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THE HYPERBOLIC SYSTEM OF EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. The initial value problem for the hyperbolic system of equations with nonlocal boundary conditions is studied. The positivity of the space operator A generated by this problem in interpolation spaces is established. The structure interpolation spaces of this state operator is studied. The positivity of this space operator in Hölder spaces is established. In applications, the stability estimates for the hyperbolic system of equations with nonlocal boundary conditions are obtained.
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1. Introduction.

Various local and nonlocal boundary value problems for partial differential equations can be considered as an abstract boundary value problem for the ordinary differential equation in a Banach space with a densely defined unbounded space operator. As is well-known that the study of the various properties of partial differential equations is based on the positivity property of the differential operators in Banach spaces [1]-[3].

The method of operators as a tool for the investigation of the solution hyperbolic differential equations in Hilbert and Banach spaces, has been systematically developed by several authors (see, e.g., [1]- [11]). It is known that (see, e.g., [12]- [16] and the references given therein) many applied problems in fluid mechanics, other areas of physics and mathematical biology were formulated into nonlocal mathematical models. However, such problems were not well investigated in general.

In present paper, we consider the initial value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + a(x)\frac{\partial u(t,x)}{\partial x} + \delta(u(t,x) - v(t,x)) = f_1(t,x), \\ 0 < x < l, \quad 0 < t < T, \\ \\ \frac{\partial v(t,x)}{\partial t} - a(x)\frac{\partial v(t,x)}{\partial x} + \delta v(t,x) = f_2(t,x), \\ 0 < x < l, \quad 0 < t < T, \\ \\ u(t,0) = \gamma u(t,l), 0 \le \gamma \le 1, \quad \beta v(t,0) = v(t,l), 0 \le \beta \le 1, \\ 0 \le t \le T, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad 0 \le x \le l \end{cases} \tag{1}$$

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for the hyperbolic system of equations with nonlocal boundary conditions. Here

$$a(x) \geq a > 0, \tag{2}$$

$u_0(x)$, $v_0(x)$, $(x \in [0,l])$, $f_1(t,x)$, $f_2(t,x)$, $((t,x) \in [0,T] \times [0,l])$ are given smooth functions and they satisfy every compatibility conditions which guarantees the problem (1) has a smooth solution $u(t,x)$ and $v(t,x)$. Note that the problem of sound waves (see [19]) and the problem on expansion of electricity oscillations (see [20]) can be replaced to the problem (1).

Let E be a Banach space and $A : D(A) \subset E \to E$ be a linear unbounded operator densely defined in E. We call A positive operator A in the Banach space if the operator $(\lambda I + A)$ has a bounded in E inverse and for any $\lambda \geq 0$, and the following estimate holds:

$$||(\lambda I + A)^{-1}||_{E \to E} \leq \frac{M}{\lambda + 1}. \tag{3}$$

Throughout the present paper, M denotes positive constants, which may differ in time and thus is not a subject of precision. However, we will use $M(\alpha, \beta, \cdots)$ to stress the fact that the constant depends only on $M(\alpha, \beta, \cdots)$.

For a positive operator A in the Banach space E, let us introduce the fractional spaces $E_\alpha = E_\alpha(E, A)$ $(0 < \beta < 1)$ consisting of those $v \in E$ for which the norm

$$\|v\|_{E_\alpha} = \sup_{\lambda > 0} \lambda^\alpha \|A(\lambda + A)^{-1} v\|_E + \|v\|_E$$

is finite.

Let us introduce the Banach space $\mathbb{C}^\alpha[0,l] = C^\alpha([0,l], R) \times C^\alpha([0,l], R)$ $(0 \leq \alpha \leq 1)$ of all continuous vector functions $u = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$ defined on $[0,l]$ and satisfying a Hölder condition for which the following norm is finite

$$\|u\|_{\mathbb{C}^\alpha[0,l]} = \|u\|_{\mathbb{C}[0,l]}$$

$$+ \sup_{\substack{x, x+\tau \in [0,l] \\ \tau \neq 0}} \frac{|u_1(x+\tau) - u_1(x)|}{|\tau|^\alpha} + \sup_{\substack{x, x+\tau \in [0,l] \\ \tau \neq 0}} \frac{|u_2(x+\tau) - u_2(x)|}{|\tau|^\alpha}.$$

