SOME RESULTS ON A SUBCLASS OF MULTIVALENT HARMONIC MAPPINGS

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Abstract

In this thesis, the fundamental function classes and their properties are introduced. Some results that we can generalize to multivalent functions are obtained.

Next, as the main part of the study, a new subclass of sense-preserving multivalent harmonic mappings in the open unit disc is defined. Furthermore, distortion inequalities and growth theorems for the functions of this new class are determined.

MULTIVALENT HARMONİK TASVİRLERİN BİR ALTSINIFINDA BAZI SONUÇLAR

Özet

Bu tezde, temel fonksiyon sınıfları ve özellikleri tanıtıldı. Multivalent fonksiyonlara genelleştirebileceğimiz bazı sonuçlar elde edildi.

Daha sonra, çalışmanın ana kısmı olarak, birim diskte yön koruyan multivalent harmonik tasvirlerin yeni bir alt sınıfı tanımlandı. Ayrıca bu yeni sınıfın fonksiyonları için distorsiyon eşitsizlikleri ve genişleme teoremleri belirlendi.

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Chapter 1
Introduction

1.1 Literature Review

Complex analysis, also known as the theory of functions of a complex variable, is a branch of mathematics which has a great importance in physisc, engineering and other sciences with its explanatoriness in the theory and applications.

By the introduction of the notation "i" for the square root of -1 in 1777 by Euler, and the idea of Hamilton to consider a complex number as a pair of real numbers, complex numbers became more comprehensible and useable. By this way, many mathematicians like Weierstrass, Schwarz and Poincaré who started to make great contributions in the theory of complex numbers together with geometry provided new fields under complex analysis to be generated. Geometric function theory is one of these branches that studies the geometry of analytic functions.

Riemann who laid the foundations of complex-valued function theory made his studies mostly on analytic functions and he was the first mathematician to study on conformal mappings. In 1851, he showed in his graduate work, also known as the Riemann Mapping Theory, that a simply connected domain of complex plane can be mapped conformally onto the unit disc by a function f analytic and one-to-one [28]. Combining the studies of Riemann and the idea of using power series of Weierstrass, Bieberbach started to examine the functions in both ways and he set some problems about the coefficients of functions under certain normalizations.

Koebe, who was also inspired by Riemann's studies, improved the Riemann Mapping Theorem and showed that the mapping is unique under certain conditions and he introduced the univalent functions in 1907 [20]. By Koebe's definition of univalent function, there have been lots of studies in this field. Many mathematicians made great contributions to univalent function theory by introducing certain definitions of the images of functions such as starlikeness, convexity and by specifying boundaries for the modulus of the coefficients of functions.

In 1984, De Branges proved the conjecture of Bieberbach about the coefficients of the normalized univalent functions that was revealed in 1916 [7]. Thereby, new problems were produced and it was proved that the results provided in univalent function theory can be extended to the harmonic mappings by Clunie and Sheil-Small in 1984 [6]. By this way, in harmonic mapping field, problems were set to obtain similar results in univalent function theory and conformal mappings.

Multivalent function theory became a study field after Montel introduced the term of multivalent function in his book in 1933. If a complex-valued analytic function f takes each of its values at most p times then the functions is called multivalent of order p. In case of $p=1$, it is said that the function is univalent. Thus, we can say that the multivalent functions are a natural generalization of univalent functions [17], [30].

As the other branches of geometric function theory, introducing new subclasses of $p-$valent functions and examining the properties, determining the bounds of the coefficients and giving distortion inequalities are some of the problems of the multivalent function theory. These problems were studied by many mathematicians including Fekete, Szegö, Goodman and Hummel. Also, new subclasses of multivalent functions have been introduced and studied by Silverman, Owa, Altıntaş, Aouf and Liu.

1.2 Purpose of Thesis

In this thesis, a certain subclass of multivalent functions will be investigated. To have a general knowledge about the subject, analytic functions and some necessary and sufficient conditions for analyticity will be introduced in the second chapter. Next, in the third chapter some properties of conformal mappings will be given.

In the fourth section of the thesis, the class of univalent functions is defined. Growth and distortion theorems of univalent functions class and some of the subclasses of this class and some properties and theorems that we can extend to harmonic mappings will be given. Then, we will introduce the results generalized to harmonic mappings in the fifth chapter.

Finally, in the sixth section, by giving definitions and some properties that the class satisfies, multivalent functions will be introduced. In addition, as the main subject of this thesis, we will introduce a subclass of multivalent functions and certain results on this subclass will be investigated.

Chapter 2
Analytic Functions

This chapter is on complex-valued functions which are analytic in a simply-connected subdomain of the complex plane. These functions may be one-variable or several-variables. However, our focus is on functions of one-variable. By simply-connected domain, we mean a region $D \in \mathbb{C}$ where every closed curve in D can be smoothly contracted to a point without leaving D.

A complex-valued function $w = f(z)$ is said to be analytic at a point z_0 if f is differentiable at z_0 and in an open neighborhood of z_0. By the term differentiable, we mean that the limit

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists. The definition of derivative can be written in the form

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

This limit must be regardless of how Δz approaches to zero, along a horizontal line or along a vertical line. If a function $w = f(z)$ is differentiable at every point in a domain D, then f is analytic in D.

The meaning of the derivative of a function in complex-analysis is different from the meaning in real-calculus. In real calculus, the derivative represents informations such as the difference in the function, velocity or slope. However, the main concern in complex-valued functions is whether the derivative exists or not. The existence of the derivative gives informations about the structural properties of the function.

The meaning of the existence of the derivative at the point z_0 of the complex-valued function $w = f(z)$ differs depending on whether the point z_0 is an interior point or a boundary point of D. To avoid this confusion, all analytic functions are defined on open sets. In this study, the complex-valued functions $w = f(z)$ are defined on simply-connected domains.

Since the limit definition of derivative for an analytic function must be regardless of the way Δz approaches to 0, the limits of two ways must be equal. Considering the complex-valued function $w = f(z)$ can be written $f(z) = u(x,y) + iv(x,y)$ where the functions $u(x,y)$ and $v(x,y)$ are real-valued functions of two real-variables, the derivative can be expressed as:

$$
\begin{aligned}
f'(z) &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\
&= \lim_{\Delta x + i\Delta y \to 0} \frac{\left[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\right] - \left[u(x,y) + iv(x,y)\right]}{\Delta x + i\Delta y} \\
&= \lim_{\Delta x + i\Delta y \to 0} \frac{[u(x+\Delta x, y+\Delta y) - u(x,y)] + i[v(x+\Delta x, y+\Delta y) - iv(x,y)]}{\Delta x + i\Delta y} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left[\frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} + i \frac{v(x+\Delta x, y+\Delta y) - iv(x,y)}{\Delta x + i\Delta y} \right] \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} + i \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{v(x+\Delta x, y+\Delta y) - iv(x,y)}{\Delta x + i\Delta y}
\end{aligned}
$$

Case 1: Consider $\Delta y = 0$. Then the limit becomes

$$
\begin{aligned}
f'(z) &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - iv(x,y)}{\Delta x} \\
&= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.
\end{aligned}
$$

Case 2: Consider $\Delta x = 0$. Then the limit becomes

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - iv(x, y)}{i\Delta y}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Since we consider that the function f is analytic, then it is differentiable which means that these two limits must be equal. Then we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{2.1}$$

These equations are called as Cauchy-Riemann equations. Real and imaginary parts of an analytic function $f(z) = u(x, y) + iv(x, y)$ satisfy these equations. Again, if u and v satisfy Cauchy-Riemann equations in a simply-connected domain and the partial derivatives u_x, u_y, v_x, v_y are continuous functions, then $f(z)$ is an analytic function in this domain. Therefore, we can say that satisfying Cauchy-Riemann equations is a necessary and sufficient condition for analyticity of a function.

Complex-valued functions are also said to be holomorphic or regular if they are analytic. Since these functions are differentiable, they have derivatives of all orders. This property tells us that a complex-valued analytic function can be expressed as Taylor series. These functions can be written as:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad a_n = \frac{f^{(n)}}{n!}. \tag{2.2}$$

Here, to calculate the coefficients a_n, Cauchy Integral Formula for derivatives is used.

Cauchy Integral Formula.

Suppose $f(z)$ is analytic in a simply-connected domain D and Γ is a simple closed curve lying entirely in D. Then, for any point z_0 in Γ,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz. \tag{2.3}$$

We can suppose that the centre z_0 may be shifted to origin without loss of generality and the Taylor series of the function $f(z)$ can be expressed with the coefficients $a_n = \frac{f^{(n)}(0)}{n!}$ where $f^{(n)}(0) = \frac{n!}{2\pi i}\oint_{\Gamma} \frac{f(z)}{z^{n+1}} \mathrm{d}z.$

Following theorems are some consequences of Cauchy Integral Formula.

Gauss Mean Value Theorem.

If $f(z)$ is analytic inside and on a simple closed curve Γ with center z_0 and radius r, then $f(z_0)$ is the mean value of the values of $f(z)$ on Γ, i.e.,

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta}) d\theta.$$

This result is known as Gauss' Mean Value Theorem and shows that the value of f at the center z_0 of a circle is the average of all values of f on the boundary of Γ.

