

Harmonic Mappings Related to Janowski Convex Functions of Complex Order b

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Abstract

Let S_H be the class of all sense-preserving harmonic mappings in the open unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$.

In the present paper the authors investigate the properties of the class of harmonic mappings which is based on the generalized of R. J. Libera Theorem [7].

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1 Introduction

Let F be the class of analytic functions in D , and let S denote those functions in F that are univalent and normalized by $h(0) = 0$, $h'(0) = 1$. Furthermore, let Ω be the family of functions $\phi(z)$ regular in D and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for every $z \in D$.

Next, for arbitrary fixed numbers A, B , $-1 < A \leq 1$, $-1 \leq B < A$ denote by $P(A, B)$, the family of functions $p(z) = 1 + p_1z + p_2z^2 + \dots$ regular in D and such that $p(z)$ is in $P(A, B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)} \quad (1)$$

for some $\phi(z) \in \Omega$ and every $z \in D$. This class was introduced by Janowski [6].

Moreover, let $S^*(A, B, b)$ be denote by the family of functions $h(z) \in S$ such that $h(z)$ is in $S^*(A, B, b)$ if and only if $(\frac{f(z)}{z}) \neq 0$,

$$1 + \frac{1}{b}z \frac{h'(z)}{h(z)} - 1 = p(z), z \in D, (b \neq 0, \text{complex}) \quad (2)$$

for some function $p(z) \in P(A, B)$ and all $z \in D$. Let $C(A, B, b)$ denote the family of functions which are regular, such that $h(z)$ is in $C(A, B, b)$ if and only if

$$1 + \frac{1}{b}z \frac{h''(z)}{h'(z)} = p(z), z \in D, (b \neq 0, \text{complex}) \quad (3)$$

for some function $p(z) \in P(A, B)$ and all $z \in D$. Let $s_1(z) = z + d_2z^2 + \dots$ and $s_2(z) = z + e_2z^2 + \dots$ be elements of F . If there exists $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$, then we say that $s_1(z)$ is subordinate to $s_2(z)$ and we write $s_1(z) \prec s_2(z)$. Specially if $s_2(z)$ is univalent in D , then $s_1(z) \prec s_2(z)$ if and only if $S_1(D) \subset S_2(D)$ and $S_1(0) = S_2(0)$ implies $S_1(D_r) \subset S_2(D_r)$ where $D_r = \{z \mid |z| < r, 0 < r < 1\}$. (Subordination and Lindelof principle [1], [5])

Finally, a planar harmonic mapping in the open disc D is a complex-valued harmonic function f , which maps D onto some planar domain $f(D)$. Since D is simply-connected domain, the mapping f has a canonical decomposition $f = h + \bar{g}$, where $h(z)$ and $g(z)$ are analytic in D and have the following power series

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where $a_n, b_n \in \mathbb{C}$, $n = 0, 1, 2, 3, \dots$ as usual we call $h(z)$ analytic part of $f(z)$ and $g(z)$ co-analytic part of f , an elegant and complete treatment of the

theory of harmonic mapping in given Duren’s monograph [4]. Lewy proved in 1936 [4] that the harmonic mapping f locally univalent in D if and only if its Jacobian $J_f = |h'(z)|^2 - |g'(z)|^2$ is different from zero in D . In view of this result locally univalent harmonic mapping in the open unit disc D are either sense-preserving if $|h'(z)| > |g'(z)|$ in D , or sense-reversing if $|g'(z)| > |h'(z)|$ in D . Through this paper we will restrict ourselves to the study sense-preserving harmonic mappings. We also note that $f = h(z) + \overline{g(z)}$ is sense-preserving in D if and only if $h'(z)$ does not vanish in D , and the second dilatation $w(z) = (\frac{h'(z)}{g'(z)})$ has the property $|w(z)| < 1$ for all $z \in D$. Therefore the class of all sense-preserving harmonic mappings in the open unit disc with $a_0 = b_0 = 0$, and $a_1 = 1$ will be denoted by S_H . Thus S_H contains standard class S of univalent functions. The family of all mappings $f \in S_H$ with the additional property $g'(0) = 0$, i.e, $b_1 = 0$ is denoted by S_H^0 . Thus it is clear that $S \subset S_H^0 \subset S_H$. The aim of this paper is to investigate some properties of the subclass

$$S_{HC(A,B,b)} = \left\{ f = h(z) + \overline{g(z)} \mid 1 + \frac{1}{b} \left(\frac{g'(z)}{h'(z)} - b_1 \right) \prec \frac{1 + Az}{1 + Bz}, b, b_1 \in \mathbb{C}, b \neq 0, |b_1| < 1, \right. \\ \left. h(z) \in C(A, B, b) \right\}.$$

For the aim of this paper we will need the following lemma and theorem.

