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# MODULATIONAL INSTABILITY OF THREE DIMENSIONAL WAVES IN A PLASMA WITH VORTEX ELECTRON DISTRIBUTION

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ABSTRACT. In the present work, employing the three dimensional equations of a plasma composed of a cold electron fluid, hot electrons obeying a trapped /vortex-like distribution, and stationary ions, we study the amplitude modulation of an electron-acoustic waves by use of the conventional reductive perturbation method. Employing the field equations with fractional power type of nonlinearity, we obtained the three dimensional form of the modified nonlinear Schrödinger equation as the evolution equation of the same order of nonlinearity. The modulational instability of the homogeneous harmonic solution is investigated and the criteria for the instability is discussed as a function of the obliqueness angle.The numerical calculations show that the critical value of the wave number of the envelop wave increases with the wave number $k$ of the carrier wave and the obliqueness angle $\gamma$.

Keywords:Modified nonlinear Schrödinger equation, modulational instability, solitary waves.

AMS Subject Classification: 83-02, 35Q51

## 1. Introduction

The concept of electro-acoustic mode had been conceived by Fried and Gould[1] during the numerical solutions of the linear electrostatic dispersion equation in an unmagnetized, homogeneous plasma. Besides the well-known Langmuir and ion-acoustic waves, they noticed the existence of a heavily damped acoustic-like solution of dispersion equation. It was later shown that in the presence of two distinct groups (cold and hot) of electrons and immobile ions, one indeed obtains a weakly damped electron-acoustic mode ( Watanabe and Taniuti[2] ), the properties of which significantly differ from those of the Langmuir waves.

To study the properties of electron-acoustic solitary wave structure Dubouloz et al.[3] considered a one-dimensional, unmagnetized collisionless plasma consisting of cold electrons, Maxwellian hot electrons and stationary ions. However, in practice, the hot electrons may not follow a Maxwellian distribution, due to the formation of phase space holes caused by the trapping of hot electrons in a wave potential. Accordingly, in most space plasma the hot electrons follow the trapped/vortex- like distribution (Abbasi et al.[4], Schamel[5,6]). Therefore, in the present work, we shall consider a plasma model consisting of a cold electron fluid, hot electrons obeying a non-isothermal (trapped/vortex-like) distribution, and stationary ions.

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The propagation of small-but-finite amplitude waves in a one-dimensional ion-acoustic model had been studied by several researchers (see, for instance, Washimi and Taniuti[7] and one dimensional electron-acoustic model by Schamel[5, 6], Mamun and Shukla[8] by use of the classical reductive perturbation method (Taniuti [9] and Demiray [10, 11] by use of the modified PLK (Poincaré, Lighthill-Kuo) method, wherein the contribution of higher order terms is also investigated.

Due to its central importance to the theory of quantum mechanics, the nonlinear equations of Schrödinger type are of great interest. They arise in many nonlinear problems such as water waves [12-14], waves in plasmas [15-19], nonlinear waves in fluid-filled elastic or viscoelastic tubes [20-22] and other nonlinear waves of similar nature. In all these works the nonlinear equations with integer power had been taken into consideration. However, when the nonlinearity is of the type of fractional order of certain field quantities much more careful analysis of the problem has to be made.

In the present work, employing the three dimensional equations of a plasma composed of a cold electron fluid, hot electrons obeying a trapped /vortex-like distribution, and stationary ions, we study the amplitude modulation of three dimensional nonlinear electron-acoustic waves through the use the reductive perturbation method. Due to nature of the problem, the field equations involve fractional nonlinearity (3/2) and they cause serious difficulties in amplitude modulation problems. Following the procedure outlined in Demiray [19], we expanded this nonlinear term of fractional order into Fourier cosine series of the phase function and obtained the three dimensional form of the modified nonlinear Schrödinger equation.The modulational instability of the homogeneous harmonic solution is investigated and the criteria for the instability is discussed as a function of the obliqueness angle. The numerical calculations show that the critical value of the wave number of the envelop wave increases with the wave number $k$ of the carrier wave and the obliqueness angle $\gamma$.

## 2. Govening Equations

We consider a three-dimensional collisionless plasma consisting of stationary ions, a cold electron fluid and hot electrons obeying a trapped/vortex-like distribution. The dynamics of electron-acoustic waves is governed by the following equations:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) + \frac{\partial}{\partial y}(nv) + \frac{\partial}{\partial z}(nw) = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - \alpha\frac{\partial \phi}{\partial x} = 0, \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} - \alpha\frac{\partial \phi}{\partial y} = 0, \tag{3}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - \alpha\frac{\partial \phi}{\partial z} = 0, \tag{4}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{n}{\alpha} + n_h - (1 + \frac{1}{\alpha}), \tag{5}$$

where $n$ is the normalized cold electron number density, $n_h$ is the normalized hot electron number density, $(u, v, w)$ are the cold electron fluid velocity components in cartesian coordinates, $\phi$ is the electrostatic potential and the coefficient $\alpha$ is defined by $\alpha = n_{h0}/n_0$ ; where $n_{h0}$ and $n_0$ are the equilibrium values of the hot and cold electron number densities, respectively. The hot electron number density $n_h$, for $\beta < 0$ and small $\phi$ ( $\phi << 1$, can be expressed by Schamel[5]

$$n_h = 1 + \phi - \frac{4(1-\beta)}{3\sqrt{\pi}}\phi^{3/2} + \frac{1}{2}\phi^2 + ... \tag{6}$$