Here $\mathbb{C}[0,l] = C([0,l], R) \times C([0,l], R)$ is the Banach space of all continuous vector functions $u = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$ defined on $[0,l]$ with norm

$$\|u\|_{\mathbb{C}[0,l]} = \max_{x \in [0,l]} |u_1(x)| + \max_{x \in [0,l]} |u_2(x)|.$$

We consider the state operator A generated by problem (1) defined by the formula

$$Au = \begin{pmatrix} a(x)\frac{du_1(x)}{dx} + \delta u_1(x) & -\delta u_2(x) \\ 0 & -a(x)\frac{du_2(x)}{dx} + \delta u_2(x) \end{pmatrix} \tag{4}$$

with domain

$$D(A) = \left\{ \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} : u_m(x), \frac{du_m(x)}{dx} \in C([0,l], R), m = 1, 2; \right.$$

$$\left. u_1(0) = \gamma u_1(l), \beta u_2(0) = u_2(l) \right\}.$$

We will study the resolvent of the operator $-A$, i.e.

$$A \begin{pmatrix} u \\ v \end{pmatrix} + \lambda \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \tag{5}$$

or

$$\begin{cases} a(x)\frac{du(x)}{dx} + (\delta+\lambda)\, u(x) - \delta v(x) = \varphi(x),\ 0 < x < l, \\ -a(x)\frac{dv(x)}{dx} + (\delta+\lambda)\, v(x) = \psi(x), 0 < x < l, \\ u(0) = \gamma u(l), \beta v(0) = v(l). \end{cases} \tag{6}$$

will be investigated. The Green's matrix function of A is constructed. The positivity of the operator A in the Banach space $\mathbb{C}([0,l])$ is established. It is proved that for any $\alpha \in (0,1)$ the norms in spaces $E_\alpha(\mathbb{C}[0,l], A)$ and $\overset{\circ}{\mathbb{C}}{}^{\alpha}[0,l]$ are equivalent. The positivity of A in the Hölder spaces of $\overset{\circ}{\mathbb{C}}{}^{\alpha}[0,l]$, $\alpha \in (0,1)$ is proved. In applications, the stability estimates for the solution of the problem (1) for the hyperbolic system of equations with nonlocal boundary conditions are obtained.

2. The Green's Matrix Function of A and Positivity of A in $\mathbb{C}$

Lemma 2.1. *For any $\lambda \geq 0$, equation (6) is uniquely solvable and the following formula holds:*

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = (A+\lambda)^{-1} \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} = \int_0^l G(x,s;\lambda) \begin{pmatrix} \varphi(s) \\ \psi(s) \end{pmatrix} ds, \tag{7}$$

where

$$G(x,s;\lambda) = \begin{pmatrix} G_{11}(x,s;\lambda) & G_{12}(x,s;\lambda) \\ 0 & G_{22}(x,s;\lambda) \end{pmatrix}.$$

Here

$$G_{11}(x,s;\lambda) = Q \cdot \begin{cases} \frac{1}{a(s)} \exp\left\{ -(\delta+\lambda) \int_s^x \frac{dy}{a(y)} \right\}, 0 \leq s \leq x, \\ \frac{\gamma}{a(s)} \exp\left\{ -(\delta+\lambda) \left(\int_0^x \frac{dy}{a(y)} + \int_s^l \frac{dy}{a(y)} \right) \right\}, x \leq s \leq l, \end{cases} \tag{8}$$

$$G_{22}(x,s;\lambda) = P \cdot \begin{cases} \frac{\beta}{a(s)} \exp\left\{ -(\delta+\lambda) \left(\int_x^l \frac{dy}{a(y)} + \int_0^s \frac{dy}{a(y)} \right) \right\}, 0 \leq s \leq x, \\ \frac{1}{a(s)} \exp\left\{ -(\delta+\lambda) \int_x^s \frac{dy}{a(y)} \right\}, x \leq s \leq l. \end{cases} \tag{9}$$

$$G_{12}(x,s;\lambda) = \delta \cdot \int_0^l G_{11}(x,p;\lambda) \cdot G_{22}(p,s;\lambda) dp, \tag{10}$$

$$P = \left(1 - \beta \cdot \exp\left\{ -(\delta+\lambda) \int_0^l \frac{dy}{a(y)} \right\} \right)^{-1},$$

$$Q = \left(1 - \gamma \cdot \exp\left\{ -(\delta+\lambda) \int_0^l \frac{dy}{a(y)} \right\} \right)^{-1}.$$

