0^*),$$
$$u_1=\theta_1^1+\theta_2^2,\qquad u_2=\theta_1^1\wedge\theta_2^1\wedge\theta_1^2,\qquad \omega=\theta_1^1\wedge\theta_2^1\wedge\theta_1^2\wedge\theta_2^2.$$

The Vey basis, [4], for $W(\mathfrak{g}_0)_{[4]}$ is then introduced by

$$\big\{1,\, c_1^2\otimes u_1,\, c_2\otimes u_1,\, c_2\otimes u_2,\, c_1^2\otimes\omega,\, c_2\otimes\omega\big\}. \tag{4.17}$$

Next, the Poincaré duality yields the 6 cocycles in the complex $C(\mathfrak{g}_0, V)$:

$$\begin{aligned}
&\mathfrak{D}_{\rm P}(1) = 1\otimes Y_1^1\wedge Y_1^2\wedge Y_2^1\wedge Y_2^2,\\
&\mathfrak{D}_{\rm P}(c_2\otimes u_1) = c_2\otimes\left(Y_1^2\wedge Y_2^1\wedge Y_2^2 - Y_1^1\wedge Y_1^2\wedge Y_2^1\right),\\
&\mathfrak{D}_{\rm P}(c_1^2\otimes u_1) = c_1^2\otimes\left(Y_1^2\wedge Y_2^1\wedge Y_2^2 - Y_1^1\wedge Y_1^2\wedge Y_2^1\right),\\
&\mathfrak{D}_{\rm P}(c_2\otimes u_2) = c_2\otimes Y_2^2, \quad \mathfrak{D}_{\rm P}(c_1^2\otimes\omega) = c_1^2,\\
&\mathfrak{D}_{\rm P}(c_2\otimes\omega) = c_2.
\end{aligned}$$

Let us label Y_i^j as $Y_1 := Y_1^1,\ \ Y_2 := Y_1^2,\ \ Y_3 := Y_2^1,\ \ Y_4 := Y_2^2$.

Proposition 4.2. *The Hopf-cyclic cohomology $HC(\mathcal{K}, V)$ is generated by the classes*

$$\begin{aligned}
&[\mathscr{T}\mathscr{F}] \in HC^4(\mathcal{K}, V)\\
&[\mathscr{G}\mathscr{V}] := \Big[\sum_{\sigma\in S_3}(-1)^\sigma c_1^2\otimes\Big(Y_{\sigma(2)}\otimes Y_{\sigma(3)}\otimes Y_{\sigma(4)}\\
&\qquad\qquad\qquad\qquad - Y_{\sigma(1)}\otimes Y_{\sigma(2)}\otimes Y_{\sigma(3)}\Big)\Big] \in HC^3(\mathcal{K}, V),\\
&[\mathscr{R}_1] := \Big[\sum_{\sigma\in S_3}(-1)^\sigma c_2\otimes\Big(Y_{\sigma(2)}\otimes Y_{\sigma(3)}\otimes Y_{\sigma(4)}\\
&\qquad\qquad\qquad\qquad - Y_{\sigma(1)}\otimes Y_{\sigma(2)}\otimes Y_{\sigma(3)}\Big)\Big] \in HC^3(\mathcal{K}, V),\\
&[\mathscr{R}_2] := [c_2\otimes Y_4] \in HC^1(\mathcal{K}, V),\\
&[\mathscr{R}_3] := [c_1^2] \in HC^0(\mathcal{K}, V),\\
&[\mathscr{R}_4] := [c_2] \in HC^0(\mathcal{K}, V).
\end{aligned} \tag{4.18}$$

Proof. It is straightforward to check that $\mathscr{R}_1, \ldots, \mathscr{R}_4$ and $\mathscr{G}\mathscr{V}$ are Hopf-cyclic cocycles and that

$$\begin{aligned}
&\mu(\mathscr{R}_1) = \mathfrak{D}_{\rm P}(c_2\otimes u_1), \quad \mu(\mathscr{R}_2) = \mathfrak{D}_{\rm P}(c_2\otimes u_2),\\
&\mu(\mathscr{R}_3) = \mathfrak{D}_{\rm P}(c_1^2\otimes\omega), \quad \mu(\mathscr{R}_4) = \mathfrak{D}_{\rm P}(c_2\otimes\omega),\\
&\mu(\mathscr{G}\mathscr{V}) = \mathfrak{D}_{\rm P}(c_1^2\otimes u_1).
\end{aligned}$$