## 3. Modulation of nonlinear waves

In this section we shall study the amplitude modulation of three dimensional nonlinear waves propagating in such a plasma medium. For that purpose we introduce the following slow variables

$$\xi=\epsilon(x-\lambda t),\quad \eta=\epsilon y,\quad \zeta=\epsilon z,\quad \tau=\epsilon^2 t, \tag{7}$$

where $\epsilon$ is a small parameter measuring the band width of superposed waves and $\lambda$ is an unknown constant to be determined from the solution. The field variables are assumed to be functions of the fast $(x,t)$ as well as the slow variables $(\xi,\eta,\zeta,\tau)$. Then, the following differential relations hold true

$$\frac{\partial}{\partial t}\to\frac{\partial}{\partial t}-\epsilon\lambda\frac{\partial}{\partial\xi}+\epsilon^2\frac{\partial}{\partial\tau},\quad \frac{\partial}{\partial x}\to\frac{\partial}{\partial x}+\epsilon\frac{\partial}{\partial\xi},\quad \frac{\partial}{\partial y}\to\epsilon\frac{\partial}{\partial\eta},\quad \frac{\partial}{\partial z}\to\epsilon\frac{\partial}{\partial\zeta}. \tag{8}$$

For our future purposes it is convenient to assume that the field quantities can be expanded into a power series of $\epsilon$ in the following form

$$\begin{aligned} n&=1+\epsilon^4(n_1+\epsilon n_2+\epsilon^2 n_3+...),\\ u&=\epsilon^4(u_1+\epsilon u_2+\epsilon^2 n_3+...),\\ v&=\epsilon^5(v_1+\epsilon v_2+\epsilon^2 v_3+...),\\ w&=\epsilon^5(w_1+\epsilon w_2+\epsilon^2 w_3+...)\\ \phi&=\epsilon^4(\phi_1+\epsilon\phi_2+\epsilon^2\phi_3+...). \end{aligned} \tag{9}$$

Introducing the expansions (8) and (9) into the field equations (1)-(5) and setting the coefficients of like powers of $\epsilon$ equal to zero we obtain the following sets of differential equations:

$O(\epsilon^4)$ equations:

$$\frac{\partial n_1}{\partial t}+\frac{\partial u_1}{\partial x}=0,\quad \frac{\partial u_1}{\partial t}-\alpha\frac{\partial\phi_1}{\partial x}=0,\quad \frac{\partial^2\phi_1}{\partial x^2}-\frac{n_1}{\alpha}-\phi_1=0. \tag{10}$$

$O(\epsilon^5)$ equations:

$$\begin{gathered}\frac{\partial n_2}{\partial t}+\frac{\partial u_2}{\partial x}-\lambda\frac{\partial n_1}{\partial\xi}+\frac{\partial u_1}{\partial\xi}=0,\quad \frac{\partial u_2}{\partial t}-\alpha\frac{\partial\phi_2}{\partial x}-\lambda\frac{\partial u_1}{\partial\xi}-\alpha\frac{\partial\phi_1}{\partial\xi}=0,\\ \frac{\partial v_1}{\partial t}-\alpha\frac{\partial\phi_1}{\partial\eta}=0,\quad \frac{\partial w_1}{\partial t}-\alpha\frac{\partial\phi_1}{\partial\zeta}=0,\quad \frac{\partial^2\phi_2}{\partial x^2}+2\frac{\partial^2\phi_1}{\partial x\partial\xi}-\frac{n_2}{\alpha}-\phi_2=0.\end{gathered} \tag{11}$$