Proof. Using the resolvent equation (6), we get

$$-a(x)\frac{dv(x)}{dx}+(\delta+\lambda)\,v(x)=\psi(x),0<x<l,\beta v(0)=v(l).$$

Applying the Cauchy formula, we get

$$v(x)=\exp\left\{-(\delta+\lambda)\int\limits_{x}^{l}\frac{dy}{a(y)}\right\}v(l)$$

$$+\int\limits_{x}^{l}\frac{1}{a(s)}\exp\left\{-(\delta+\lambda)\int\limits_{x}^{s}\frac{dy}{a(y)}\right\}\psi(s)ds.$$

From this formula and nonlocal boundary condition $\beta v(0)=v(l)$ it follows that

$$v(x)=\beta\cdot P\cdot\exp\left\{-(\delta+\lambda)\int\limits_{x}^{l}\frac{dy}{a(y)}\right\}$$

$$\times\int\limits_{0}^{l}\frac{1}{a(s)}\exp\left\{-(\delta+\lambda)\int\limits_{0}^{s}\frac{dy}{a(y)}\right\}\psi(s)ds$$

$$+\int\limits_{x}^{l}\frac{1}{a(s)}\exp\left\{-(\delta+\lambda)\int\limits_{x}^{s}\frac{dy}{a(y)}\right\}\psi(s)ds$$

$$=\beta\cdot P\cdot\exp\left\{-(\delta+\lambda)\int\limits_{x}^{l}\frac{dy}{a(y)}\right\}$$

$$\times\int\limits_{0}^{x}\frac{1}{a(s)}\exp\left\{-(\delta+\lambda)\int\limits_{0}^{s}\frac{dy}{a(y)}\right\}\psi(s)ds$$

$$+P\cdot\int\limits_{x}^{l}\frac{1}{a(s)}\exp\left\{-(\delta+\lambda)\int\limits_{x}^{s}\frac{dy}{a(y)}\right\}\psi(s)ds$$

$$=\int\limits_{0}^{l}G_{22}(x,s;\lambda)\psi(s)ds.$$

Using the resolvent equation (6), we get

$$a(x)\frac{du(x)}{dx}+(\delta+\lambda)\,u(x)-\delta v(x)=\varphi(x),\ 0<x<l,u(0)=\gamma u(l).$$

Applying the Cauchy formula, we get

$$u(x)=\exp\left\{-(\delta+\lambda)\int\limits_{0}^{x}\frac{dy}{a(y)}\right\}u(0)$$

$$+\int\limits_{0}^{x}\frac{1}{a(s)}\exp\left\{-(\delta+\lambda)\int\limits_{s}^{x}\frac{dy}{a(y)}\right\}(\delta v(s)+\varphi(s))\,ds.$$

From this formula and nonlocal boundary condition $u(0) = \gamma u(l)$ it follows that

$$u(x) = \gamma \cdot Q \cdot \exp\left\{ -(\delta+\lambda)\int_0^x \frac{dy}{a(y)} \right\}$$

$$\times \int_0^l \frac{1}{a(s)} \exp\left\{ -(\delta+\lambda)\int_s^l \frac{dy}{a(y)} \right\} (\delta v(s) + \varphi(s))\, ds$$

$$+ \int_0^x \frac{1}{a(s)} \exp\left\{ -(\delta+\lambda)\int_s^x \frac{dy}{a(y)} \right\} (\delta v(s) + \varphi(s))\, ds$$

$$= \gamma \cdot Q \cdot \exp\left\{ -(\delta+\lambda)\int_0^x \frac{dy}{a(y)} \right\}$$

$$\times \int_x^l \frac{1}{a(s)} \exp\left\{ -(\delta+\lambda)\int_s^l \frac{dy}{a(y)} \right\} (\delta v(s) + \varphi(s))\, ds$$

$$+ \int_0^x \frac{1}{a(s)} \exp\left\{ -(\delta+\lambda)\int_s^x \frac{dy}{a(y)} \right\} (\delta v(s) + \varphi(s))\, ds$$

$$= \int_0^l G_{11}(x,s;\lambda)\,(\delta v(s) + \varphi(s))\, ds$$

$$= \int_0^l G_{11}(x,s;\lambda)\varphi(s)ds + \int_0^l G_{11}(x,p;\lambda)\delta v(p)dp$$

$$= \int_0^l G_{11}(x,s;\lambda)\varphi(s)ds$$

$$+\delta \int_0^l G_{11}(x,p;\lambda) \int_0^l G_{22}(p,s;\lambda)\psi(s)dsdp$$

$$= \int_0^l G_{11}(x,s;\lambda)\varphi(s)ds + \delta \int_0^l G_{12}(x,s;\lambda)\psi(s)ds.$$