Proof. *By Cauchy integral formula*

$$f(z_0) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{z - z_0} \mathrm{d}z$$

If Γ *has radius* r *and center* z_0, *the equation of* Γ *is* $|z - z_0| = r$ *or* $z = z_0 + re^{i\theta}$ *where* $0 \le \theta \le 2\pi$. *Noting that* $dz = ire^{i\theta} d\theta$, *then the equation becomes*

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})ire^{i\theta}}{re^{i\theta}} d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

Maximum Modulus Theorem.

Given $f(z)$ analytic on some domain D, if f is non-constant on D then the maximum value of $|f(z)|$ for $z \in D$ will occur on the boundary of D. (Alternatively, if $|f(z)|$ is maximized by some value not on the boundary of D, then f is constant on D.)

Schwarz Lemma.

A very useful consequence of Maximum principle, Schwarz lemma states that for unit disc $U = \{z : |z| < 1\}$ in the complex plane and analytic function $\phi : U \to U$ such that $\phi(0) = 0$ ensures $|\phi(z)| \le |z|$ for all z in U and $|\phi'(0)| \le 1$.

Equality is only valid for the function $\phi(z) = kz$ where $|k| = 1$.

The functions $\phi(z)$ which satisfy the conditions of Schwarz lemma are called Schwarz functions.

Jack Lemma.

[18] Let $w(z)$ be a non-constant analytic function in the open unit disc U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at z_0, then $z_0 w'(z_0) = m w(z_0)$, $m \ge 1$.

Subordination Principle.

[8] Let $f(z)$ and $g(z)$ be the functions defined in the unit disc $U = \{z : |z| < 1\}$ and analytic. If $f(z)$ can be written in the form $f(z) = g(\phi(z))$ for a function $\phi(z)$ defined in U, analytic and satisfies the conditions of Schwarz Lemma, then $f(z)$ is subordinate to $g(z)$. Subordination is expressed as $f(z) \prec g(z)$.

Lindelöf Principle.

[8] Let $s_1(z) = z + d_2 z^2 + \ldots$ and $s_2(z) = z + e_2 z^2 + \ldots$ be analytic functions in the unit disc U. If $s_2(z)$ is univalent in U then $s_1(z) \prec s_2(z)$ if and only if $s_1(U) \subset s_2(U)$ and $s_1(0) = s_2(0)$ implies $s_1(U_r) \subset s_2(U_r)$, where $U_r = \{z : |z| < r, 0 < r < 1\}$.

The Argument Principle.

Let Γ be a simple closed contour lying entirely within a domain D. Suppose

- $f(z)$ is analytic in D except for finite number of poles inside Γ.
- $f(z) \neq 0$ on Γ.

Then

$$\frac{1}{2\pi i}\oint_{\Gamma}\frac{f'(z)}{f(z)}dz = N_0 - N_p$$

where N_p is the total number of poles of f inside Γ and N_0 is the total number of zeros of f inside Γ counted according to their multiplicities.

Chapter 3
Conformal Mapping

Non-constant linear mappings rotate, translate and magnify the points in the complex plane. If there is an angle-preserving property, then this mapping is called conformal mapping. In this section, we will give some basic informations about conformal mappings.

Let $w = f(z) = u(x,y) + iv(x,y)$ be a complex mapping defined in a domain D. Then, the equations $u(x,y)$ and $v(x,y)$ define a transformation or mapping between $xy-$plane and $uv-$plane.

Suppose that Γ_1 and Γ_2 are intersecting curves at (x_0, y_0) in the $xy-$plane and the curves $\Gamma_1^{'}$ and $\Gamma_2^{'}$ are intersecting at the point (u_0, v_0) in $uv-$plane. By the transformation $w = f(z)$, the point (x_0, y_0) is mapped into (u_0, v_0) and the curves Γ_1 and Γ_2 are mapped into the curves $\Gamma_1^{'}$ and $\Gamma_2^{'}$, respectively. If for every Γ_1 and Γ_2 intersecting at (x_0, y_0), the angle between these curves and the angle between image curves $\Gamma_1^{'}$ and $\Gamma_2^{'}$ are equal in both magnitude and sense, then $w = f(z)$ is said to be a conformal mapping.

Theorem 3.1. If $f(z)$ is analytic and $f'(z)$ does not vanish in a domain D, then the mapping $w = f(z)$ is conformal at all points of D.

The modulus $|f'(z)|^2$ defines a magnification factor for area. Small figures in the neighborhood of a point z_0 of $z-$plane are mapped into $w-$plane as similar images by magnifying approximately $|f'(z_0)|^2$. Small distances transformed by conformal mappings from $z-$plane to $w-$plane are magnified by an amount of $|f'(z)|$, which is called linear magnification factor.

There are four operations for transformations. For given complex constants α, β and real constants a, θ_0, we can express these operations as follows:

i. *Translation.* $w = z + \beta$. By this transformation, figures in the $z-$plane are translated in the direction of vector β.
ii. *Rotation.* $w = e^{i\theta_0} z$. By this transformation, figures in the $z-$plane are rotated through an angle θ_0.
iii. *Streching.* $w = az$. By this transformation, figures in the $z-$plane are streched in the direction z if $a > 1$.
iv. *Inversion.* $w = \dfrac{1}{z}$.

3.1 Linear Transformations

The transformation $w = az + b$, $a \neq 0$ where a, b constants in $\mathbb{C}$ is called a linear transformation. We can write this linear transformation in terms of successive transformations of the transformations above. So, we see that a general linear transformation is combination of the transformations of translation, rotation and stretching.

3.2 Linear Fractional Transformations

The transformation

$$w = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0 \tag{3.1}$$

is called a fractional transformation for complex constants a, b, c, d. This transformation is also knows as the Möbius transformation. It is considered as combinations of the transformations of translation, rotation, stretching and inversion.

Let f_1, f_2, f_3 and f_4 be the transformations as following:

- $f_1(z) = z + \frac{d}{c}$ gives a translation by $\frac{d}{c}$.
- $f_2(z) = \frac{1}{z}$ gives the inversion of $f_2(z)$ with respect to the real axis.
- $f_3(z) = \frac{bc - ad}{c^2} z$ gives a rotation.
- $f_4(z) = z + \frac{a}{c}$ gives a translation by $\frac{a}{c}$.

Composition of these functions gives the fractional transformation

$$f_4 \circ f_3 \circ f_2 \circ f_1(z) = w = \frac{az + b}{cz + d}.$$

If $c = 0$, then the transformation (3.1) is a linear mapping, as a special case of Möbius transformation. Möbius transformations are conformal on their domains since $ad - bc \neq 0$.

Möbius transformations map the circles into circles or lines and lines into lines or circles. The image of a line ℓ under the transformation (3.1) is a circle if and only if $c \neq 0$ and the pole $z = -\frac{d}{c}$ is not on the line ℓ. Likewise, the transformation of a circle by (3.1) is a line if and only if $c \neq 0$ and the pole $z = -\frac{d}{c}$ is on the circle.

We mostly use the transformation

$$w = f(z) = \frac{z + a}{1 + \overline{a}z}$$

to map the circle $|z| = r$ onto a circle with radius $\rho(r) = \dfrac{r(1-|a|^2)}{1-|a|^2 r^2}$ and the center at

$$C(r) = \frac{|a|(1-r^2)}{1-|a|^2 r^2}.$$

The transformation that maps the unit disc onto a unit disc is given as

$$w = f(z) = e^{i\theta} \frac{z - z_0}{1 - \overline{z_0} z}.$$

Chapter 4
Univalent Functions

In this chapter, we aim to introduce the class S of univalent functions and give some general results of the class to understand the latter results.

An analytic function $f(z)$ defined in a simply-connected domain D of the complex plane which is single-valued is said to be univalent if it does not take the same value more than once. It means that a single-valued analytic function $f(z)$ is univalent if it is one-to-one, i.e., for $z_1, z_2 \in D$,

$$z_1 = z_2 \Leftrightarrow f(z_1) = f(z_2).$$

Locally univalency is defined with the univalency in a neighborhood of a point $z_0 \in D$. If $f(z)$ is univalent in a neighborhood of z_0, then the function is said to be locally univalent at the point z_0 in D. Locally univalency of analytic function $f(z)$ at z_0 is also equivalent to the expression $|f'(z_0)| \neq 0$.

The derivative of $f(z)$ gives some informations about the geometric behavior of $f(z)$ at the point that $f(z)$ is locally univalent. The modulus of the derivative of $f(z)$, i.e., $|f'(z)|$, is the local magnification factor of lengths and $\arg f(z)$ represents the local orientation factor where $f'(z) = 0$.

For an analytic function $f(z) = u(x,y) + iv(x,y)$, the Jacobian is defined as

$$J_f = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = |f'(z)|^2 \tag{4.1}$$

since the function $f(z)$ satisfies the Cauchy-Riemann equations. Because the Jacobian does not vanish for locally univalent functions, $f(z)$ preserves angles and orientations. Then we can refer a univalent function as a conformal mapping.