$O(\epsilon^6)$ equations:

$$\begin{gathered}\frac{\partial n_3}{\partial t}+\frac{\partial u_3}{\partial x}-\lambda\frac{\partial n_2}{\partial\xi}+\frac{\partial u_2}{\partial\xi}+\frac{\partial n_1}{\partial\tau}+\frac{\partial v_1}{\partial\eta}+\frac{\partial w_1}{\partial\zeta}=0,\\ \frac{\partial u_3}{\partial t}-\alpha\frac{\partial\phi_3}{\partial x}-\lambda\frac{\partial u_2}{\partial\xi}-\alpha\frac{\partial\phi_2}{\partial\xi}+\frac{\partial u_1}{\partial\tau}=0,\\ \frac{\partial v_2}{\partial t}-\alpha\frac{\partial\phi_2}{\partial\eta}-\lambda\frac{\partial v_1}{\partial\xi}=0,\quad \frac{\partial w_2}{\partial t}-\alpha\frac{\partial\phi_2}{\partial\zeta}-\lambda\frac{\partial w_1}{\partial\xi}=0,\\ \frac{\partial^2\phi_3}{\partial x^2}+2\frac{\partial^2\phi_2}{\partial x\partial\xi}+\frac{\partial^2\phi_1}{\partial\xi^2}+\frac{\partial^2\phi_1}{\partial\eta^2}+\frac{\partial^2\phi_1}{\partial\zeta^2}-\frac{n_3}{\alpha}-\phi_3+\frac{4(1-\beta)}{3\sqrt{\pi}}\phi_1^{3/2}=0.\end{gathered} \tag{12}$$

3.1. **Solution of the field equations.** For the solution of the set (10) we write

$$(n_1, u_1, \phi_1) = Re[(N_1, U_1, \varphi_1)\exp(i\theta)], \quad \theta = \omega t - kx, \tag{13}$$

where $N_1, U_1, \varphi_1$ are some complex functions depending on the slow variables $\xi, \eta, \zeta$ and $\tau$ and Re[.] stands for the real part of the corresponding complex variable. Introducing (13) into (10) we obtain

$$U_1 = -\alpha\frac{k}{\omega}\varphi_1, \quad N_1 = -\alpha\frac{k^2}{\omega^2}\varphi_1, \quad \phi_1 = \varphi_1(\xi, \eta, \zeta, \tau), \tag{14}$$

provided that the following dispersion relation holds true

$$\omega^2 = \frac{k^2}{1+k^2}, \tag{15}$$

where $\varphi_1(\xi, \eta, \zeta, \tau)$ is an unknown complex function whose governing equation will be obtained later.

Introducing the solution (14) into the differential equation (11) we get

$$\frac{\partial n_2}{\partial t} + \frac{\partial u_2}{\partial x} + \alpha[\frac{k}{\omega}(\lambda\frac{k}{\omega} - 1)\frac{\partial \varphi_1}{\partial \xi}\exp(i\theta) + c.c.] = 0,$$
$$\frac{\partial u_2}{\partial t} - \alpha\frac{\partial \phi_2}{\partial x} + \alpha[(\lambda\frac{k}{\omega} - 1)\frac{\partial \varphi_1}{\partial \xi}\exp(i\theta) + c.c.] = 0,$$
$$\frac{\partial v_1}{\partial t} - \alpha[\frac{\partial \varphi_1}{\partial \eta}\exp(i\theta) + c.c.] = 0, \quad \frac{\partial w_1}{\partial t} - \alpha[\frac{\partial \varphi_1}{\partial \zeta}\exp(i\theta) + c.c.] = 0,$$
$$\frac{\partial^2 \phi_2}{\partial x^2} - \frac{n_2}{\alpha} - \phi_2 - [2ik\frac{\partial \varphi_1}{\partial \xi}\exp(i\theta) + c.c.] = 0, \tag{16}$$

where *c.c.* stands for the complex conjugate of the corresponding quantity. The form of equation(16) suggests us to seek the following type of solution

$$(u_2, n_2, \phi_2, v_1, w_1) = Re[(U_2, N_2, \varphi_2, V_1, W_1)\exp(i\theta)], \tag{17}$$

where $U_2, N_2, \varphi_2, V_1, W_1$ are some unknown complex functions of the slow variables $(\xi, \eta, \zeta, \tau)$. Introducing (17) into the equation (16) we have

$$i\omega N_2 - ikU_2 + \alpha\frac{k}{\omega}(\lambda\frac{k}{\omega} - 1)\frac{\partial \varphi_1}{\partial \xi} = 0, \quad i\omega U_2 + i\alpha k\varphi_2 + \alpha(\lambda\frac{k}{\omega} - 1)\frac{\partial \varphi_1}{\partial \xi} = 0,$$
$$N_2 = -\alpha(1+k^2)\varphi_2 - 2i\alpha k\frac{\partial \varphi_1}{\partial \xi}, \quad V_1 = -i\frac{\alpha}{\omega}\frac{\partial \varphi_1}{\partial \eta}, \quad W_1 = -i\frac{\alpha}{\omega}\frac{\partial \varphi_1}{\partial \zeta}. \tag{18}$$

Eliminating $U_2, N_2, \varphi_2$ between the equation (18) and utilizing the dispersion relation we obtain

$$[k\omega + \frac{k}{\omega}(\lambda\frac{k}{\omega} - 1)]\frac{\partial \varphi_1}{\partial \xi} = 0. \tag{19}$$