Lemma 2.1 is proved. □

Lemma 2.2. *The following pointwise estimates hold:*

$$|P|\,, |Q| \le \frac{1}{1 - \exp\left\{ -\frac{(\delta+\lambda)l}{a} \right\}}, \tag{11}$$

$$|G_{11}(x,s;\lambda)| \le \frac{1}{a\left(1 - \exp\left\{ -\frac{(\delta+\lambda)l}{a} \right\}\right)} \tag{12}$$

$$\times \begin{cases} \exp\left\{-\frac{(\delta+\lambda)(x-s)}{a}\right\}, 0 \le s \le x, \\ \exp\left\{-\frac{(\delta+\lambda)(l+x-s)}{a}\right\}, x \le s \le l, \end{cases}$$

$$|G_{22}(x,s;\lambda)| \le \frac{1}{a\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)} \tag{13}$$

$$\times \begin{cases} \exp\left\{-\frac{(\delta+\lambda)(l-x+s)}{a}\right\}, 0 \le s \le x, \\ \exp\left\{-\frac{(\delta+\lambda)(s-x)}{a}\right\}, x \le s \le l, \end{cases}$$

$$|G_{12}(x,s;\lambda)| \le \frac{1}{a\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)} \tag{14}$$

$$\times \begin{cases} \exp\left\{-\frac{(\delta+\lambda)(x-s)}{a}\right\}, 0 \le s \le x, \\ \exp\left\{-\frac{(\delta+\lambda)(s-x)}{a}\right\}, x \le s \le l. \end{cases}$$

Proof. It is easy to see that estimates (11), (12) and (13) follow from the triangle inequality and the condition (2). Applying the triangle inequality and the condition (2), we get

$$|G_{12}(x,s;\lambda)| \le \delta \cdot \int_0^l |G_{11}(x,p;\lambda)| \cdot |G_{22}(p,s;\lambda)|\, dp.$$

If $0 \le s \le x$. Then, using estimates (11), (12) and (13), we get

$$|G_{12}(x,s;\lambda)| \le \delta \int_0^s |G_{11}(x,p;\lambda)| \cdot |G_{22}(p,s;\lambda)|\, dp$$

$$+\delta \int_s^x |G_{11}(x,p;\lambda)| \cdot |G_{22}(p,s;\lambda)|\, dp + \delta \int_x^l |G_{11}(x,p;\lambda)| \cdot |G_{22}(p,s;\lambda)|\, dp$$

$$\le \frac{\delta}{a^2\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)^2} \left(\int_0^s \exp\left\{-\frac{(\delta+\lambda)(x+s-2p)}{a}\right\} dp \right.$$

$$+\int_s^x \exp\left\{-\frac{(\delta+\lambda)(l+x+s-2p)}{a}\right\} dp$$

$$\left. +\int_x^l \exp\left\{-\frac{(\delta+\lambda)(2l+x+s-2p)}{a}\right\} dp \right)$$

$$\le \frac{\delta}{2a(\delta+\lambda)\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)^2} \left(\exp\left\{-\frac{(\delta+\lambda)(x-s)}{a}\right\} \right.$$

$$-\exp\left\{-\frac{(\delta+\lambda)(x+s)}{a}\right\} + \exp\left\{-\frac{(\delta+\lambda)(l+s-x)}{a}\right\}$$

$$-\exp\left\{-\frac{(\delta+\lambda)(l+x-s)}{a}\right\}$$

$$\left. +\exp\left\{-\frac{(\delta+\lambda)(x+s)}{a}\right\} - \exp\left\{-\frac{(\delta+\lambda)(2l-x+s)}{a}\right\} \right)$$

$$= \frac{\delta \exp\left\{-\frac{(\delta+\lambda)(x-s)}{a}\right\}}{2a(\delta+\lambda)\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)^2}\left(1-\exp\left\{-\frac{(\delta+\lambda)\,l}{a}\right\}\right)$$

$$\times\left(1+\exp\left\{-\frac{(\delta+\lambda)\,(l+2s-2x)}{a}\right\}\right)$$

$$\leq \frac{\exp\left\{-\frac{(\delta+\lambda)(x-s)}{a}\right\}}{a\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)}. \tag{15}$$