On the other hand,

$$[\mathscr{T}\mathscr{F}] = \Big[\sum_{\sigma\in S_4}(-1)^\sigma 1\otimes Y_{\sigma(1)}\otimes Y_{\sigma(2)}\otimes Y_{\sigma(3)}\otimes Y_{\sigma(4)}\Big] \in E_1^{2,2}(\mathcal{K}, V) \tag{4.19}$$

is a cyclic cocycle in the E_1 level of the spectral sequence that corresponds to the natural filtration of V, [13, Thm. 6.2]. Hence

$$\mu(\mathscr{T}\mathscr{F}) = \mathfrak{D}_{\mathrm{P}}(1).$$

Therefore, the claim follows from [13, Thm. 6.2]. $\square$

In this paper we do not complete the fundamental cocycle as we know its counterpart as a cyclic cocycle over $\mathcal{A}$ by the following argument. Let us recall the characteristic map

$$\chi_\varphi : C^\bullet(\mathcal{K}, V) \longrightarrow C^{\bullet+2}(\mathcal{A}) \tag{4.20}$$

for the SAYD-twisted cyclic 2-cocycle defined by the Theorem 3.13. To this end, we first prove a generalization of [3, Prop. 18]. In view of [10], $\mathcal{H} := \mathcal{H}_n$ is realized as a bicrossed product Hopf algebra $\mathcal{U} \blacktriangleright\!\!\vartriangleleft \mathcal{F}^{\mathrm{cop}}$. Here $\mathcal{F}$ is the commutative algebra of regular functions on the group of diffeomorphisms which preserve the origin and with identity Jacobian at the origin, and $\mathcal{U} = U(g\ell_2^{\mathrm{affine}})$. The coaction involed in this bicrossed product realization is recalled below

$$\begin{aligned} &\nabla : \mathcal{U} \to \mathcal{F}^{\mathrm{cop}} \otimes \mathcal{U}, \\ &X_k \mapsto 1 \otimes X_k + \delta^i_{jk} \otimes Y^j_i, \qquad Y^j_i \mapsto 1 \otimes Y^j_i. \end{aligned} \tag{4.21}$$

In the following proposition, for any $1 \le j \le m := n^2 + n$,

$$\nabla^j(Z) = Z_{<-j>} \otimes \cdots \otimes Z_{<-1>} \otimes Z_{<0>} \otimes 1 \otimes \cdots \otimes 1 \in \mathcal{H}^{\otimes\, m+1}. \tag{4.22}$$

Proposition 4.3. *The $m := n^2 + n$-cochain*

$$\mathrm{TF} := (-1)^{(m-1)!} \sum_{\sigma \in S_m} (-1)^\sigma \nabla^m(Z^{\sigma(1)}) \cdots \nabla(Z^{\sigma(m)}) \in \mathcal{H}^{\otimes\, m+1} \tag{4.23}$$

is a cyclic m-cocycle whose class $[\mathrm{TF}] \in HC^m(\mathcal{H})$ corresponds, by the Connes-Moscovici characteristic map, to the transverse fundamental class $[TF] \in HC^m(\mathcal{A})$.