In order to have a function $\varphi_1$ depending on the variable $\xi$, the coefficient of $\partial\varphi_1/\partial\xi$ must vanish

$$k\omega + \frac{k}{\omega}(\lambda\frac{k}{\omega} - 1) = 0, \quad \text{or} \quad \lambda = \frac{\omega^3}{k^3}, \tag{20}$$

where $\lambda = d\omega/dk$ is the group velocity. Hence, from the solution of the equations (18) we obtain

$$U_2 = -\alpha\frac{k}{\omega}\varphi_2 + i\frac{\alpha}{\omega}(\lambda\frac{k}{\omega} - 1)\frac{\partial \varphi_1}{\partial \xi}, \quad N_2 = -\alpha\frac{k^2}{\omega^2}\varphi_2 - 2i\alpha k\frac{\partial \varphi_1}{\partial \xi},$$
$$V_1 = -i\frac{\alpha}{\omega}\frac{\partial \varphi_1}{\partial \eta}, \quad W_1 = -i\frac{\alpha}{\omega}\frac{\partial \varphi_1}{\partial \zeta}, \tag{21}$$

where $\phi_2 = \varphi_2(\xi, \eta, \zeta, \tau)$ is another unknown function whose governing equation will be obtained from the higher order expansion.

Introducing the solutions given in (14) and (21) into the equation (12) we obtain

$$\frac{\partial n_3}{\partial t}+\frac{\partial u_3}{\partial x}+\alpha\Big\{\frac{k}{\omega}(\lambda\frac{k}{\omega}-1)\frac{\partial \varphi_2}{\partial \xi}+i[2\lambda k+\frac{1}{\omega}(\lambda\frac{k}{\omega}-1)]\frac{\partial^2\varphi_1}{\partial\xi^2}$$
$$-\frac{k^2}{\omega^2}\frac{\partial\varphi_1}{\partial\tau}-\frac{i}{\omega}\frac{\partial^2\varphi_1}{\partial\eta^2}-\frac{i}{\omega}\frac{\partial^2\varphi_1}{\partial\zeta^2}\Big\}\exp(i\theta)+c.c.=0,$$
$$\frac{\partial u_3}{\partial t}-\alpha\frac{\partial\phi_3}{\partial x}+\alpha\Big\{(\lambda\frac{k}{\omega}-1)\frac{\partial\varphi_2}{\partial\xi}-i\frac{\lambda}{\omega}(\lambda\frac{k}{\omega}-1)\frac{\partial^2\varphi_1}{\partial\xi^2}-\frac{k}{\omega}\frac{\partial\varphi_1}{\partial\tau}\Big\}\exp(i\theta)+c.c.=0,$$
$$\frac{\partial v_2}{\partial t}+\alpha[i\frac{\lambda}{\omega}\frac{\partial^2\varphi_1}{\partial\xi\partial\eta}-\frac{\partial\varphi_2}{\partial\eta}]\exp(i\theta)+c.c.=0,$$
$$\frac{\partial w_2}{\partial t}+\alpha[i\frac{\lambda}{\omega}\frac{\partial^2\varphi_1}{\partial\xi\partial\zeta}-\frac{\partial\varphi_2}{\partial\zeta}]\exp(i\theta)+c.c.=0,$$
$$\frac{\partial^2\phi_3}{\partial x^2}-\frac{n_3}{\alpha}-\phi_3+\frac{4(1-\beta)}{3\sqrt{\pi}}\phi_1^{3/2}$$
$$+[-2ik\frac{\partial\varphi_2}{\partial\xi}+\frac{\partial^2\varphi_1}{\partial\xi^2}+\frac{\partial^2\varphi_1}{\partial\eta^2}+\frac{\partial^2\varphi_1}{\partial\zeta^2}]\exp(i\theta)+c.c.=0. \tag{22}$$

For our future purposes we need only the equations related to the coefficients of $\exp(i\theta)$ terms. In order to obtain such an equation we have to examine the term $(\phi_1)^{3/2}$, appearing in the last expression of equation (22). Denoting the modulus of $\varphi_1$ by $|\varphi_1|$ and the argument of it by $s$, $\phi_1$ can be expressed as

$$\phi_1^{3/2}=\Big\{|\varphi_1|[\frac{e^{i(s+\theta)}+e^{-i(s+\theta)}}{2}]\Big\}^{3/2}=|\varphi_1|^{3/2}|\cos^{3/2}(s+\theta). \tag{23}$$