Lemma 1.1 ([2]) *Let $\phi(z)$ be a non-constant analytic function in the unit disc D with $\phi(0) = 0$. If $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_0 then $z_0\phi'(z_0) = k\phi(z_0)$, $k \geq 1$.*

Theorem 1.2 ([7]) *Let $h(z)$ be an element of $C(A, B, b)$, then*

$$2\left[1 + \frac{1}{b} \left(z \frac{h'(z)}{h(z)} - 1 \right)\right] - 1 = \frac{1 + A\phi(z)}{1 + B\phi(z)}$$

2 Main Results

Theorem 2.1 *Let $f = (h(z) + \overline{g(z)})$ be an element of $S_{HC(A,B,b)}$, then*

$$\frac{g(z)}{h(z)} \prec b_1 \frac{1 + Az}{1 + Bz}, \tag{4}$$

Proof 2.2 *Since $f = (h(z) + \overline{g(z)})$ be an element of $S_{HC(A,B,b)}$, then*

$$1 + \frac{1}{b} \left(\frac{g'(z)}{h'(z)} - b_1 \right) \prec \frac{1 + Az}{1 + Bz} \Rightarrow 1 + \frac{1}{b} \left(\frac{g'(z)}{h'(z)} - b_1 \right) = \frac{1 + A\phi(z)}{1 + B\phi(z)} \Rightarrow \tag{5}$$

$$\frac{g'(z)}{h'(z)} = \frac{b_1 + (b_1B + b(A - B))\phi(z)}{1 + B\phi(z)} \Rightarrow \frac{g'(z)}{h'(z)} \prec \frac{b_1 + (b_1B + b(A - B))z}{1 + Bz} \quad (6)$$

On the other hand the transformation

$$\left(\frac{b_1 + (b_1B + b(A - B))z}{1 + Bz} \right)$$

maps $|z| = r$ onto the disc with the centre

$$C(r) = \frac{b_1 + (b(B^2 - AB) - b_1B^2)r^2}{1 - B^2r^2}$$

and the radius $\rho(r) = \frac{|b|(A-B)r}{1-B^2r^2}$, therefore using the subordination principle we can write

$$\left| \frac{g'(z)}{h'(z)} - \frac{b_1 + (b(B^2 - AB) - b_1B^2)r^2}{1 - B^2r^2} \right| \leq \frac{|b|(A - B)r}{1 - B^2r^2} \quad (7)$$

The inequality (7) shows that all the values of $\left(\frac{g'(z)}{h'(z)}\right)$ are in the disc

$$W(D_r) = \left\{ z \left| \left| \frac{g'(z)}{h'(z)} - \frac{b_1 + (b(B^2 - AB) - b_1B^2)r^2}{1 - B^2r^2} \right| \leq \frac{|b|(A - B)r}{1 - B^2r^2} \right. \right\}. \quad (8)$$

Now, we define the function $\phi(z)$ by,

$$\frac{g(z)}{h(z)} = b_1 \frac{1 + A\phi(z)}{1 + B\phi(z)}$$

then we see that $\phi(z)$ is analytic in D and

$$\frac{g(z)}{h(z)}_{z=0} = b_1 = b_1 \frac{1 + A\phi(0)}{1 + B\phi(0)} \Rightarrow \phi(0) = \frac{0}{(A - B)} = 0,$$

$$w(z) = \frac{g'(z)}{h'(z)} = b_1 \left(\frac{1 + A\phi(z)}{1 + B\phi(z)} + \frac{(A - B)z\phi'(z)}{(1 + B\phi(z))^2} \frac{h(z)}{zh'(z)} \right) \quad (9)$$

Using Theorem 1.2 in the equality (9), this equality can be written in the following form:

$$w(z) = \frac{g'(z)}{h'(z)} = b_1 \left(\frac{1 + A\phi(z)}{1 + B\phi(z)} + \frac{2(A - B)z\phi'(z)}{(1 + B\phi(z))(2 + (b(A - B) + 2B)\phi(z))} \right) \quad (10)$$