If $x \leq s \leq l$. Then, using estimates (11), (12) and (13), we get

$$|G_{12}(x,s;\lambda)| \leq \delta \int_0^x |G_{11}(x,p;\lambda)| \cdot |G_{22}(p,s;\lambda)|\, dp$$

$$+\delta \int_x^s |G_{11}(x,p;\lambda)| \cdot |G_{22}(p,s;\lambda)|\, dp + \delta \int_s^l |G_{11}(x,p;\lambda)| \cdot |G_{22}(p,s;\lambda)|\, dp$$

$$\leq \frac{\delta}{a^2\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)^2}\left(\int_0^x \exp\left\{-\frac{(\delta+\lambda)\,(x+s-2p)}{a}\right\} dp\right.$$

$$+\int_x^s \exp\left\{-\frac{(\delta+\lambda)\,(l+x+s-2p)}{a}\right\} dp$$

$$\left.+\int_s^l \exp\left\{-\frac{(\delta+\lambda)\,(2l+x+s-2p)}{a}\right\} dp\right)$$

$$\leq \frac{\delta}{2a(\delta+\lambda)\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)^2}\left(\exp\left\{-\frac{(\delta+\lambda)\,(s-x)}{a}\right\}\right.$$

$$-\exp\left\{-\frac{(\delta+\lambda)\,(x+s)}{a}\right\}+\exp\left\{-\frac{(\delta+\lambda)\,(l+x-s)}{a}\right\}$$

$$-\exp\left\{-\frac{(\delta+\lambda)\,(l+s-x)}{a}\right\}$$

$$\left.+\exp\left\{-\frac{(\delta+\lambda)\,(x+s)}{a}\right\}-\exp\left\{-\frac{(\delta+\lambda)\,(2l+x-s)}{a}\right\}\right)$$

$$= \frac{\delta \exp\left\{-\frac{(\delta+\lambda)(s-x)}{a}\right\}}{2a(\delta+\lambda)\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)^2}\left(1-\exp\left\{-\frac{(\delta+\lambda)\,l}{a}\right\}\right)$$

$$\times\left(1+\exp\left\{-\frac{(\delta+\lambda)\,(l+2x-2s)}{a}\right\}\right)$$

$$\leq \frac{\exp\left\{-\frac{(\delta+\lambda)(s-x)}{a}\right\}}{a\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)}. \tag{16}$$

Estimate (14) follows from estimates (15) and (16). Lemma 2.2 is proved. □

Theorem 2.1. *The operator $(\lambda I + A)$ has a bounded in $\mathbb{C}[0,l]$ inverse for any $\lambda \geq 0$ and the following estimate holds:*

$$\| (\lambda + A)^{-1} \|_{\mathbb{C}[0,l] \to \mathbb{C}[0,l]} \leq \frac{M_1}{1+\lambda}. \tag{17}$$

Proof. Using the formula (7) and the triangle inequality, we get

$$|u(x)| \leq \int_0^l |G_{11}(x,s;\lambda)|\, ds \max_{0 \leq s \leq l} |\varphi(s)|$$

$$+ \int_0^l |G_{12}(x,s;\lambda)|\, ds \max_{0 \leq s \leq l} |\psi(s)|, \tag{18}$$

$$|v(x)| \leq \int_0^l |G_{22}(x,s;\lambda)|\, ds \max_{0 \leq s \leq l} |\psi(s)|. \tag{19}$$

for any $x \in [0,l]$. Using estimate (12), we get

$$\int_0^l |G_{11}(x,s;\lambda)|\, ds \leq \frac{1}{a\left(1 - \exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)} \tag{20}$$

$$\times \left(\int_0^x \exp\left\{ -\frac{(\delta+\lambda)(x-s)}{a} \right\} ds + \int_x^l \exp\left\{ -\frac{(\delta+\lambda)(l+x-s)}{a} \right\} ds \right)$$

$$\leq \frac{1}{a\frac{(\delta+\lambda)}{a}\left(1 - \exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)}$$

$$\times \left(1 - \exp\left\{ -\frac{(\delta+\lambda)\,x}{a} \right\} + \exp\left\{ -\frac{(\delta+\lambda)\,x}{a} \right\} - \exp\left\{ -\frac{(\delta+\lambda)\,l}{a} \right\} \right)$$

$$= \frac{1}{\delta+\lambda}.$$

Using estimate (13), we get

$$\int_0^l |G_{22}(x,s;\lambda)|\, ds \leq \frac{1}{a\left(1 - \exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)} \tag{21}$$