Proof. Let $a^i := f^i U^*_{\psi_i} \in \mathcal{A}$, where $0 \le i \le m$, $f^i \in C^\infty_c(F^+\mathbb{R}^2)$ and $\psi \in \Gamma$. Without loss of generality we assume that $\psi_m \ldots \psi_0 = \mathrm{Id}$. The cyclic cocycle $TF \in HC^m(\mathcal{A})$ is given by the m-cocycle

$$TF(a^0 \otimes \cdots \otimes a^m) = \int_{F^+\mathbb{R}^n} a^0 da^1 \cdots da^m, \qquad da^i = df^i U^*_{\psi_i}.$$

In order to prove the claim, we need to find $h^0, \ldots, h^m \in \mathcal{H}$ such that

$$TF(a^0 \otimes \cdots \otimes a^m) = \tau(h^0(a^0) \cdots h^m(a^m)). \tag{4.24}$$

Indeed,

$$\int_{F^+\mathbb{R}^n} a^0 da^1 \cdots da^m = \int_{F^+\mathbb{R}^n} f^0 {\psi_0}^*(df^1) \cdots ({\psi_0}^* \ldots {\psi_{m-1}}^*)(df^m) =$$
$$\int_{F^+\mathbb{R}^n} h^0(f^0){\psi_0}^*(h^1(f^1)) \cdots ({\psi_0}^* \ldots {\psi_{m-1}}^*)(h^m(f^m))\varpi =$$
$$\int_{F^+\mathbb{R}^n} (\mathrm{Id} \otimes {\psi_0}^* \otimes \cdots \otimes {\psi_0}^* \ldots {\psi_{m-1}}^*)(h^0 \otimes \cdots \otimes h^m)(f^0 \otimes \cdots \otimes f^m)\varpi, \tag{4.25}$$

where the volume form on the frame bundle is

$$\varpi = \bigwedge_{i=1}^{n} \theta^i \wedge \bigwedge_{1 \le i,j \le n} \omega^i_j \quad \text{(ordered lexicographically)} \tag{4.26}$$

In the above computation we use the notations

$$(h^0 \otimes \cdots \otimes h^m)(f^0 \otimes \cdots \otimes f^m) = h^0(f^0) \ldots h^m(f^m),$$

and similarly for any $g^0, \ldots, g^m \in C_c^\infty(F^+\mathbb{R}^n)$,

$$(\mathrm{Id} \otimes {\psi_0}^* \otimes \cdots \otimes {\psi_0}^* \ldots {\psi_{m-1}}^*)(g^0 \otimes \cdots \otimes g^m)$$
$$= g^0 {\psi_0}^*(g^1) \ldots {\psi_0}^* \ldots {\psi_{m-1}}^*(g^m).$$

Here $\psi^*(g)(x,y) = g(\psi(x), \psi'(x) \cdot y)$.

For any $f \in C_c^\infty(F^+\mathbb{R}^n)$ we have

$$df = \frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial y^j_i} dy^j_i = X_i(f)\theta^i + Y_i^j(f)\omega^i_j. \tag{4.27}$$

Therefore, for $\psi_0 \in \mathrm{Diff}(\mathbb{R}^n)$, and $f^0, f^1 \in C_c^\infty(F^+\mathbb{R}^n)$ we have

$$f^0{\psi_0}^*(df^1) = f^0{\psi_0}^*(X_i(f^1)){\psi_0}^*(\theta^i) + f^0{\psi_0}^*(Y_i^j(f^1)){\psi_0}^*(\omega^i_j)$$
$$= f^0{\psi_0}^*(X_i(f^1))\theta^i + f^0{\psi_0}^*(Y_i^j(f^1))(\gamma^i_{jk}(\psi_0)\theta^k + \omega^i_j)$$
$$= (\mathrm{Id} \otimes {\psi_0}^*)\big[(1 \otimes X_k + \delta^i_{jk} \otimes Y_i^j)(f^0 \otimes f^1)\theta^k + (1 \otimes Y_i^j)(f^0 \otimes f^1)\omega^i_j\big]$$
$$= (\mathrm{Id} \otimes {\psi_0}^*)((X_k)_{<-1>} \otimes (X_k)_{<0>})(f^0 \otimes f^1)\theta^k +$$
$$(\mathrm{Id} \otimes {\psi_0}^*)((Y_i^j)_{<-1>} \otimes (Y_i^j)_{<0>})(f^0 \otimes f^1)\omega^i_j.$$

On the second equality we used [3, (2.16)], and on the third equality we used (3.22). On the forth equality, the left coaction is (4.21).