Now, it is easy to realize that the subordination (4) is equivalent to $|\phi(z)| < 1$ for all $z \in D$. Indeed we assume the contrary; then

there is a $z_0 \in D$ such that $|\phi(z_0)| = 1$ so by I. S. Jack's Lemma (Lemma 1.1), $z_0\phi'(z_0) = k\phi(z_0)$ for some $k \geq 1$ and for such z_0 we have

$$\begin{aligned} w(z) &= \frac{g'(z_0)}{h'(z_0)} = b_1 \left(\frac{1 + A\phi(z_0)}{1 + B\phi(z_0)} + \frac{2k(A - B)\phi(z_0)}{(1 + B\phi(z_0))(2 + (b(A - B) + 2B))\phi(z_0)} \right) \\ &= w(\phi(z_0)) \notin w(D_r) \end{aligned}$$

but this contradicts to (4); so our assumption is wrong, i. e., $|\phi(z)| < 1$ for all $z \in D$.

Lemma 2.3 Let $h(z)$ be an element of $C(A, B, b)$ then

$$rF(A, B, |b|, Reb, -r) \leq |h(z)| \leq rF(A, B, |b|, Reb, r), B \neq 0;$$

$$re^{-A|b|r} \leq |h(z)| \leq re^{A|b|r}, B = 0.$$

$$G(A, B, |b|, Reb, -r) \leq |h'(z)| \leq G(A, B, |b|, Reb, r), B \neq 0;$$

$$e^{-Abr} \leq |h'(z)| \leq e^{Abr}, B = 0.$$

where

$$F(A, B, |b|, Reb, r) = \frac{(1 + Br)^{\frac{A-B}{4B}(|b|+Reb)}}{(1 - Br)^{\frac{A-B}{4B}(|b|-Reb)}}$$

$$G(A, B, |b|, Reb, r) = \frac{(1 + Br)^{\frac{A-B}{2B}(|b|+Reb)}}{(1 - Br)^{\frac{A-B}{2B}(|b|-Reb)}}$$

These inequalities are sharp.

Proof 2.4 Using Theorem 1.2 and subordination principle, then we write

$$\left| \left(2 \left(1 + \frac{1}{b} \left(z \frac{h'(z)}{h(z)} - 1 \right) \right) - 1 \right) - \frac{1 + AB r^2}{1 - B^2 r^2} \right| \leq \frac{(A - B)r}{1 - B^2 r^2}, B \neq 0;$$

$$\left| \left(2 \left(1 + \frac{1}{b} \left(z \frac{h'(z)}{h(z)} - 1 \right) \right) - 1 \right) - 1 \right| \leq Ar, B = 0.$$

After the straightforward calculations, we obtain

$$\left| z \frac{h'(z)}{h(z)} - \frac{2 - (2B^2 - b(B^2 - AB))r^2}{2(1 - B^2 r^2)} \right| \leq \frac{|b|(A - B)r}{2(1 - B^2 r^2)}, B \neq 0;$$

$$\left| z \frac{h'(z)}{h(z)} - 1 \right| \leq \frac{|b|Ar}{2}, B = 0.$$

These inequalities can be written in the following form,

$$\frac{2 - |b|(A - B)r - (M - Nx)r^2}{2r(1 + Br)(1 - Br)} \leq \frac{\partial}{\partial r} \log |h(z)| \leq \frac{2 + |b|(A - B)r + (M - Nx)r^2}{2r(1 + Br)(1 - Br)}, B \neq 0;$$

$$\frac{1}{r} - \frac{A|b|}{2} \leq \frac{\partial}{\partial r} \log |h(z)| \leq \frac{1}{r} + \frac{A|b|}{2}, B = 0.$$

where $M = 2B^2$, $N = (B^2 - AB)$, $x = Reb$. Then after integration from the last inequality, we get the result.

Similarly, using subordination principle nad the definition of the class $C(A, B, b)$, then we write

$$\left| \left(1 + \frac{1}{b} \frac{h''(z)}{h'(z)}\right) - \frac{1 - ABr^2}{1 - Br^2} \right| \leq \frac{(A - B)r}{1 - B^2r^2}, B \neq 0;$$

$$\left| \frac{1}{b} \frac{h''(z)}{h'(z)} \right| \leq Ar, B = 0.$$

These inequalities can be written in the following form,

$$\frac{rK_1 - L_1}{(1 - Br)(1 + Br)} \leq \frac{\partial}{\partial r} \log |h'(z)| \leq \frac{rK_1 + L_1}{(1 - Br)(1 + Br)}, B \neq 0;$$

$$-|b|A \leq \frac{\partial}{\partial r} \log |h'(z)| \leq |b|A, B = 0.$$

where $K_1 = (B^2 - AB)Reb$, $L_1 = |b|(A - B)$. After the integration from here, we get the result.