Our main concern in this chapter is the univalent functions. To reduce the complications, we normalize the functions without loss of generality. Univalent functions class S is the class of functions $f(z)$ which are analytic, univalent, defined in the unit disk and normalized by $f(0)=0$ and $f'(0)=1$. Then the class S contains the functions with the Taylor expansion

$$f(z)=z+a_2z^2+a_3z^3+\ldots.$$

In this thesis and our future studies, we will consider the domain of the analytic function $f(z)$ as the unit disk $U=\{z:\ |z|<1\}$. This restriction is guaranteed by Bernard Riemann in 1851.

Original version of Riemann Mapping Theorem.

Let D_1 and D_2 be two simply connected proper subdomains of the complex plane $\mathbb{C}$. Given $z_0\in D_1$, $\xi_0\in\partial D_1$, $\omega_0\in D_2$ and $\zeta\in\partial D_2$, there exists a unique mapping f from D_1 onto D_2 which is analytic and injective in D_1 and applies z_0 into ω_0 and ξ_0 into ζ_0.

The Riemann Mapping Theorem.

Let D be a simply connected domain in $\mathbb{C}$ with $D\neq\mathbb{C}$ and let z_0 be a point in D. Then there exists a unique mapping f from D onto the unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ which is analytic and injective in D with $f(z_0)=0$ and $f'(z_0)>0$.

Riemann's mapping theorem demonstrates the existence of a mapping function, it does not actually produce this function.

4.1 Properties of Univalent Functions

Univalency need not be preserved under algebraic operations. However, there exist some transformations that are invariant in the class of S.

i. *Rotation:* If $f(z)\in S$, $\theta\in\mathbb{R}$ and $g(z)=e^{-i\theta}f(e^{i\theta}z)$, then $g(z)\in S$.

ii. *Dilation:* If $f(z)\in S$ and $g(z)=r^{-1}f(rz)$ where $0<r<1$ then $g(z)\in S$.

iii. *Conjugation:* If $f(z)\in S$ and $g(z)=\overline{f(\overline{z})}=z+\overline{a_2}z^2+\overline{a_3}z^3+\ldots$, then $g(z)\in S$.

iv. *Disk Automorphism:* If $f(z)\in S$ and

$$g(z)=\frac{f\left(\dfrac{z+a}{1+\overline{a}z}\right)-f(a)}{(1-|a|^2)f'(a)}, \qquad |a|<1,$$

then $g(z)\in S$.

v. *Range transformation:* If $f(z)\in S$ and $\psi(z)$ is a function analytic and univalent on the range of $f(z)$, with $\psi(0)=0$ and $\psi'(0)=1$, then $g(z)=\psi\circ f(z)\in S$.

vi. *Omitted value transform:* If $f(z)\in S$, $z\in U$ and $f(z)\neq\alpha$, then

$$g(z)=\frac{\alpha f(z)}{\alpha-f(z)}\in S.$$

vii. *Square-root transformation:* If $f(z)\in S$ and $g(z)=\sqrt{f(z^2)}$, then $g(z)\in S$.

viii. *$n-th$ root transformation:* If $f(z)\in S$ and $h(z)$ is defined by

$$h(z)=\sqrt[n]{f(z^n)}=z\left(\frac{f(z^n)}{z^n}\right)^{1/n},$$

where $z\in U$, and $n=2,3,\ldots$ then $h(z)\in S$.

4.2 Coefficient Estimates

The coefficient estimates of the functions in the class of S have an important place in univalent function theory. In 1916, Bieberbach first predicted that for a function $f(z) \in S$ the coefficients in the Taylor series expansion of $f(z)$ has the inequality such that $|a_n| \leq n$ for $n = 2,3,\ldots$ and showed that $|a_2| \leq 2$. This prediction has the name Bieberbach Conjecture. The equality in the Bieberbach conjecture holds only for the Koebe function and its rotations.

Until De Branges proved that conjecture in 1985, many mathematician developed new important methods to show the inequality holds. Until the proof of Branges, only the inequalities for the coefficient a_2, a_3, a_4, a_5 and a_6 had been shown. In 1923, Loewner proved that $|a_3| \leq 3$. 32 years later, Shiffer and Garabedian followed him by showing that $|a_4| \leq 4$. Before Pederson and Schiffer had a proof that $|a_5| \leq 5$, Pederson showed that $|a_6| \leq 6$ in 1969.

Koebe Function.
An important example of univalent function class is the Koebe function

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \ldots.$$

The Koebe function maps the disc U onto the entire plane minus the part of the negative real axis from $-\frac{1}{4}$ to infinity. This is best seen by writing

$$k(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}$$

and observing that the function

$$w = \frac{1+z}{1-z}$$

maps U conformally onto the right half plane $Re w > 0$.

Koebe function and its suitable rotations play a fundamental role as the extremal function for many extremal problems in the class of univalent functions S.

Rotations of the Koebe function can be written as

$$e^{-i\theta}k(e^{i\theta}z) = \frac{z}{(1-e^{i\theta}z)^2} = \sum_{n=1}^{\infty} ne^{i(n-1)\theta}z^n.$$

Koebe function is the largest function in S.

Koebe One-Quarter Theorem.

Let the function $w = f(z) = z + a_2z^2 + a_3z^3 + \ldots \in S$. The image of U transformed by $f(z)$ contains the disc with center at origin and radius $\frac{1}{4}$.

Proof. *Let ξ be a point outside of the unit disc U.*

The function

$$F(z) = \frac{\xi f(z)}{\xi - f(z)} = z + \left(a_2 + \frac{1}{\xi}\right)z^2 + \ldots$$

belongs to S. Since $\xi \notin U$, $\xi - f(z) \neq 0$. Then $F(z)$ is analytic in the open unit disc U. The function $F(z)$ satisfies the normalization conditions of the class S:

$$F(z) = z + \left(a_2 + \frac{1}{\xi}\right)z^2 + \ldots \Rightarrow F(0) = 0,$$

$$F'(z) = 1 + 2\left(a_2 + \frac{1}{\xi}\right)z + \ldots \Rightarrow F'(0) = 1.$$

On the other hand, since $f(z) \in S$, it is one-to-one, i.e., for $z_1 \neq z_2$ in U, $f(z_1) \neq f(z_2)$. This provides that $F(z)$ is one-to-one in the open unit disc.

Then we can see that $F(z)$ is also in S. Using the Bieberbach conjecture, we can say that the modulus of the coefficient of a univalent function satisfies $|a_n| < n$.

Then, for the second coefficient of $F(z)$, we have

$$\left|a_2 + \frac{1}{\xi}\right| \leq 2.$$

To obtain the inequality for $|\xi|$, we can write

$$\left|\frac{1}{\xi}\right| = \left|\frac{1}{\xi} + a_2 - a_2\right|$$

$$\Rightarrow \left|\frac{1}{\xi}\right| \leq \left|\frac{1}{\xi} + a_2\right| + |-a_2| \leq 2 + 2$$

$$\Rightarrow \left|\frac{1}{\xi}\right| \leq 4$$

$$\Rightarrow |\xi| \geq \frac{1}{4}.$$

Bieberbach inequality $|a_2| \leq 2$ has further implications in the geometric theory of conformal mapping. The idea of growth of analytic function $f(z)$ refers to the size of the image domain, that is $|f(z)|$. These concepts tell much about the boundedness of these functions and their derivatives.

Growth Theorem.

For each $f \in S$,

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \qquad |z| = r < 1.$$

One important consequence is the Koebe distortion theorem which provides sharp upper and lower bounds for $|f'(z)|$ as $f(z)$ ranges over the class S. The term distortion arises from the geometric interpretation of $|f'(z)|$ as the infinitesimal magnification factor of arc length under the mapping f or from that of the Jacobian as the infinitesimal magnification factor of area.

Distortion Theorem.

For each $f \in S$

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \qquad |z| = r < 1.$$

4.3 Subclasses of Univalent Functions

The name of the geometric function theory actually describes the significance of relation between analytic structure of the functions and geometric behavior. In univalent function theory, we try to explain the ranges of the functions by geometry and to describe these geometries by formula. The range of the functions may represent starlike, convex, close-to-convex, spirallike geometries. Now, we will define some geometries, give some result about them and identify the subclasses of univalent function class.

4.3.1 Functions with Positive Real Parts (Carathéodory Class)

Some basic properties of the class of functions with positive real part in the unit disc U are given in this section. This class was first introduced by Carathéodory.

Let P denote the class of analytic functions $p(z)$ in the unit disc U which satisfy the conditions

$$p(0) = 1 \qquad \text{and} \qquad Re\, p(z) > 0$$

for all $z \in U$. This class is called the Carathéodory class. The functions in the Carathéodory class have the following Taylor series expansion

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots.$$

The extremal function of P is $p(z) = \dfrac{1+z}{1-z}$, $z \in U$. This extremal function has an important role in the class P, as the Koebe function's role in the univalent function class.