On the other hand we have

$$
\begin{aligned}
&(-1)^{(m-1)!}\ f^0\psi_0{}^*(df^1)\psi_0{}^*\psi_1{}^*(df^2)\dots\psi_0{}^*\cdots\psi_{m-1}{}^*(df^m)\\
&=\psi_0{}^*\cdots\psi_{m-1}{}^*(df^m)\dots\psi_0{}^*\psi_1{}^*(df^2)\psi_0{}^*(df^1)f^0\\
&=\big(\psi_{m-1}\cdots\psi_0{}^*(X_i(f^m))\theta^i+\\
&\psi_{m-1}\cdots\psi_0{}^*(Y_i^j(f^m))(\gamma^i_{jk}(\psi_{m-1}\cdots\psi_1)\theta^k+\omega^i_j)\big)\\
&\dots\\
&\Big(\psi_1\psi_0{}^*(X_i(f^2))\theta^i+\psi_1\psi_0{}^*(Y_i^j(f^2))(\gamma^i_{jk}(\psi_2\psi_1)\theta^k+\omega^i_j)\Big)\cdot\\
&\Big(\psi_0{}^*(X_i(f^1))\theta^i+\psi_0{}^*(Y_i^j(f^1))(\gamma^i_{jk}(\psi_1)\theta^k+\omega^i_j)\Big)\cdot f^0
\end{aligned}
$$

$$
\begin{aligned}
&=(\mathrm{Id}\otimes\psi_0{}^*\otimes\cdots\otimes\psi_0{}^*\dots\psi_{m-1}{}^*)\Big\{\\
&\Big[((X_i)_{<-m>}\otimes\cdots\otimes(X_i)_{<0>})\theta^i+((Y_i^j)_{<-m>}\otimes\cdots\otimes(Y_i^j)_{<0>})\omega^i_j\Big]\cdot\\
&\dots\\
&\big[((X_i)_{<-2>}\otimes(X_i)_{<-1>}\otimes(X_i)_{<0>}\otimes 1\otimes\cdots\otimes 1)\theta^i+\\
&\qquad((Y_i^j)_{<-2>}\otimes(Y_i^j)_{<-1>}\otimes(Y_i^j)_{<0>}\otimes 1\otimes\cdots\otimes 1)\omega^i_j\big]\cdot\\
&\big[((X_i)_{<-1>}\otimes(X_i)_{<0>}\otimes 1\otimes\cdots\otimes 1)\theta^i+\\
&((Y_i^j)_{<-1>}\otimes(Y_i^j)_{<0>}\otimes 1\otimes\cdots\otimes 1)\omega^i_j\Big]\Big\}\big(f^0\otimes\cdots\otimes f^m\big)\,.
\end{aligned}
\tag{4.28}
$$

On the third equality, we used the cocycle identity [10, (1.16)] in order to obtain the expressions in $\mathcal{H}^{\otimes\, m+1}$ in the range of the coaction (4.21). We reversed the order of the multiplication in (4.28) in order to avoid obtaining elements in $\mathcal{H}^{\otimes\, m+1}$ involving $Y_i^j\delta^p_{qr}\in\mathcal{H}$ which do not belong the PBW basis of $\mathcal{H}$, [3, Prop. 3].

The coefficient of the volume form (4.26), which is an element $H\in\mathcal{H}^{\otimes\, m+1}$, can now be expressed by carrying out the multiplication in (4.28). Let $(Z^1,\dots,Z^m)=(X_1,\dots,X_n,Y_1^1,\dots,Y_n^n)$, where the right hand side is ordered lexicographically. Then

$$
H=\sum_{\sigma\in S_m}(-1)^\sigma\nabla^m(Z^{\sigma(1)})\cdots\nabla(Z^{\sigma(m)}). \tag{4.29}
$$