In order to ensure that $(\phi_1)^{3/2}$ remains real we must restrict the variation of $(s+\theta)$ as $|(s+\theta)|\le\pi/2$. Since $\cos(s+\theta)$ is a periodic function in $(s+\theta)$ we can expand the expression of $(\phi_1)^{3/2}$ into a Fourier cosine series of the following form

$$\phi_1^{3/2}=\sum_{n=0}^{\infty}a_n\cos n(s+\theta), \tag{24}$$

where the coefficient $a_n$ is defined by

$$a_n=\frac{4}{\pi}|\varphi_1|^{3/2}d_n,\quad d_n=\int_0^{\pi/2}\cos^{3/2}(s+\theta)\cos n(s+\theta)d(s+\theta). \tag{25}$$

For our future purposes we need only the coefficient of $a_1$ and the corresponding term of the Fourier series, which reads

$$\frac{4}{\pi}|\varphi_1|^{3/2}d_1\cos(s+\theta)=\frac{2}{\pi}|\varphi_1|^{3/2}\Big\{\exp[i(s+\theta)]+\exp[-i(s+\theta)]\Big\}. \tag{26}$$

Noting $\varphi_1=|\varphi_1|\exp(is)$, the equation (26) can be written in the following form

$$\frac{4}{\pi}|\varphi_1|^{3/2}d_1\cos(s+\theta)=\frac{2d_1}{\pi}|\varphi_1|^{1/2}[\varphi_1\exp(i\theta)+c.c.],$$
$$d_1=\int_0^{\pi/2}(\cos x)^{5/2}dx. \tag{27}$$

Inserting (27) into the equation (22), the equations related to the $\exp(i\theta)$ terms take the following form

$$i\omega N_3^{(1)}-ikU_3^{(1)}+\alpha\Big\{\frac{k}{\omega}(\lambda\frac{k}{\omega}-1)\frac{\partial\varphi_2}{\partial\xi}$$
$$+i[2\lambda k+\frac{1}{\omega}(\lambda\frac{k}{\omega}-1)]\frac{\partial^2\varphi_1}{\partial\xi^2}-i\frac{1}{\omega}(\frac{\partial^2\varphi_1}{\partial\eta^2}+\frac{\partial^2\varphi_1}{\partial\zeta^2})-\frac{k^2}{\omega^2}\frac{\partial\varphi_1}{\partial\tau}\Big\}=0,$$

$$i\omega U_3^{(1)} + i\alpha k\phi_3^{(1)} + \alpha\Big\{(\lambda\frac{k}{\omega} - 1)\frac{\partial\varphi_2}{\partial\xi} - i\frac{\lambda}{\omega}(\lambda\frac{k}{\omega} - 1)\frac{\partial^2\varphi_1}{\partial\xi^2} - \frac{k}{\omega}\frac{\partial\varphi_1}{\partial\tau}\Big\} = 0,$$

$$N_3^{(1)} = \alpha\Big\{ - (k^2+1)\phi_3^{(1)} - 2ik\frac{\partial\varphi_2}{\partial\xi} + \frac{\partial^2\varphi_1}{\partial\xi^2}$$

$$+\frac{\partial^2\varphi_1}{\partial\eta^2} + \frac{\partial^2\varphi_1}{\partial\zeta^2} + \frac{8(1-\beta)d_1}{3\pi\sqrt{\pi}}|\varphi_1|^{1/2}\varphi_1\Big\},$$

$$i\omega V_2 + \alpha[i\frac{\lambda}{\omega}\frac{\partial^2\varphi_1}{\partial\xi\partial\eta} - \frac{\partial\varphi_2}{\partial\eta}] = 0, \quad i\omega W_2 + \alpha[i\frac{\lambda}{\omega}\frac{\partial^2\varphi_1}{\partial\xi\partial\zeta} - \frac{\partial\varphi_2}{\partial\zeta}] = 0, \tag{28}$$

where for $v_2, w_2, n_3, u_3$ and $\phi_3$ we have sought solutions of the following form

$$(v_2, w_2) = Re[(V_2, W_2)\exp(i\theta)],$$

$$(n_3, u_3, \phi_3) = Re[\sum_{n=1}^{\infty}(N_3^{(n)}, U_3^{(n)}, \phi_3^{(n)})\exp(in\theta)]. \tag{29}$$

Eliminating $N_3^{(1)}, U_3^{(1)}, \phi_3^{(1)}$ between the equations (28) we obtain the following evolution equation

$$i\frac{\partial\varphi_1}{\partial\tau} + \mu_1\frac{\partial^2\varphi_1}{\partial\xi^2} + \mu_2|\varphi_1|^{1/2}\varphi_1 - \mu_3(\frac{\partial^2\varphi_1}{\partial\eta^2} + \frac{\partial^2\varphi_1}{\partial\zeta^2}) = 0, \tag{30}$$

where the coefficients $\mu_1, \mu_2$ and $\mu_3$ are defined by

$$\mu_1 = \frac{3\lambda k}{2(1+k^2)}, \quad \mu_2 = \frac{4\lambda k d_1(1-\beta)}{3(\pi)^{3/2}}, \quad \mu_3 = \frac{\lambda}{2k}. \tag{31}$$

Carrying out the integral for $d_1$, given in (27) numerically, the result is found to be $d_1 \approx 0.72$ .