- The function $p(z)=\dfrac{1+z}{1-z}$ maps the unit disc U onto the right-half plane $Re\, p(z)>0$.
- The circle $|z|=r$ where $0<r<1$ is mapped onto the circle with radius $\rho(r)=\dfrac{2r}{1-r^2}$ and the center at $C(r)=\left(\dfrac{1+r^2}{1-r^2},0\right)$ by the extremal function $p(z)=\dfrac{1+z}{1-z}$.

Functions with positive real parts can be represented by subordination and integral representation.

Theorem 4.1. Let $p(z)$ be an analytic function defined in the unit disc U and satisfy the conditions $p(0)=1$ and $Re\, p(z)>0$. Then there is a correspondence between the functions $p(z)$ and $\phi(z)$ as following:

$$p(z)=\frac{1+\phi(z)}{1-\phi(z)}$$

where $\phi(z)$ is a function that satisfies the conditions of Schwarz Lemma.

Proof. *The function* $g(z)=\dfrac{1+z}{1-z}$, $z\in U$ *maps the unit disc onto the right-half plane.*

On the other hand, for two analytic functions $f(z)$ *and* $g(z)$ *in the unit disc, we say that* $f(z)$ *is subordinate to* $g(z)$ *if there exists a Schwarz function* $\phi(z)$ *such that* $f(z)=g\circ\phi(z)$. *Then if* $f(z)$ *is subordinate to* $g(z)$ *then* $f(0)=g(0)$ *and* $f(U)\subset g(U)$.

Consequently, the univalent function $g(z)$ *satisfies* $g(0)=\dfrac{1+0}{1-0}=0$ *and* $p(U)\subset g(U)$. *These expressions show that the conditions of subordination principle are satisfied. So, we can write*

$$p(z) \prec \frac{1+z}{1-z}$$

and see that

$$p(z) = \frac{1+z}{1-z} \Rightarrow p(z) = \frac{1+\phi(z)}{1-\phi(z)}.$$

We can derive the Herglotz representation formula. This formula was found in 1922 by Herglotz.

Theorem 4.2. Let f be an analytic function defined in the unit disc U. Then $Re\, f(z) \geq 0$ if and only if there exists a non-decreasing function μ on the closed interval $[0,2\pi]$ such that $\mu(2\pi) - \mu(0) = Re\, f(0)$ and

$$f(z) = \int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}} d\mu(t) + iIm\, f(z), \qquad z \in U. \tag{4.2}$$

This equation is called the Herglotz formula.

We can have the following corollary by considering the function $f(z)$ as a function with positive real part.

Corollary 4.1. Let the analytic function $p(z)$ of the unit disc U satisfy $p(0) = 1$. Then $p(z) \in \mathrm{P}$ if and only if there exists a non-decreasing function μ on the closed interval $[0,2\pi]$ such that $\mu(2\pi) - \mu(0) = 1$ and such that

$$p(z) = \int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}} d\mu(t), \qquad z \in U. \tag{4.3}$$

Using the Herglotz formula, we can derive the growth and distortion results for the functions with positive real parts.

Theorem 4.3. If a function $p(z)$ belongs to the class P, then the following sharp estimates are obtained:

$$\frac{1-r}{1+r} \le |p(z)| \le \frac{1+r}{1-r},$$

$$\frac{1-r}{1+r} \le Re\, p(z) \le \frac{1+r}{1-r},$$

$$|p'(z)| \le \frac{2 Re\, p(z)}{1-r^2} \le \frac{2r}{(1-r)^2}.$$

By Herglotz formula, we can also have the following inequalities for the coefficients of the functions in P. This result is due to Carathéodory.

Theorem 4.4. If $p(z)$ with the power series $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $z \in U$ belongs to the class P, then the coefficients of the power series of the function $p(z)$ are bounded with $|p_n| \le 2$ for $n = 1, 2, \ldots$. These estimates are sharp.

4.3.2 Starlike Functions

Starlike functions class is one of the most significant subclasses of univalent functions class. The class was introduced by Robertson in 1936. Starlike functions class is denoted by S^*. S^* consists of functions that map the open unit disc U onto a starlike domain. These functions are called starlike functions.

A domain D is said to be starlike with respect to a point $z_0 \in D$ if the linear line segment that joins z_0 to each z in D lies entirely in D. If the point z_0 is origin, then the domain D is said to be starlike with respect to the origin.

Geometrically, this means that every point in D can be seen from the origin. Analytically, a starlike function $f(z)$ is described by

$$Re\left(z \frac{f'(z)}{f(z)} \right) > 0$$

where $z \in U$ and $f(z)$ is analytic.

Koebe function and its rotations are an example of starlike functions and the function is extremal for this class.

Starlike functions have the same upper and lower bounds for distortion and growth of the functions as the class of univalent functions S since the Koebe function is starlike and extremal in S.

Growth and distortion theorems for the class S^* is given as following.

Theorem 4.5. Let $f(z) \in S^*$ and $|z| < 1$, then

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2},$$

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}.$$

These two estimates are sharp. The equality is valid only for Koebe function and its rotations.

Bounds for the coefficients of the Taylor expansion of the function $f(z) \in S^*$ is again obtained as the same as in the class S.

Theorem 4.6. The coefficients of the functions $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \in S^*$ satisfy $|a_n| \leq n$ for $n \geq 2$. Equality $|a_n| = n$ holds for $n = 2,3,\ldots$ holds if the function $f(z)$ is a rotation of Koebe function.

Radius of Starlikeness.

For every radius $r \leq \rho = \tanh\dfrac{\pi}{4}$, each function $f \in S$ maps the unit disk $|z| < r$ onto a domain starlike with respect to the origin.

The radius of starlikeness ρ, is the biggest positive number that each function $f(z) \in S$ maps the disk $|z| \leq \rho$ onto a starlike domain. Here, the Koebe functions is not extremal since it is starlike in the full unit disk.

4.3.3 Convex Functions

Convex functions class is again introduced by Robertson. A domain D is said to be convex if the linear segment joining any two points $z_0, z_1 \in D$ lies entirely in D. This means that a domain is convex if it is starlike with respect to each points of D. A convex function is the function that maps the unit disk U conformally onto a convex domain.

The convex functions can be represented analytically as

$$Re\left(1 + z\frac{f''(z)}{f'(z)}\right) > 0,$$

where $z \in U$ and $f(z) \in S$. The class of convex functions is denoted by $\mathcal{C}$.

While starlike function class S^* satisfy the same growth, distortion and coefficient bounds inequalities with the class of univalent functions, convex functions have stronger bounds. The following growth and distortion theorems are given for normalized convex functions:

Theorem 4.7. Let $f(z)$ be a function of $\mathcal{C}$ and $|z| < 1$. Then

$$\frac{r}{1+r} \leq |f(z)| \leq \frac{r}{1-r},$$

$$\frac{1}{(1+r)^2} \leq |f'(z)| \leq \frac{1}{(1-r)^2}.$$

Equality holds for the function $f(z) = \dfrac{z}{1-\lambda z}$ where $|\lambda| = 1$ and $\lambda \in \mathbb{C}$.

Convex functions class is contained in the class of starlike functions. Analytic expressions of starlike and convex functions result in a connection between that classes S^* and $\mathcal{C}$. This connection was first observed by Alexander in 1915 and stated by Alexander's theorem:

Alexander's Theorem.

Let $f(z)$ be a normalized analytic function in the unit disc U with $f(0)=0$ and $f'(0)=1$. Then $f(z)$ is in $\mathcal{C}$ if and only if $zf'(z)\in S$.

Following theorem proved by Loewner is a consequence of Alexander's theorem and gives the bounds for the coefficients of convex functions.

Theorem 4.8. If $f(z)=z+a_2z^2+a_3z^3+\dots$ is a convex function in the unit disc U, then the bounds for the coefficients of the Taylar series expansion is given by $|a_n|\le 1$ for $n\ge 2$.

The equality holds for the function $f(z)=\dfrac{z}{1-\lambda z}$ and its rotations where $|\lambda|=1$ and $\lambda\in\mathbb{C}$.

Radius of Convexity.

For a positive number $\rho\le 2-\sqrt{3}$, each function $f(z)\in S$ maps the unit disc $|z|<\rho$ onto a convex domain. The same is wrong for $\rho>2-\sqrt{3}$.

4.3.4 Starlike and Convex Functions of Order α

In this section, we introduce two subclasses of starlike and convex functions on the unit disc U. The classes of starlike and convex functions of order α were first introduced by Robertson.

For an analytic function $f:U\to\mathbb{C}$, starlikeness of order α is defined as follows:

Definition 4.1. Let $f : U \to \mathbb{C}$ be an analytic function. $f(z)$ is called starlike of order α for $0 \le \alpha < 1$ if the followings are satisfied

i. $f(0) = 0$, $f'(0) \neq 0$,

ii. $Re\left(\dfrac{zf'(z)}{f(z)}\right) > \alpha, \qquad z \in U.$

The class of starlike functions of order α is denoted by $S^*(\alpha)$.

Definition 4.2. $f(z)$ is said to be convex of order α, $0 \le \alpha < 1$ if

$$f'(0) \neq 0 \quad \text{and} \quad Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$$

where $z \in U$. $\mathcal{C}(\alpha)$ denotes the class of convex functions of order α.