Finally, as a result of (4.25), (4.28) and (4.29), we have the element

$$\mathrm{TF} := (-1)^{(m-1)!} \sum_{\sigma \in S_m} (-1)^{\sigma} \nabla^m (Z^{\sigma(1)}) \cdots \nabla(Z^{\sigma(m)}) \in \mathcal{H}^{\otimes m+1} \tag{4.30}$$

such that $\chi_\tau(\mathrm{TF}) = TF \in C^m(\mathcal{A})$. □

Let us illustrate the proposition for $n = 1$. We have $(Z^1, Z^2) = (X, Y)$, $m = 1^2 + 1 = 2$ and

$$\begin{aligned}
\mathrm{TF} &= (-1)^{1!} \sum_{\sigma \in S_2} (-1)^{\sigma} \nabla^2 (Z^{\sigma(1)}) \nabla(Z^{\sigma(2)}) \\
&= -\Big((X_{<-2>} \otimes X_{<-1>} \otimes X_{<0>})(Y_{<-1>} \otimes Y_{<0>} \otimes 1) - \\
&\quad (Y_{<-2>} \otimes Y_{<-1>} \otimes Y_{<0>})(X_{<-1>} \otimes X_{<0>} \otimes 1) \Big) \\
&= -1 \otimes Y \otimes X - 1 \otimes \delta_1 Y \otimes Y - \delta_1 \otimes Y \otimes Y + 1 \otimes X \otimes Y + \delta_1 \otimes Y \otimes Y \\
&= 1 \otimes X \otimes Y - 1 \otimes Y \otimes X - 1 \otimes \delta_1 Y \otimes Y.
\end{aligned}$$

Next we recall the isomorphism

$$\begin{aligned}
&\Psi_{\bowtie} : C^{\bullet}(\mathcal{H}, \mathbb{C}_\delta) \to D^{\bullet}(\mathcal{U}, \mathcal{F}, \mathbb{C}_\delta) \\
&\Psi_{\bowtie}(u^1 \bowtie f^1 \otimes \ldots \otimes u^n \bowtie f^n) = \\
&{u^1}_{<-n>} f^1 \otimes \cdots \otimes {u^1}_{<-1>} \ldots {u^n}_{<-1>} f^n \otimes {u^1}_{<0>} \otimes \cdots \otimes {u^n}_{<0>}
\end{aligned} \tag{4.31}$$

defined in [10] that identifies the Hopf-cyclic complex $C^{\bullet}(\mathcal{H}, \mathbb{C}_\delta)$ of the Hopf algebra $\mathcal{H} = \mathcal{U} \bowtie \mathcal{F}$ with the diagonal subcomplex

$$D^{\bullet}(\mathcal{U}, \mathcal{F}, \mathbb{C}_\delta) := \mathbb{C}_\delta \otimes \mathcal{F}^{\otimes \bullet} \otimes \mathcal{U}^{\otimes \bullet}. \tag{4.32}$$

Remark 4.4. The transverse fundamental class $[\mathrm{TF}] \in HC^{n^2+n}(\mathcal{H}_n, \mathbb{C}_\delta)$ defined in (4.23) corresponds to the class

$$[1 \otimes X_1 \wedge \cdots \wedge X_n \wedge Y_1^1 \wedge \cdots \wedge Y_n^n], \tag{4.33}$$

in the total complex $C^{\bullet,\bullet}(\mathfrak{g}, \mathcal{F}, \mathbb{C}_\delta)$ [10, (3.37)], by the composition of (4.31) and (4.14).

On the next move, we introduce the commutative diagram

$$\begin{array}{ccc}
C^j_{\mathcal{K}}(\mathcal{K}, V) & \xrightarrow{\overline{\chi}_\varphi} & C^{j+k}(\mathcal{H}, \mathbb{C}_\delta) \\
& \searrow^{\chi_\varphi} & \downarrow T^{\natural} \\
& & C^{j+k}(\mathcal{A})
\end{array} \tag{4.34}$$

induced by (a decomposition of) the cup product (4.5) via a cyclic cocycle $\varphi \in C^k_{\mathcal{K}}(\mathcal{A}, V)$ in the image of (3.30). Here $T^\natural : \mathcal{H}^{\otimes j} \to C^j(\mathcal{A})$ is the isomorphism defined in [3, (3.12)].

We are now ready to prove our claim. On the following proposition, $\varphi \in C^2_{\mathcal{K}}(\mathcal{A}, V)$ is the cyclic cocycle defined in Theorem 3.13.