3.2. **Modulational instability.** In this sub-section we shall study modulational instability of the plasma wave described by the 3D NLS equation (30). The NLS equation (30) has the trivial homogeneous solution

$$\varphi_1 = \varphi_1^0\exp(i\mu_2|\varphi_1^0|^{1/2}\tau), \tag{32}$$

where $\varphi_1^0$ is a real constant representing the amplitude of the carrier wave. Now, we investigate the evolution of the small modulation $\delta\varphi_1$ defined by

$$\varphi_1 = [\varphi_1^0 + \delta\varphi_1(\xi, \eta, \zeta, \tau)]\exp(i\mu_2|\varphi_1^0|^{1/2}\tau). \tag{33}$$

Substituting (33) into (30) and collecting the first order terms in $\delta\varphi_1$ we have

$$i\frac{\partial}{\partial\tau}(\delta\varphi_1) + \mu_1\frac{\partial^2}{\partial\xi^2}(\delta\varphi_1) + \frac{\mu_2}{4}|\varphi_1^0|^{1/2}(\delta\varphi_1 + \delta\varphi_1{}^*)$$

$$-\mu_3[\frac{\partial^2}{\partial\eta^2}(\delta\varphi_1) + \frac{\partial^2}{\partial\zeta^2}(\delta\varphi_1)] = 0, \tag{34}$$

where $\delta\varphi_1{}^*$ is the complex conjugate of $\delta\varphi_1$. Letting $\delta\varphi_1 = U + iV$ and $(U, V) = (U_0, V_0)\exp[i(\Omega\tau - K_1\xi - K_2\eta - K_3\zeta)] + c.c.$ in equation (34) and separating the real and imaginary parts, we obtain the following coupled equations

$$[\frac{\mu_2|\varphi_1^0|^{1/2}}{2} + \mu_3(K_2^2 + K_3^2) - \mu_1 K_1^2]U_0 - i\Omega V_0 = 0,$$

$$i\omega U_0 + [\mu_3(K_2^2 + K_3^2) - \mu_1 K_1^2]V_0 = 0, \tag{35}$$

where $\Omega$ is the modulational frequency, $K_1, K_2, K_3$ are the components of the propagation direction vector and $K = \sqrt{K_1^2 + K_2^2 + K_3^2}$ is the modulational wave number. The following dispersion equation may be obtained from the equation (35)

$$\Omega^2 = [\mu_1 K_1^2 - \mu_3(K_2^2 + K_3^2)]^2 \Big\{1 - \frac{\mu_2 |\varphi_1^0|^{1/2}}{2[\mu_1 K_1^2 - \mu_3(K_2^2 + K_3^2)}\Big\}. \tag{36}$$

The modulational instability will occur when the following condition is satisfied

$$\frac{\mu_2 |\varphi_1^0|^{1/2}}{[\mu_1 K_1^2 - \mu_3(K_2^2 + K_3^2)]} > 2. \tag{37}$$

Denoting the tangent of the angle between the propagation direction and the $\xi$ axis by $\gamma$ we can write $\gamma = \sqrt{K_2^2 + K_3^2}/K_1$ and the dispersion equation (36) becomes

$$\Omega^2 = K^4 [\frac{\mu_1 - \mu_3 \gamma^2}{1 + \gamma^2}]^2 [1 - \frac{|\varphi_1^0|^{1/2}(1+\gamma^2)}{2K^2} \frac{\mu_2/\mu_1}{1 - \gamma^2 \mu_3/\mu_1}]. \tag{38}$$

The dispersion relation for one dimensional case may be obtained from (36), by setting $K_2 = K_3 = 0$, as

$$\Omega^2 = (\mu_1 K^2)^2 [1 - \frac{|\varphi_1^0|^{1/2} \mu_2}{2\mu_1 K^2}]. \tag{39}$$

This is exactly the same with that of found in [18].

For the present problem $\mu_2/\mu_1 > 0$ for all values of the wave number $k$ , therefore, the modulational instability occurs when

$$K^2 < K_c^2 = \frac{|\varphi_1^0|^{1/2} \mu_2}{2\mu_1}. \tag{40}$$

For the general case the modulational instability may occur when the conditions $0 < \gamma < \gamma_c$ and $K < K_c$, $\gamma_c$ and $K_c$ are defined by

$$\gamma_c = (\mu_1/\mu_3)^{1/2}, \quad K_c = \Big[\frac{|\varphi_1^0|^{1/2}(1+\gamma^2)\mu_2/\mu_1}{2(1 - \frac{\gamma^2}{\gamma_c^2})}\Big]^{1/2}. \tag{41}$$