Corollary 2.5 Let $f = (h(z) + \overline{g(z)})$ be an element of $S_{HC(A,B,b)}$, then

$$rF(A, B, |b|, Reb, -r) \frac{|b_1| - |b|(A - B)r - |b_1B^2 - bB^2 + bAB|r^2}{1 - B^2r^2} \leq |g(z)| \leq$$

$$rF(A, B, |b|, Reb, r) \frac{|b_1| + |b|(A - B)r + |b_1B^2 - bB^2 + bAB|r^2}{1 - B^2r^2}, B \neq 0;$$

$$re^{-A|b|r}[|b_1| - |b|Ar] \leq |g(z)| \leq re^{A|b|r}[|b_1| + |b|Ar], B = 0.$$

$$G(A, B, |b|, Reb, -r) \frac{|b_1| - |b|(A - B)r - |b_1B^2 - bB^2 + bAB|r^2}{1 - B^2r^2} \leq |g'(z)| \leq$$

$$G(A, B, |b|, Reb, r) \frac{|b_1| + |b|(A - B)r + |b_1B^2 - bB^2 + bAB|r^2}{1 - B^2r^2}, B \neq 0;$$

$$G(A, B, |b|, Reb, -r)[|b_1| - |b|Ar] \leq |g'(z)| \leq G(A, B, |b|, Reb, r)[|b_1| + |b|Ar], B = 0.$$

Proof 2.6 This corollary is a simple consequence of Theorem 1.2 and Lemma 2.2.

Lemma 2.7 Let $f = (h(z) + \overline{g(z)})$ be an element of $S_{HC(A,B,b)}$, then

$$\frac{|b_1| - r}{1 + |b_1|r} \leq |w(z)| = \left| \frac{g'(z)}{h'(z)} \right| \leq \frac{|b_1| + r}{1 + |b_1|r} \tag{11}$$

$$\frac{(1 - r^2)(1 - |b_1|^2)}{(1 + |b_1|r)^2} \leq (1 - |w(z)|^2) \leq \frac{(1 - r^2)(1 - |b_1|^2)}{(1 - |b_1|r)^2} \tag{12}$$

$$\frac{(1 - r)(1 + |b_1|)}{(1 - |b_1|r)} \leq (1 + |w(z)|) \leq \frac{(1 + r)(1 + |b_1|)}{(1 + |b_1|r)} \tag{13}$$

$$\frac{(1 - r)(1 - |b_1|)}{(1 + |b_1|r)} \leq (1 - |w(z)|) \leq \frac{(1 + r)(1 - |b_1|)}{(1 - |b_1|r)} \tag{14}$$

Proof 2.8 Since $f = (h(z) + \overline{g(z)}) \in S_{HC(A,B,b)}$, then

$$w(z) = \frac{g'(z)}{h'(z)} = \frac{b_1 + 2b_2z + \dots}{1 + 2a_2z + \dots} \Rightarrow w(0) = b_1, |w(z)| < 1$$

so the function

$$\phi(z) = \frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)} = \frac{w(z) - b_1}{1 - \overline{b_1}w(z)}$$

satisfies the conditions of Schwarz Lemma. Therefore we have

$$w(z) = \frac{g'(z)}{h'(z)} = \frac{b_1 + \phi(z)}{1 + \overline{b_1}\phi(z)} \tag{15}$$

This shows that,

$$w(z) = \frac{g'(z)}{h'(z)} \prec \frac{b_1 + z}{1 + \overline{b_1}z}$$

On the other hand, the linear transformation $(\frac{b_1+z}{1+\overline{b_1}z})$ maps $|z| = r$ onto the disc with the center

$$C(r) = \left(\frac{(1 - r^2)Reb_1}{1 - r^2}, \frac{(1 - r^2)Imb_1}{1 - r^2} \right)$$

at the radius

$$\rho(r) = \frac{(1 - |b_1|)^2 r}{1 - r^2}$$

then we have the results, easily.