Proposition 4.5. *The cyclic cohomology class* $[\mathscr{T}\mathscr{F}] \in HC^4(\mathcal{K}, V)$ *is mapped by* $\chi_\varphi : HC^4(\mathcal{K}, V) \to HC^6(\mathcal{A})$ *to the transverse characteristic class* $[TF] \in HC^6(\mathcal{A})$.

Proof. By the diagram (4.34) we understand that it is enough to observe $[\overline{\chi}_\varphi(\mathscr{T}\mathscr{F})] = [\mathrm{TF}] \in HC^6(\mathcal{H})$. This, in turn, follows from

$$\mu \circ \psi_{\bowtie}([\overline{\chi}_\varphi(\mathscr{T}\mathscr{F})]) = \mu \circ \psi_{\bowtie}([\mathrm{TF}]) = [1 \otimes X_1 \wedge X_2 \wedge Y_1^1 \wedge \cdots \wedge Y_2^2], \tag{4.35}$$

thanks to the large kernel of (4.14). Hence the result follows since $\mu \circ \psi_{\bowtie}$ is an isomorphism on the level of cohomologies. □

In the following we present the images of the cyclic cocycles $\mathscr{G}\mathscr{V}$, $\mathscr{R}_1$, $\mathscr{R}_2$, $\mathscr{R}_3$ and $\mathscr{R}_4$ under the characteristic map $\chi_\varphi : C^\bullet(\mathcal{K}, V) \to C^{\bullet+2}(\mathcal{A})$. We do not display the detailed account of the computation as it is lengthy and straightforward.

$$\chi_\varphi(\mathscr{G}\mathscr{V})(a_0\otimes\cdots\otimes a_5)=\sum_{k=1}^{3}\sum_{1\le i,j\le 2}\sum_{\sigma,\gamma,\eta\in S_2}2\cdot(-1)^\sigma(-1)^\gamma(-1)^{k-1}$$

$$\Big\{-\tau\left(a_0\delta^i_{i\eta(1)}(a_1)\delta^j_{j\eta(2)}(a_2)Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_3)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_4)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_5)\right)$$
$$+\tau\left(a_0\delta^i_{i\eta(1)}(a_1)Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_2)\delta^j_{j\eta(2)}(a_3)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_4)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_5)\right)$$
$$-\tau\left(a_0\delta^i_{i\eta(1)}(a_1)Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_2)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_3)\delta^j_{j\eta(2)}(a_4)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_5)\right)$$
$$+\tau\left(a_0\delta^i_{i\eta(1)}(a_1)Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_2)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_3)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_4)\delta^j_{j\eta(2)}(a_5)\right)$$
$$-\tau\left(a_0Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_1)\delta^i_{i\eta(1)}(a_2)\delta^j_{j\eta(2)}(a_3)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_4)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_5)\right)$$
$$+\tau\left(a_0Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_1)\delta^i_{i\eta(1)}(a_2)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_3)\delta^j_{j\eta(2)}(a_4)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_5)\right)$$
$$-\tau\left(a_0Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_1)\delta^i_{i\eta(1)}(a_2)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_3)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_4)\delta^j_{j\eta(2)}(a_5)\right)$$
$$-\tau\left(a_0Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_1)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_2)\delta^i_{i\eta(1)}(a_3)\delta^j_{j\eta(2)}(a_4)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_5)\right)$$
$$+\tau\left(a_0Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_1)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_2)\delta^i_{i\eta(1)}(a_3)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_4)\delta^j_{j\eta(2)}(a_5)\right)$$
$$-\tau\left(a_0Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_1)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_2)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_3)\delta^i_{i\eta(1)}(a_4)\delta^j_{j\eta(2)}(a_5)\right)\Big\}.\tag{4.36}$$