Growth and distortion theorems for starlike and convex functions of order α are given due to Robertson:

Theorem 4.9. If $f(z)$ is a function of the class $\mathcal{C}(\alpha)$ where $0 \le \alpha < 1$ and $|z| = r < 1$, then

$$\frac{1}{(1+r)^{2(1-\alpha)}} \le |f'(z)| \le \frac{1}{(1-r)^{2(1-\alpha)}},$$

$$\frac{(1+r)^{2\alpha-1} - 1}{2\alpha - 1} \le |f(z)| \le \frac{1-(1-r)^{2\alpha} - 1}{2\alpha - 1}, \qquad \alpha \neq \frac{1}{2},$$

$$\log(1+r) \le |f(z)| \le -\log(1-r), \qquad \alpha = \frac{1}{2}.$$

These inequalities are sharp. The function

$$f(z) = \begin{cases} \dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1}, & \alpha \neq 1/2, \\ -\log(1-z), & \alpha = 1/2 \end{cases}$$

satisfy the equality.

Theorem 4.10. Growth theorem for functions $f(z)$ in the class $S^*(\alpha)$, $0 \le \alpha < 1$ is

$$\frac{r}{(1+r)^{2(1-\alpha)}} \le |f(z)| \le \frac{r}{(1-r)^{2(1-\alpha)}}.$$

The function

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}}, \quad z \in U$$

satisfies the equality.

4.3.5 Close-to-Convex Functions

These functions were identified by Kaplan in 1952 [19]. Let $f(z)$ be analytic in the unit disk U. $f(z)$ is called close-to-convex function if there exists a convex function $g(z)$ such that

$$Re\left(\frac{f'(z)}{g'(z)}\right) > 0$$

for all $z \in D$. Here the function $g(z)$ does not have to be normalized. The class of close-to-convex functions is denoted by $\mathcal{K}$.

Chapter 5

Harmonic Functions

5.1 Harmonic Functions

A complex function of two real variables, $f(z)=u(x,y)+iv(x,y)$, is harmonic in its domain $D\subset\mathbb{C}$, if u and v are real harmonic functions in D. A real-valued function u of two variables x and y which has continuous first and second order partial derivatives in an open set is said to be harmonic if it satisfies the Laplace equation :

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0. \qquad (5.1)$$

For an analytic function $f(z)=u(x,y)+iv(x,y)$ which is complex-valued, the functions $u(x,y)$ and $v(x,y)$ are harmonic in its domain D. Since an analytic function $f(z)$ satisfies Cauchy-Riemann equations, we have

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

If we differentiate the first equality with respect to x and the second one with respect to y, then we have

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial y\partial x}, \qquad \frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial x\partial y}. \qquad (5.2)$$

By Schwarz theorem, we know that the mixed partial differentials are equal since we assume that the functions $u(x,y)$ and $v(x,y)$ are continuous. By the addition of the functions in (5.2), it is easy to see that

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

Then the function $u(x,y)$ is harmonic. The same is valid for the function $v(x,y)$. Here, the functions $u(x,y)$ and $v(x,y)$ are harmonic conjugate since the function $f(z)$ is analytic.

Conversely, if we know that a function $u(x,y)$ as the real part of the complex-function $f(z)$ is harmonic in a domain D, then we can construct a unique harmonic conjugate $v(x,y)$. By this way, we can have a function $f(z)$ analytic in D.

We can also represent the real-valued functions of two real-variables $u(x,y)$ and $v(x,y)$ in terms of conjugate coordinates. Since we express a complex variable with a pair of real-variables x and y as $z = x + iy$ and the conjugate of z as $\overline{z} = x - iy$, one can write:

$$x = \frac{z+\overline{z}}{2} \qquad y = \frac{z-\overline{z}}{2}.$$

Then the functions become

$$u(x,y) = u\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2}\right),$$
$$v(x,y) = v\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2}\right)$$

and we can express the partial derivatives of $f(z)$ with respect to z and $\overline{z}$ as:

$$\frac{\partial f}{\partial z} = f_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right),$$
$$\frac{\partial f}{\partial \overline{z}} = f_{\overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Using the derivatives with respect to z and $\overline{z}$, the Laplacian of the function f can be written as :

$$\Delta f = 4\frac{\partial^2 f}{\partial z \partial \overline{z}} = 4\frac{\partial^2 f}{\partial \overline{z} \partial z}. \tag{5.3}$$

By direct calculations, these equalities can be seen easily.

A necessary condition for analyticity of a function $f(z)=u(x,y)+iv(x,y)$ can be obtained from the conjugate derivatives. If a function $f(z)$ is analytic, then it is independent of $\bar{z}$.

$$\frac{\partial f}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right)$$

$$=\frac{1}{2}\left(\frac{\partial(u+iv)}{\partial x}+i\frac{\partial(u+iv)}{\partial y}\right)$$

$$=\frac{1}{2}\left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}+i\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\right]$$

$$=\frac{1}{2}\left[\frac{\partial u}{\partial x}-i\frac{\partial v}{\partial y}+i\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] \tag{5.4}$$

Since the function $f(z)$ analytic, it satisfies the Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$. Then the equation (5.4) becomes

$$\frac{\partial f}{\partial \bar{z}}=0.$$

Another necessary and sufficient condition for a function $f(z)$ to be harmonic is the derivative of the function with respect to the variable z to be analytic.

As we can see, there is a relationship between analytic functions and harmonic functions. An analytic function $f(z)$ is also a harmonic function. However, harmonic functions need not to be analytic. Compositions of analytic functions are again analytic, while harmonic functions are not preserved under composition. The real and imaginary parts of a harmonic functions do not have to be conjugate.

5.2 Harmonic Mappings

The functions

$$u = u(x,y)$$
$$v = v(x,y)$$

transform the points from $xy-$plane to $uv-$plane. If there is a one-to-one correspondence between the points of $xy-$plane and the points of $uv-$plane and the transformation functions $u(x,y)$ and $v(x,y)$ are harmonic, then we say that the mapping is a harmonic mapping. We refer to a univalent complex-valued harmonic function of one complex variable by harmonic mapping.

A complex-valued harmonic function is a harmonic mapping of a domain $D \subset \mathbb{C}$ if and only if it is univalent in D. The real and imaginary parts of a harmonic mapping need not be conjugate. However, the real and imaginary parts of a conformal mapping are harmonic conjugate and they satisfy the Cauchy-Riemann equations. As it is seen, harmonic mappings are generalization of conformal mappings.

Although the harmonic mappings were area of interest of differential geometers, it got the attention of complex analysts in the 1980s. The most significant contributions were provided by a paper of J. Clunie and Sheil-Small which can be referred as milestone in harmonic mapping studies. This paper pointed out the most of the properties of conformal mapping can be extended to the harmonic mappings.

$$J_f = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = u_x v_y - u_y v_x .$$

The determinant of the Jacobian of transformation by the mapping $f(z)$ measures how much the area in $xy-$plane is distorted by the given transformation.

If the given function $f(z)$ is analytic, then the Jacobian becomes

$$J_f = u_x^2 + v_x^2 = |f'(z)|^2 .$$

For analytic functions $f(z)$, the Jacobian does not vanish at a point z, if and only if the function f is locally univalent at z. By the theorem of Lewy which he showed in 1936, we can see that this result remains true also for harmonic mappings.

Lewy's Theorem.
If $f(z)$ is a complex-valued harmonic function that is locally univalent in a domain $D \subset \mathbb{C}$, then its Jacobian $J_f(z)$ is different from zero for all $z \in D$.

Since the Jacobian does not vanish, it takes positive or negative values. We say that a harmonic mapping is sense-preserving if $J_f(z) > 0$ in the domain D or sense-reversing if $J_f(z) < 0$. If the function $f(z)$ is sense-preserving harmonic function, the conjugate of the function f, i.e., $\overline{f}$ is sense-reversing [8], [21].

Jacobian of a function can be expressed in terms of the partial derivatives with respect to conjugate coordinates as :

$$J_f = |f_z|^2 - |f_{\bar{z}}|^2 .$$

Then, the sense-preserving and sense-reversing conditions can be stated as follows:

- $f(z)$ is sense-preserving where $|f_z(z)| > |f_{\bar{z}}(z)|$,
- $f(z)$ is sense-reversing where $|f_z(z)| < |f_{\bar{z}}(z)|$.

For sense-preserving mappings $w = f(z)$ we can write the inequality

$$(|f_z| - |f_{\bar{z}}|)|dz| \leq |dw| \leq (|f_z| + |f_{\bar{z}}|)|dz| .$$

Interpreting these inequalities geometrically, one can see that

$$D_f = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}$$

is the ratio of the major and minor axes which f maps an infinitesimal circle onto an infinitesimal ellipse with. D_f is called the first dilatation of f. It is obvious that $1 \le D_f < \infty$. If the first dilatation of the sense-preserving function f is bounded above with a real number K in the given region, then f is said to be $K-$ quaziconformal. In case of $K=1$, f is reduced to conformal mapping.