Corollary 2.9 *Let $f = (h(z) + \overline{g(z)})$ be an element of $S_{HC(A,B,b)}$, then*

$$(G(A, B, |b|, \text{Re}b, -r))^2 \frac{(1-r^2)(1-|b_1|^2)}{(1+|b_1|r)^2} \leq J_f \leq (G(A, B, |b|, \text{Re}b, r))^2 \frac{(1-r^2)(1-|b_1|^2)}{(1-|b_1|r)^2},$$

$$e^{-2A|b|r} \frac{(1-r^2)(1-|b_1|^2)}{(1+|b_1|r)^2} \leq J_f \leq e^{2A|b|r} \frac{(1-r^2)(1-|b_1|^2)}{(1-|b_1|r)^2}, B = 0$$

Proof 2.10 *Since*

$$J_f = |h'(z)|^2 - |g'(z)|^2 = |h'(z)|^2 - |h'(z)|^2 |w(z)|^2$$

$$= |h'(z)|^2 (1 - |w(z)|^2)$$

Using Lemma 2.7, we get the result.

Corollary 2.11 *Let $f = (h(z) + \overline{g(z)})$ be an element of $S_{HC(A,B,b)}$, then*

$$\int_0^r G(A, B, |b|, \text{Re}b, -r) \frac{(1-r)(1-|b_1|)}{(1+|b_1|r)} dr \leq |f| \leq \int_0^r G(A, B, |b|, \text{Re}b, r) \frac{(1+r)(1+|b_1|)}{(1+|b_1|r)} dr,$$

$$\int_0^r e^{-A|b|r} \frac{(1-r)(1-|b_1|)}{(1+|b_1|r)} dr \leq |f| \leq \int_0^r e^{A|b|r} \frac{(1+r)(1+|b_1|)}{(1+|b_1|r)} dr, B = 0.$$

Proof 2.12 *Since*

$$(|h'(z)| - |g'(z)|) |dz| \leq |df| \leq (|h'(z)| + |g'(z)|) |dz| \Rightarrow$$

$$|h'(z)| (1 - |w(z)|) |dz| \leq |df| \leq |h'(z)| (1 + |w(z)|) |dz| \tag{16}$$

Using Lemma 2.7 and after integration we obtain the result.

Theorem 2.13 *Let $f = (h(z) + \overline{g(z)})$ be an element of $S_{HC(A,B,b)}$, then*

$$\sum_{k=2}^n k^2 |b_k - b_1 a_k|^2 \leq |1 - b_1^2|^2 + \sum_{k=2}^{n+1} k^2 |a_k - b_1 k|^2 \tag{17}$$

Proof 2.14 *Using Lemma 2.7 we can write*

$$w(z) = \frac{g'(z)}{h'(z)} \prec \frac{b_1 + z}{1 + \overline{b_1}z} \Rightarrow \frac{g'(z)}{h'(z)} = \frac{b_1 + \phi(z)}{1 + b_1 \phi(z)} \Rightarrow$$

$$g'(z)(1 + \overline{b_1} \phi(z)) = h'(z)(b_1 + \phi(z)) \Rightarrow (g'(z) - b_1 h'(z)) = (h'(z) - \overline{b_1} g'(z)) \phi(z) \Rightarrow$$

$$\left(\sum_{n=1}^{\infty} b_n z^n\right)' - b_1 \left(z + \sum_{n=2}^{\infty} a_n z^n\right)' = \left[\left(z + \sum_{n=2}^{\infty} a_n z^n\right)' - b_1 \left(\sum_{n=1}^{\infty} b_n z^n\right)'\right] \phi(z) \Rightarrow$$

$$\sum_{k=2}^n k(b_k - b_1 a_k)z^{k-1} + \sum_{k=n+1}^{\infty} d_k z^{k-1} = [(1 - b_1^2) + \sum_{k=2}^n k(a_k - b_1 b_k)z^{k-1}] \phi(z) \tag{18}$$

Since the last inequality has the form $f_1(z) = f_2(z)\phi(z)$ with $|\phi(z)| < 1$, it follows that

$$\frac{1}{2\pi} \int_0^{2\pi} |f_1(re^{i\theta})|^2 d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |f_2(re^{i\theta})|^2 d\theta \tag{19}$$

for each r , ($0 < r < 1$). Expressing (19) in terms of coefficient in (17) we obtain the inequality,

$$\sum_{k=2}^n k |b_k - b_1 a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2k} \leq [|1 - b_1^2|^2 + \sum_{k=2}^{n+1} k^2 |a_k - b_1 b_k|^2] r^{2k} \tag{20}$$

By letting $r \rightarrow 1^-$ in (20), we obtain desired result. The Proof of this method is due to Clunie [1].

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