$$\chi_\varphi(\mathscr{R}_2)(a_0\otimes\cdots\otimes a_3)$$
$$=\sum_{\sigma\in S_2}(-1)^\sigma\Big\{-\tau(a_0\delta^1_{2\sigma(1)}(a_1)\delta^2_{1\sigma(2)}(a_2)Y^2_2(a_3))$$
$$-\tau(a_0\delta^2_{1\sigma(1)}(a_1)\delta^1_{2\sigma(2)}(a_2)Y^2_2(a_3))+\tau(a_0\delta^1_{2\sigma(1)}(a_1)Y^2_2(a_2)\delta^2_{1\sigma(2)}(a_3))$$
$$+\tau(a_0\delta^2_{1\sigma(1)}(a_1)Y^2_2(a_2)\delta^1_{2\sigma(2)}(a_3))-\tau(a_0Y^2_2(a_1)\delta^1_{2\sigma(1)}(a_2)\delta^2_{1\sigma(2)}(a_3))$$
$$-\tau(a_0Y^2_2(a_1)\delta^2_{1\sigma(1)}(a_2)\delta^1_{2\sigma(2)}(a_3))\Big\}.\tag{4.37}$$

$$\chi_\varphi(\mathscr{R}_3)(a_0\otimes a_1\otimes a_2)=\sum_{1\le i,j\le 2}\sum_{\sigma\in S_2}2\cdot(-1)^\sigma\tau(a_0\delta^i_{i\sigma(1)}(a_1)\delta^j_{j\sigma(2)}(a_2)).\tag{4.38}$$

$$
\begin{aligned}
&\chi_\varphi(\mathscr{R}_1)(a_0\otimes\cdots\otimes a_5)=\sum_{k=1}^{3}\sum_{\sigma,\gamma,\eta\in S_2}(-1)^{\sigma}(-1)^{\gamma}(-1)^{k-1}\Big\{\\
&-\tau\left(a_0\delta^1_{2\eta(1)}(a_1)\delta^2_{1\eta(2)}(a_2)Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_3)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_4)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_5)\right)\\
&-\tau\left(a_0\delta^2_{1\eta(1)}(a_1)\delta^1_{2\eta(2)}(a_2)Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_3)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_4)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_5)\right)\\
&+\tau\left(a_0\delta^1_{2\eta(1)}(a_1)Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_2)\delta^2_{1\eta(2)}(a_3)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_4)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_5)\right)\\
&+\tau\left(a_0\delta^2_{1\eta(1)}(a_1)Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_2)\delta^1_{2\eta(2)}(a_3)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_4)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_5)\right)\\
&-\tau\left(a_0\delta^1_{2\eta(1)}(a_1)Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_2)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_3)\delta^2_{1\eta(2)}(a_4)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_5)\right)\\
&-\tau\left(a_0\delta^2_{1\eta(1)}(a_1)Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_2)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_3)\delta^1_{2\eta(2)}(a_4)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_5)\right)\\
&+\tau\left(a_0\delta^1_{2\eta(1)}(a_1)Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_2)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_3)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_4)\delta^2_{1\eta(2)}(a_5)\right)\\
&+\tau\left(a_0\delta^2_{1\eta(1)}(a_1)Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_2)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_3)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_4)\delta^1_{2\eta(2)}(a_5)\right)\\
&-\tau\left(a_0Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_1)\delta^1_{2\eta(1)}(a_2)\delta^2_{1\eta(2)}(a_3)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_4)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_5)\right)\\
&-\tau\left(a_0Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_1)\delta^2_{1\eta(1)}(a_2)\delta^1_{2\eta(2)}(a_3)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_4)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_5)\right)\\
&+\tau\left(a_0Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_1)\delta^1_{2\eta(1)}(a_2)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_3)\delta^2_{1\eta(2)}(a_4)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_5)\right)\\
&+\tau\left(a_0Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_1)\delta^2_{1\eta(1)}(a_2)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_3)\delta^1_{2\eta(2)}(a_4)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_5)\right)\\
&-\tau\left(a_0Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_1)\delta^1_{2\eta(1)}(a_2)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_3)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_4)\delta^2_{1\eta(2)}(a_5)\right)\\
&-\tau\left(a_0Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_1)\delta^2_{1\eta(1)}(a_2)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_3)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_4)\delta^1_{2\eta(2)}(a_5)\right)\\
&-\tau\left(a_0Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_1)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_2)\delta^1_{2\eta(1)}(a_3)\delta^2_{1\eta(2)}(a_4)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_5)\right)\\
&-\tau\left(a_0Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_1)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_2)\delta^2_{1\eta(1)}(a_3)\delta^1_{2\eta(2)}(a_4)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_5)\right)\\
&+\tau\left(a_0Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_1)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_2)\delta^1_{2\eta(1)}(a_3)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_4)\delta^2_{1\eta(2)}(a_5)\right)\\
&+\tau\left(a_0Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_1)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_2)\delta^2_{1\eta(1)}(a_3)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_4)\delta^1_{2\eta(2)}(a_5)\right)\\
&-\tau\left(a_0Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_1)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_2)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_3)\delta^1_{2\eta(1)}(a_4)\delta^2_{1\eta(2)}(a_5)\right)\\
&-\tau\left(a_0Y^{\mu^k(\sigma(2))}_{\mu^k(\sigma(1))}(a_1)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(1))}(a_2)Y^{\mu^k(\sigma(1))}_{\mu^k(\gamma(2))}(a_3)\delta^2_{1\eta(1)}(a_4)\delta^1_{2\eta(2)}(a_5)\right)\Big\}.
\end{aligned}
\tag{4.39}
$$