The ratio of $f_{\bar{z}}$ and f_z, $\mu(z) = \dfrac{f_{\bar{z}}}{f_z}$, is called the second dilatation or complex dilatation of f. Since the function $f(z)$ is sense-preserving, $|\mu(z)| < 1$ for every z in the given region. It can be observed that $D_f(z) \le K$ if and only if $|\mu(z)| \le \dfrac{K-1}{K+1}$.

Let $h(z)$ and $g(z)$ be analytic functions defined in a simply connected domain D. There is no loss of generality in taking the unit disk as the domain by Riemann mapping theorem. So, let our domain D be the unit disk $U = \{z : |z| < 1\}$. A complex-valued harmonic function can be uniquely represented as $f(z) = h(z) + \overline{g(z)}$ with $g(0) = 0$. In this canonical form, the analytic function $h(z)$ is called the analytic part of f and $g(z)$ is called as the co-analytic part of f. The analytic functions $h(z)$ and $g(z)$ have the representation in power series as:

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

The Jacobian of the function is $J_f = |h'(z)|^2 - |g'(z)|^2$. If f is sense-preserving, the inequality $|g'(z)| < |h'(z)|$ holds by Lewy's theorem.

If the function defined as $f(z) = h(z) + \overline{g(z)}$ is analytic and univalent in the unit disk, then f is said to be a harmonic univalent function.

Subclasses of harmonic univalent functions are described by normalizations will be demonstrated.

- S_H denotes the class of harmonic functions defined by

$$\begin{cases} h(z) & = z + a_2 z^2 + \dots \\ g(z) & = b_1 z + b_2 z^2 + \dots \end{cases}$$

  where $a_0 = b_0 = 0$ and $a_1 = 1$. If $g(z) = 0$, then the class is reduced t to univalent functions class S.

- The class of harmonic functions normalized by $g'(0) = 0$, i.e., $b_1 = 0$, is denoted by S_H^0.

- If the function $f(z) = h(z) + \overline{g(z)}$ transforms the unit disc U onto a starlike domain, then the harmonic function $f(z)$ is called starlike harmonic function. The class of starlike harmonic functions is denoted by S_H^*.

- The harmonic function f that transforms the open unit disc onto a convex domain is said to be a convex harmonic function. $\mathcal{C}_H$ denotes the class of convex harmonic functions.

- Let

$$h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1}, \quad g(z) = \sum_{n=1}^{\infty} b_{n+m-1} z^{n+m-1},$$

  where $|b_m| < 1$ be analytic functions in the open unit disc U.

  $f(z) = h(z) + \overline{g(z)}$ defines a multivalent harmonic function if $f(z)$ is sense-preserving in U. The class of such functions is denoted by $S_H(m)$.

Alexander's Theorem.

Let $f(z)=h(z)+\overline{g(z)}$ be a starlike harmonic function. If the functions $H(z)$ and $G(z)$ defined as

$$\begin{cases} zH'(z)=h(z)\Rightarrow H'(z)=\dfrac{h(z)}{z} \\ zG'(z)=-g(z)\Rightarrow G'(z)=-\dfrac{g(z)}{z} \end{cases}$$

are analytic in the open unit disc, then the function $F(z)=H(z)+\overline{G(z)}$ is a convex harmonic function.

In other words, the necessary and sufficient condition for $f(z)$ to be starlike is the convexity of $zf'(z)$.

5.3 Subclasses of Harmonic Functions

5.3.1 Starlike Harmonic Functions.

A sense-preserving harmonic mapping $f\in S_H$ is said to be starlike if it transforms the unit disc onto a starlike domain. $arg\left\{f(e^{i\theta})\right\}$ should be a non-decreasing function of θ, which is,

$$\frac{d}{d\theta}arg\left\{f(e^{i\theta})\right\}\geq 0.$$

For analytic functions f, this requirement takes the form:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>0, \qquad z\in U.$$

Theorem 5.1. The coefficients of every starlike function $f\in S_H^0$ satisfy the sharp inequalities

$$|a_n|\leq\frac{1}{6}(2n+1)(n+1), \qquad |b_n|\leq\frac{1}{6}(2n-1)(n-1),$$

and

$$\left||a_n|-|b_n|\right| \le n, \qquad n = 2,3,\ldots.$$

Theorem 5.2. The coefficients of starlike functions f of the class S_H satisfy the inequalities

$$|a_n| \le \frac{1}{3}(2n^2+1), \qquad |b_n| \le \frac{1}{3}(2n^2+1),$$

for $n = 2,3,,\ldots$.

Theorem 5.3. The growth for starlike functions in the class S_H^0 is shown with the sharp inequality

$$|f(z)| \le \frac{1}{3}\cdot\frac{3r+r^3}{(1-r)^3}, \qquad |z| = r < 1.$$

The harmonic Koebe function satisfies the equality.

Subordination Principle for Harmonic Mappings.

Let $f(z) = h(z) + \overline{g(z)}$ and $F(z) = H(z) + \overline{G(z)}$ be harmonic functions in the open unit disc $U = \{z : |z| < 1\}$ and let $\omega_1(z)$, $\omega_2(z)$ be the second dilatations of the functions f and F, respectively. Then we say that f is subordinate to F if

$$h(z) = H(\phi(z)), \qquad g(z) = G(\phi(z)), \qquad \omega_1(z) = \omega_2(\phi(z))$$

are satisfied where $\phi(z)$ is an analytic function that satisfies $\phi(0) = 0$ and $|\phi(z)| < 1$ for all z in the unit disc.

Chapter 6
Multivalent Functions

In this chapter of the thesis, multivalent functions and some results on a subclass of multivalent functions will be introduced. Multivalent functions are considered as a generalization of univalent functions. Thus, we will obtain distortion and growth theorems by extending the results in the earlier chapters to multivalent harmonic mappings.

Let $f(z)$ be an analytic function in a subset D of the complex plane and the number of the roots of $w = f(z)$ be $n(w)$ for any w in D. Then we have the following cases:

i. $p-$**valent:** If the function $w = f(z)$ takes each of its values at most p times, i.e., if $n(w) \le p$, where p is a positive integer, $f(z)$ is called $p-$valent in D. In the case when $p = 1$, $p-$valent function $f(z)$ is univalent. Geometrically this means that above each point of the $w-$plane lie at most p points of the Riemann surface into which $w = f(z)$ maps D.

ii. $p-$**valent in the mean over circles:** The function $f(z)$ is called $p-$valent in the mean over circles in D if for all $R > 0$

$$\frac{1}{2\pi}\int_0^{2\pi} n(Re^{i\phi})d\phi \le p.$$

Geometrically this means that the linear measure of the arc on the Riemann surface to which $w = f(z)$ maps D and projecting to the circle $|w| = R$ does not exceed p times the length of this circle.

iii. $p-$**valent in the mean over areas:** A function $f(z)$ is called $p-$ valent in the mean over areas in D if

$$\int_0^R \left(\int_0^{2\pi} n(\rho e^{i\phi}) d\phi \right) \rho d\rho \leq p\pi R^2$$

for all $R > 0$. Geometrically this means that the area of a part of the Riemann surface to which $w = f(z)$ maps D and projecting to a disc $|w| < R$ does not exceed p times the area of this disc.

Then we can see that if a function $w = f(z)$ is $p-$valent in the mean over circles in a domain, then it is also $p-$valent in the mean over areas. However the converse does not hold. Moreover, if a function which is p-valent in a domain D is also $p-$ valent in the mean over circles in it, and a function which is $p-$valent in the mean over circles is $p-$valent in the mean over areas.

These definitions lead us to the following relation:

f is univalent $\Rightarrow f$ is $p-$valent $\Rightarrow f$ is $p-$valent in the mean over circles $\Rightarrow f$ is $p-$valent in the mean over areas

Multivalent functions have many extremal properties similar to univalent functions. Thus, some results in univalent functions such as distortion theorems, coefficient estimates can be generalized to multivalent functions.

Theorem 6.1. If a function

$$f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \ldots, \qquad p: \text{positive integer}$$

is analytic and $p-$valent in the unit disc U, then

$$\left|a_{p+1}\right| \leq 2p. \tag{6.1}$$

This inequality was proved for the functions $p-$valent in the means over areas in 1941 by Spencer [31].

The following distortion theorems were given by Hayman in 1950 for functions $p-$valent in means over circles. A generalization to $p-$valent in the means over areas was introduced by Garabedian and Royden in 1954.

Theorem 6.2. For a function $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ where p is a positive integer and $|z| = r$, $0 < r < 1$, then

$$\frac{r^p}{(1+r)^{2p}} \leq |f(z)| \leq \frac{r^p}{(1-r)^{2p}},$$

$$|f'(z)| \leq \frac{p(1+r)}{r(1-r)}|f(z)| \leq \frac{pr^{p-1}(1+r)}{(1-r)^{2p+1}}.$$

To obtain bounds for the coefficients of a $p-$valent function $f(z) = \sum_{n=1}^{\infty} b_n z^n$ in the unit disc U, Goodman gave the estimation for $n > p$ where p is a positive integer

$$|b_n| \leq \sum_{k=1}^{p} \frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)}|b_k|. \tag{6.2}$$

To simplfy the estimation of the boundaries of the coefficients, Cartwright showed that the coefficients $a_{p+1}, a_{p+2}, \ldots$ can be restricted depending on the first $p+1$ coefficient $a_0, a_1, \ldots, a_{p+1}$ of the function.