In this case the instability growth rate $\Gamma$ may be defined by

$$\Gamma = \mu_1 K^2 \frac{(1 - \frac{\gamma^2}{\gamma_c^2})}{(1+\gamma^2)} \Big[\frac{|\varphi_1^0|^{1/2}(1+\gamma^2)\mu_2/\mu_1}{2K^2(1 - \frac{\gamma^2}{\gamma_c^2})} - 1\Big]^{1/2}. \tag{42}$$

For fixed wave number $k$, the growth rate $\Gamma$ is a function of $\gamma$ and $K$ . The maximum value of the growth rate occurs at $K = K_c/\sqrt{2}$ and the result is found to be

$$\Gamma_{max} = \frac{\mu_2 |\varphi_1^{(0)}|^{1/2}}{4}. \tag{43}$$

It is seen that the maximum growth rate is independent of the obliqueness angle $\gamma$ and the wave number $K$ ; it simply is a function of the wave number $k$ .

3.3. **Numerical results and discussion.** For the illustration of the analytical results of the modulational instability problem we shall choose $\beta = -0.5$ and $\varphi_1^0 = 1$. Then, the expressions of the critical obliqueness angle $\gamma_c$, the critical wave number $K_c$ and the maximum growth rate $\Gamma_{max}$ take the following form

$$\gamma_c = \frac{\sqrt{3}k}{\sqrt{1+k^2}}, \quad K_c = 0.294 \frac{\sqrt{(1+\gamma^2)(1+k^2)}}{\sqrt{(1 - \frac{(1+k^2)}{3k^2}\gamma^2)}},$$

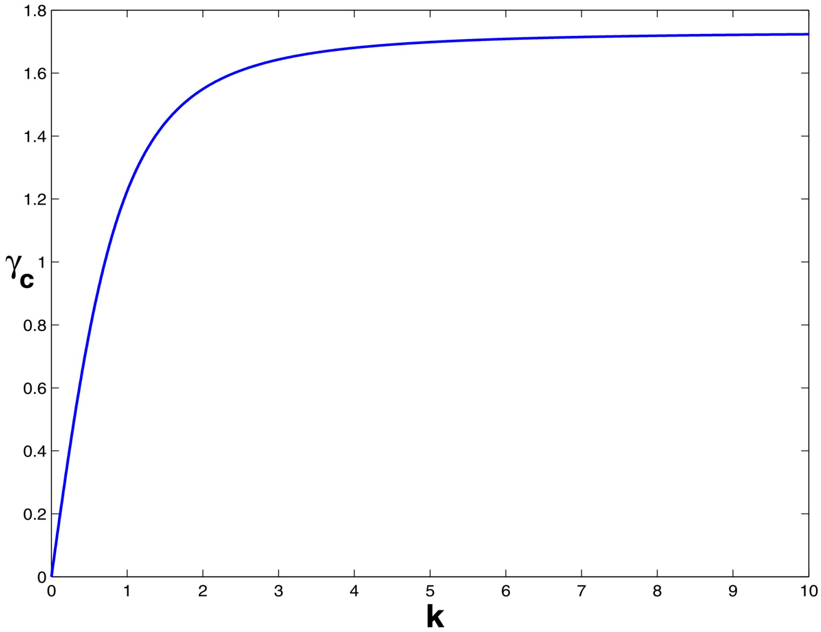


FIGURE 1. The variation of the critical obliqueness angle $\gamma_c$ with the wave number $k$.

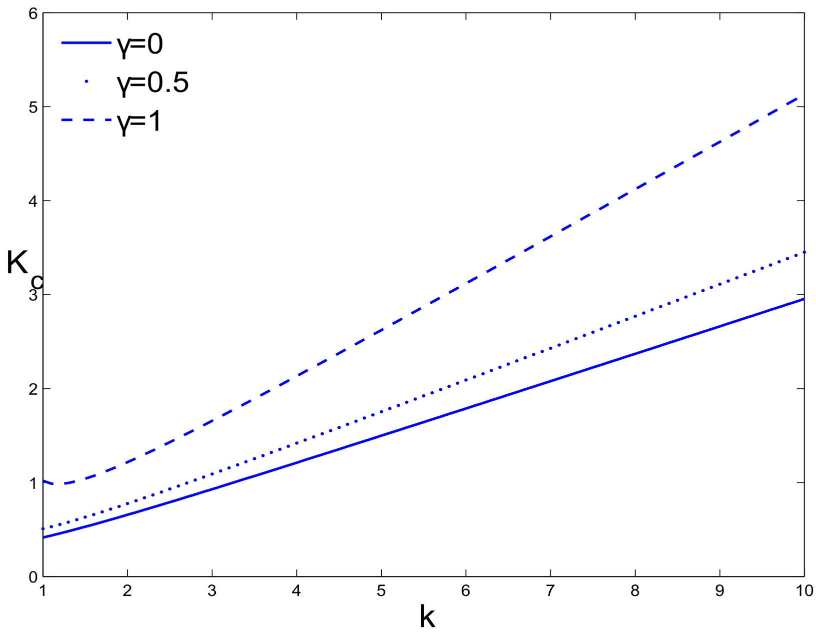


FIGURE 2. The variations of $K_c$ with the wave number $k$ for various obliqueness angle $\gamma$.