Finally,

$$\chi_\varphi(\mathscr{R}_4)(a_0\otimes a_1\otimes a_2) = \varphi(c_2\otimes a_0\otimes a_1\otimes a_2)$$
$$= \sum_{\sigma\in S_2}(-1)^\sigma\Big\{\tau(a_0\delta^1_{2\sigma(1)}(a_1)\delta^2_{1\sigma(2)}(a_2)) + \tau(a_0\delta^2_{1\sigma(1)}(a_1)\delta^1_{2\sigma(2)}(a_2))\Big\}. \tag{4.40}$$

Remark 4.6. One knows that the characteristic map $\chi_\tau : C^\bullet(\mathcal{H},\mathbb{C}_\delta)\to C^\bullet(\mathcal{A})$ is injective [2]. Since $\chi_\varphi(\mathscr{T}\mathscr{F})$, $\chi_\varphi(\mathscr{G}\mathscr{V})$, $\chi_\varphi(\mathscr{R}_1)$, $\chi_\varphi(\mathscr{R}_2)$, $\chi_\varphi(\mathscr{R}_3)$, and $\chi_\varphi(\mathscr{R}_4)$, are all in the range of χ_τ, as a byproduct of our study in this paper, one calculates cyclic cocycles representing a basis for $HP^\bullet(\mathcal{H},\mathbb{C}_\delta)$.

Remark 4.7. Proposition 4.5 holds in arbitrary codimension too, that is, the class $[\mathscr{T}\mathscr{F}]\in HC^{n^2}(\mathcal{K},V)$ is mapped by $\chi_\varphi : HC^{n^2}(\mathcal{K},V)\to HC^{n^2+n}(\mathcal{A})$ to the transverse characteristic class $[TF]\in HC^{n^2+n}(\mathcal{A})$.

Indeed, for A'_n of (3.79),

$$\varphi\in C^n_{\mathcal{K}}(\mathcal{A},V) = \chi^{\rm eq}_\tau\circ\mathcal{I}^{-1}(A'_n), \tag{4.41}$$

and by the kernel of (4.14) the only terms of $[\mathscr{T}\mathscr{F}]\in HC^{n^2}(\mathcal{K},V)$ that survive after the composition of (4.31) and (4.14), are those paired with the image of $1^*\otimes X_{\sigma(1)}\otimes\cdots\otimes X_{\sigma(n)}$ of $A'_n\in\mathcal{E}^n$. This yields,

$$\mu\circ\psi_{\blacktriangleright\!\!\blacktriangleleft}([\overline{\chi}_\varphi(\mathscr{T}\mathscr{F})]) = \mu\circ\psi_{\blacktriangleright\!\!\blacktriangleleft}([\mathrm{TF}]) = [1\otimes X_1\wedge\cdots\wedge X_n\wedge Y^1_1\wedge\cdots\wedge Y^n_n]. \tag{4.42}$$

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