This estimation says that the $n-$th coefficient can be bounded by a linear combination of the first p coefficients. If $p=1$, then de-Branges Theorem for univalent functions is obtained.

6.1 Subclasses of Multivalent Functions

6.1.1 $p-$valent Analytic Functions.

We denote by $S(p,n)$, $p\geq 1$ and $n\geq 1$ integers, the class of all regular and $p-$valent functions in U. Let

$$s(z)=z^{p}+c_{np+1}z^{np+1}+c_{np+2}z^{np+2}+\dots \tag{6.3}$$

for all $z\in U$.

It is clear that $S(p,1)\supset S(p,2)\supset S(p,3)\supset\dots\supset S(p,m)\supset\dots$.

6.1.2 $p-$valent Starlike Functions.

Let $S^{*}(p,n)$ ($p\geq 1$ and $n\geq 1$ integers,) denote the class of functions of the form (6.3) which are regular in U and satisfying

$$Re\left(z\frac{s'(z)}{s(z)}\right)>0$$

and

$$\int_{0}^{2\pi} Re\left(z\frac{s'(z)}{s(z)}\right)d\theta=2pn\pi \tag{6.4}$$

for every $z\in U$. A member of $S^{*}(p,n)$ is called $p-$valent starlike function in the unit disc U.

6.1.3 $p-$valent Convex Functions.

Let $f(z)\in S(p,n)$. If $f(z)$ satisfies the inequality

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0 \tag{6.5}$$

for every $z \in U$, then $f(z)$ is said to be a $p-$valent convex function in U and the class of such functions is denoted by $\mathcal{C}(p,n)$.

Finally, a planar harmonic mapping in the open unit disc U is a complex-valued harmonic function f, which maps U onto the some planar domain $f(U)$. Since U is a simply connected domain, the mapping f has a canonical decomposition $f(z)=h(z)+\overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in U and have the following power series expansion

$$h(z)=z^p+a_{np+1}z^{np+1}+a_{np+2}z^{np+2}+\ldots+a_{np+m}z^{np+m}+\ldots$$

and

$$g(z)=b_{np}z^{np}+b_{np+1}z^{np+1}+b_{np+2}z^{np+2}+\ldots+b_{np+m}z^{np+m}+\ldots$$

where $|b_{np}|<1$, $p\geq 1$ and $n\geq 1$ integers, $a_{np+m}, b_{np+m}\in\mathbb{C}$ and for every $z\in U$. As usual, we call $h(z)$ the analytic part and $g(z)$ the co-analytic part of f, respectively. Let the class of such harmonic mappings is denoted by $SH(p,n)$.

Throught this thesis, we restrict ourselves to the study of sense-preserving harmonic mappings. The main aim of this thesis is to investigate the some properties of the following class

$$S^*H(p,n)=\left\{f=h+\overline{g}\in SH(p,n) \,\middle|\, \omega(z)=\frac{g'(z)}{h'(z)}\prec b_{np}\frac{1+\phi(z)}{1-\phi(z)},\right.$$

$$\left.\phi(z)=z^n\psi(z),\psi\in\Omega_1,h(z)\in S^*(p,n),z\in U\right\}.$$

Following lemma is necessary to investigate some properties of this class.

Lemma 6.1. [10] Let $w(z)=a_nz^n+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+\ldots$, $a_n\neq 0$, $n\geq 1$ be analytic in U. If the maximum value of $|w(z)|$ on the circle $|z|=r<1$ is attained at $z=z_0$, then we have $z_0w'(z_0)=pw(z_0)$ where $p\geq n$ and every $z\in U$.

6.2 Some Results on a Subclass of Multivalent Harmonic Functions

Let Ω_1 be the family of functions $\varphi(z)$ which are analytic in the open unit disc and satisfying the condition $|\varphi(z)|<1$ for all $z\in U$. Let Ω_2 be the family of functions $\phi(z)$ which are regular in U and satisfying the conditions $\phi(0)=0$ and $|\phi(z)|<1$ for every $z\in U$. Also let $\mathrm{P}(p,n)$ denote the family of functions $p(z)=p+p_1z+\ldots$ which are regular in the open unit disc and satisfy the condition $\operatorname{Re} p(z)>0$ where $p\geq 1$, $n\geq 1$.

Lemma 6.2. If $p(z)\in \mathrm{P}(p,n)$ then

$$p(z)=p\frac{1+z^n\psi(z)}{1-z^n\psi(z)},\quad z\in U \tag{6.6}$$

where $\psi(z)\in\Omega_1$.

Proof. *Consider the function $H(z)$ such that*

$$H(z)=\frac{p(z)}{p}, z\in U$$

where $p(z)\in \mathrm{P}(p,n)$. So that $H(z)$ is regular and satisfies the conditions $Re\, H(z)>0$ and $H(0)=1$ in U.

Let $\varphi(z)=\dfrac{1+H(z)}{1-H(z)}$, then $\varphi(z)$ is regular and $|\varphi(z)|<1$ in the unit disk U, and also

$\varphi(z)$ has $n-th$ order zero at the origin. Hence, $\varphi(z)=z^n\psi(z)$ in Ω_1 for all $z\in U$. Expressing $H(z)$ in terms of $\varphi(z)$ we have

$$H(z)=\frac{1+\varphi(z)}{1-\varphi(z)},\quad z\in U.$$

Thus,

$$H(z)=\frac{p(z)}{p}=\frac{1+\varphi(z)}{1-\varphi(z)}=\frac{1+z^n\psi(z)}{1-z^n\psi(z)}$$

or

$$p(z) = p\frac{1+z^n\psi(z)}{1-z^n\psi(z)}$$

for all $z \in U$.

Lemma 6.3. Let $f(z) = h(z) + \overline{g(z)}$ be an element of $S^*H(p,n)$, then

$$\left|\omega(z) - \frac{b_{np}(1-r^{2m})}{1-|b_{np}|^2 r^{2m}}\right| \leq \frac{(1-|b_{np}|^2)r^m}{1-|b_{np}|^2 r^{2m}}, \quad |z| = r < 1 \tag{6.7}$$

where $m = np - p + 1$.

Proof. *Since* $f(z) = h(z) + \overline{g(z)} \in S^*H(p,n)$ *then*

$$\omega(z) = \frac{g'(z)}{h'(z)} = \frac{(b_{np}z^{np} + b_{np+1}z^{np+1} + b_{np+2}z^{np+2} + \ldots)'}{(z^p + a_{np+1}z^{np+1} + a_{np+2}z^{np+2} + \ldots)'}$$

$$= \frac{b_n p + \dfrac{(np+1)b_{np+1}}{p} z^{np+1-p} + \ldots}{1 + \dfrac{(np+1)a_{np+1}}{p} z^{np+1} + \ldots}$$

so that $\omega(0) = b_{np}$.

On the other hand, because of the sense-preserving property we have that $|\omega(z)| < 1$ *for every* $z \in U$. *Thus, the function defined by*

$$\phi(z) = \frac{\omega(z) - \omega(0)}{1 - \overline{\omega(0)}\omega(z)}, \quad z \in U$$

satisfies the conditions of Schwarz lemma. Therefore, we have the following subordination relation

$$\omega(z) = \frac{b_{np} + \phi(z)}{1 + \overline{b_{np}}\phi(z)} \Leftrightarrow \omega(z) \prec \frac{b_{np} + z^m}{1 + \overline{b_{np}}z^m}, \quad z \in U.$$

It is easy to see that the linear transformation $\frac{b_{np}+z^m}{1+\overline{b_{np}}z^m}$ *maps* $|z|=r$ *onto the circle with the center*

$$C(r)=\left(\frac{\alpha_1(1-r^{2m})}{1-|b_{np}^2r^{2m}|},\frac{\alpha_2(1-r^{2m})}{1-|b_{np}|^2r^{2m}}\right)$$

and having the radius

$$\rho(r)=\frac{(1-|b_{np}|^2)r^m}{1-|b_{np}|^2r^{2m}},$$

where $\alpha_1=Re\,b_{np}$ *and* $\alpha_2=Im\,b_{np}$ *, then we can write*

$$\left|\omega(z)-\frac{b_{np}(1-r^{2m})}{1-|b_{np}|^2r^{2m}}\right|\le\frac{(1-|b_{np}|^2)r^m}{1-|b_{np}|^2r^{2m}}$$

for all $|z|=r<1$.

As a simple consequence of Lemma 6.3, we give the following corollary.