$$\Gamma_{max} = 0.0647\frac{k}{(1+k^2)^{3/2}}. \tag{44}$$

The variations of the critical obliqueness angle $\gamma_c$ with the wave number $k$ is depicted on Figure 1. As is seen, $\gamma_c$ increases with increasing wave number. In other words, the possibility of modulational instability increases with increasing wave number.

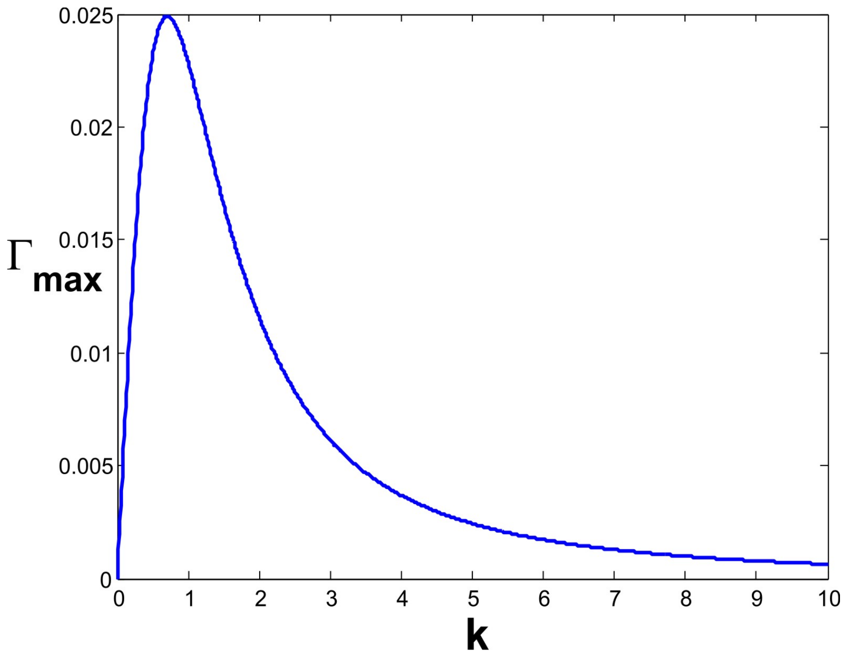


FIGURE 3. The variation of maximum growth rate $\Gamma_{max}$ with the wave number $k$ .

The variations of $K_c$ with wave number $k$ for various values of the obliqueness angle $\gamma$ are shown on Figure 2. The critical wave number $K_c$ increases with the wave number $(k)$ and the obliqueness angle $(\gamma)$. The result reveals that the most stable wave is the one dimensional wave $\gamma = 0$. As the angle $\gamma$ increases the region of stability also increases.

The variation of maximum growth rate $\Gamma_{max}$ with wave number is illustrated in Figure 3. As pointed out before, the maximum value of the growth rate is independent of the obliqueness angle $\gamma$ and it simply depends on the wave number $k$. It reaches its maximum value at about $k_c = 1/\sqrt{2}$ and in the region $0 < k < k_c$, $\Gamma_{max}$ increases with $k$ whereas in the region $k > k_c$ it decreases with the wave number.

## 4. CONCLUSION

In the present work, employing the three dimensional equations of a plasma composed of a cold electron fluid, hot electrons obeying a trapped /vortex-like distribution, and stationary ions, we studied the amplitude modulation of a three dimensional electron-acoustic waves. Due to the fractional nonlinearity (3/2) of the field equations, they cause serious difficulties in studying modulation problems. To surmount this difficulty, we expanded this nonlinear term of fractional order into Fourier cosine series of the phase function and obtained the three dimensional form of the modified nonlinear Schrödinger equation.The modulational instability of the homogeneous harmonic solution is investigated and the criteria for the instability is discussed as a function of the obliqueness angle and the wave number.

The numerical results show that the maximum obliqueness angle $\gamma_c$ for the modulational instability increases with the wave number $k$ . Another parameter $K_c$ for the modulational instability increases with increasing wave number $k$ and the obliqueness angle $\gamma$ . As a final remark we should say that, the maximum growth rate $\Gamma_{max}$ is independent of the obliqueness angle $\gamma$; it simply depends on the wave number $k$. $\Gamma_{max}$ increases with $k$ up to $k_c = 1/\sqrt{2}$ and, then start to decrease.

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