Corollary 6.1. If $f(z)=h(z)+\overline{g(z)}\in S^*H(p,n)$, then

$$\frac{|b_{np}|-r^n}{1-|b_{np}|r^n}\le|\omega(z)|\le\frac{|b_{np}|+r^n}{1+|b_{np}|r^n},$$

$$\frac{(1-r^n)(1-|b_{np}|)}{1+|b_{np}|r^n}\le1-|\omega(z)|\le\frac{(1+r^n)(1-|b_{np}|)}{1+|b_{np}|r^n},$$

$$\frac{(1-r^n)(1+|b_{np}|)}{1-|b_{np}|r^n}\le1+|\omega(z)|\le\frac{(1+r^n)(1+|b_{np}|)}{1-|b_{np}|r^n},$$

and

$$\frac{(1-|b_{np}|^2)(1-r^{2n})}{(1+|b_{np}|r^n)^2}\le1-|\omega(z)|^2\le\frac{(1-|b_{np}|^2)(1-r^{2n})}{(1-|b_{np}|r^n)^2}$$

for all $|z|=r<1$.

Theorem 6.3. Let $s(z)$ be an element of $S^*(p,n)$, then the inequalities

$$\frac{r^p}{(1+r^n)^{2p/n}} \le |s(z)| \le \frac{r^p}{(1-r^n)^{2p/n}} \tag{6.8}$$

and

$$\frac{pr^{p-1}(1-r^n)}{(1+r^n)^{(2p/n)+1}} \le |s'(z)| \le \frac{pr^{p-1}(1+r^n)}{(1-r^n)^{(2p/n)+1}} \tag{6.9}$$

hold for every $|z| = r < 1$.

Proof. *Since* $f(z) = h(z) + \overline{g(z)} \in S^*H(p,n)$ *then we have* $z\frac{s'(z)}{s(z)} \prec p\frac{1+z^n}{1-z^n}$ *for all* z *in* U. *Therefore, the inequality* $\left| z\frac{s'(z)}{s(z)} \cdot \frac{p(1+r^{2n})}{1-r^{2n}} \right| \le \frac{2pr^n}{1-r^{2n}}$ *holds for every* $|z| = r < 1$. *Thus we have*

$$\frac{p(1-r^n)}{1+r^n} \le \left| z\frac{s'(z)}{s(z)} \right| \le \frac{p(1+r^n)}{1-r^n} \tag{6.10}$$

or

$$\frac{p(1-r^n)}{1+r^n} \le Re\, z\frac{s'(z)}{s(z)} \le \frac{p(1+r^n)}{1-r^n} \tag{6.11}$$

for all $|z| = r < 1$. *It is fact that*

$$Re\, z\frac{s'(z)}{s(z)} = r\frac{\partial}{\partial r}\log|s(z)| \tag{6.12}$$

true for every $|z| = r < 1$. *Considering (6.11) and (6.12) together we obtain*

$$\frac{p(1-r^n)}{r(1+r^n)} \le \frac{\partial}{\partial r}\log|s(z)| \le \frac{p(1+r^n)}{r(1-r^n)}, \quad |z| = r < 1. \tag{6.13}$$

Integrating (6.13) we get (6.8). On the other hand the inequality (6.10) can be written in the form

$$\frac{p(1-r^n)}{r(1+r^n)}|s(z)| \le |s'(z)| \le \frac{p(1+r^n)}{r(1-r^n)}|s(z)|, \quad |z| = r < 1. \tag{6.14}$$

Using (6.8) in (6.14) we get (6.9).

Theorem 6.4. Let $f(z)=h(z)+\overline{g(z)}$ be an element of $S^*H(p,n)$, then

$$\frac{g(z)}{h(z)}=b_{np}\frac{1+\phi(z)}{1-\phi(z)}$$

where $|b_{np}|<1$, $\phi(z)=z^n\psi(z)$ and $\psi(z)\in\Omega_1$ for every $z\in U$.

Proof. *Since $f(z)=h(z)+\overline{g(z)}\in S^*H(p,n)$, we can write*

$$\omega(U_r)=\left\{z\in\mathbb{C}\,\left|\,\left|\frac{g'(z)}{h'(z)}-b_{np}\frac{1+r^{2n}}{1-r^{2n}}\right|\le\frac{2|b_{np}|r^n}{1-r^{2n}},|z|=r<1\right.\right\}.$$

On the other hand, since $h(z)$ is an element of $S^*(p,n)$, the value of $\frac{h(z)}{zh'(z)}$ at a point z_1 on the circle $|z|=r$ is

$$\frac{h(z_1)}{zh'(z_1)}=\frac{1}{p}\cdot\frac{1-r^n}{1+r^n}.$$

Now, we define the function

$$\frac{g(z)}{h(z)}=\frac{1+\phi(z)}{1-\phi(z)}, \tag{6.15}$$

where $\phi(z)=z^n\psi(z)$, $\psi(z)\in\Omega_1$ and $z\in U$, then $\phi(z)$ analytic in U and $\phi(0)=0$. We need to show that $|\phi(z)|<1$ for all $z\in U$. Assume to the contrary that there exists a $z_1\in U$ such that $|\phi(z_1)|=1$. If we take the derivative of (6.15) and after simple calculations we get

$$\omega(z)=\frac{g'(z)}{h'(z)}=b_{np}\left(\frac{1+\phi(z)}{1-\phi(z)}+\frac{2z\phi'(z)}{(1-\phi(z))^2}\cdot\frac{h(z)}{zh'(z)}\right),\quad z\in U.$$

Considering (6.11), (6.12), (6.14) and Lemma 6.1 together we obtain that

$$\omega(z_1)=\frac{g'(z_1)}{h'(z_1)}=b_{np}\left(\frac{1+\phi(z_1)}{1-\phi(z_1)}+\frac{2p\phi'(z_1)}{(1-\phi(z_1))^2}\cdot\frac{1}{p}\cdot\frac{1-r^n}{1+r^n}\right)\notin\omega(U_r),\quad |z|=r.$$

But this is a contradiction, therefore $|\phi(z)|<1$ *for all* $z\in U$ *. Thus, for a function* $f(z)=h(z)+\overline{g(z)}$ *in* $S^*H(p,n)$ *we have*

$$\frac{g(z)}{h(z)}=b_{np}\frac{1+\phi(z)}{1-\phi(z)},\quad z\in U.$$

Corollary 6.2. Let $f(z)=h(z)+\overline{g(z)}$ be an element of $S^*H(p,n)$, then

$$\frac{p|b_{np}|r^{p-1}(1-r^n)^2}{(1+r^n)^{\frac{2p}{n}+2}}\leq|g'(z)|\leq\frac{p|b_{np}|r^{p-1}(1+r^n)^2}{(1-r^n)^{\frac{2p}{n}+2}} \tag{6.16}$$

and

$$\frac{|b_{np}|r^p(1-r^n)}{(1+r^n)^{\frac{2p}{n}+1}}\leq|g(z)|\leq\frac{|b_{np}|r^p(1+r^n)}{(1-r^n)^{\frac{2p}{n}+1}} \tag{6.17}$$

for every $|z|=r<1$.

Proof. *Using the definition of the class* $S^*H(p,n)$ *and Theorem 6.4, we obtain*

$$\frac{|b_{np}|(1-r^n)}{1+r^n}|h'(z)|\leq|g'(z)|\leq\frac{|b_{np}|(1+r^n)}{1-r^n}|h'(z)|$$

and

$$\frac{|b_{np}|(1-r^n)}{1+r^n}|h(z)|\leq|g(z)|\leq\frac{|b_{np}|(1+r^n)}{1-r^n}|h(z)|$$

for all $z\in U$*. If we use theorem 6.3 in the last equation in the last inequalities we obtain (6.16) and (6.17).*

Corollary 6.3. If $f(z)=h(z)+\overline{g(z)}\in S^*H(p,n)$, then

$$\frac{p^2r^{2(p-1)}(1-r^n)^3(1+|b_{np}|^2)}{(1+r^n)^{\frac{4p}{n}+1}(1+|b_{np}|r^n)^2}\leq J_f\leq\frac{p^2r^{2(p-1)}(1+r^n)^3(1-|b_{np}|^2)}{(1-r^n)^{\frac{4p}{n}+1}(1-|b_{np}|r^n)^2},\quad |z|=r<1.$$

This corollary is a simple consequences of Corollary 6.1, Theorem 6.3 and the following equalities

$$J_f = |h'(z)|^2 - |g'(z)|^2 = |h'(z)|^2 (1 - |\omega(z)|^2), \quad z \in U.$$

Corollary 6.4. If $f(z) = h(z) + \overline{g(z)}$ be an element of $S^* H(p,n)$, then

$$p(1-|b_{np}|)\int \frac{r^{p-1}(1-r^n)^2}{(1+r^n)^{\frac{2p}{n}+1}(1+|b_{np}|r^n)}dr \leq | f | \leq p(1+|b_{np}|)\int \frac{r^{p-1}(1+r^n)^2}{(1-r^n)^{\frac{2p}{n}+1}(1+|b_{np}|r^n)}dr$$

This corollary is a simple consequence of Corollary 6.1, Theorem 6.3 and the following inequalities

$|h'(z)|(1-|\omega(z)|)|dz| \leq |df| \leq |h'(z)|(1+|\omega(z)|)|dz|, z \in U.$

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Curriculum Vitae

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