$$\times \left(\int_0^x \exp\left\{ -\frac{(\delta+\lambda)(l-x+s)}{a} \right\} ds + \int_x^l \exp\left\{ -\frac{(\delta+\lambda)(s-x)}{a} \right\} ds \right)$$

$$\leq \frac{1}{a\frac{(\delta+\lambda)}{a}\left(1 - \exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)}$$

$$\times \left(-\exp\left\{ -\frac{(\delta+\lambda)\,l}{a} \right\} + \exp\left\{ -\frac{(\delta+\lambda)(l-x)}{a} \right\} - \exp\left\{ -\frac{(\delta+\lambda)(l-x)}{a} \right\} - 1 \right)$$

$$= \frac{1}{\delta+\lambda}.$$

Using estimate (14), we get

$$\int_0^l |G_{12}(x,s;\lambda)|\,ds \leq \frac{1}{a\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)} \tag{22}$$

$$\times\left(\int_0^x \exp\left\{-\frac{(\delta+\lambda)(x-s)}{a}\right\}ds + \int_x^l \exp\left\{-\frac{(\delta+\lambda)(s-x)}{a}\right\}ds\right)$$

$$\leq \frac{1}{a\frac{(\delta+\lambda)}{a}\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)}$$

$$\times\left(1-\exp\left\{-\frac{(\delta+\lambda)x}{a}\right\}-\exp\left\{-\frac{(\delta+\lambda)(l-x)}{a}\right\}+1\right)$$

$$\leq \frac{2}{\delta+\lambda}.$$

Using estimates (18), (19), (20), (21) and (22), we get

$$\max_{x\in[0,l]}|u(x)| \leq \frac{1}{\delta+\lambda}\max_{0\leq s\leq l}|\varphi(s)| + \frac{2}{\delta+\lambda}\max_{0\leq s\leq l}|\psi(s)| \leq \frac{2}{\delta+\lambda}\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|_{\mathbb{C}[0,l]},$$

$$\max_{x\in[0,l]}|v(x)| \leq \frac{1}{\delta+\lambda}\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|_{\mathbb{C}[0,l]}.$$

Then, we have that

$$\left\|\begin{pmatrix}u\\ v\end{pmatrix}\right\|_{\mathbb{C}[0,l]} \leq \frac{3}{\delta+\lambda}\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|_{\mathbb{C}[0,l]}.$$

From that it follows estimate (17). Theorem 2.1 is proved. □

3. The Structure of Fractional Spaces Generated by A and Positivity of A in Hölder Spaces

Clearly, the operator commutes A and its resolvent $(A+\lambda)^{-1}$. By the definition of the norm in the fractional space $E_\alpha = E_\alpha(\mathbb{C}[0,l],A)$, we get

$$\|(A+\lambda)^{-1}\|_{E_\alpha\to E_\alpha} \leq \|(A+\lambda)^{-1}\|_{\mathbb{C}[0,l]\to\mathbb{C}[0,l]}.$$

Thus, from Theorem 2.1 it follows that A is a positive operator in the fractional spaces $E_\alpha(\mathbb{C}[0,l],A)$. Moreover, we have the following result

Theorem 3.1. *For $\alpha\in(0,1)$, the norms of the spaces $E_\alpha(\mathbb{C}[0,l],A)$ and the Hölder space $\overset{\circ}{\mathbb{C}}{}^\alpha[0,l]$ are equivalent. Here*

$$\overset{\circ}{\mathbb{C}}{}^\alpha[0,l] = \left\{\begin{pmatrix}\varphi(x)\\ \psi(x)\end{pmatrix}\in\mathbb{C}^\alpha[0,l]:\right.$$

$$\left.\varphi(0)=\gamma\varphi(l), 0\leq\gamma\leq 1, \beta\psi(0)=\psi(l), 0\leq\beta\leq 1\right\}.$$

Proof. For any $\lambda\geq 0$ we have the obvious equality

$$A(A+\lambda)^{-1}\begin{pmatrix}\varphi(x)\\ \psi(x)\end{pmatrix} = \begin{pmatrix}\varphi(x)\\ \psi(x)\end{pmatrix} - \lambda(A+\lambda)^{-1}\begin{pmatrix}\varphi(x)\\ \psi(x)\end{pmatrix}.$$

By formula (7), we can write

$$A(A+\lambda)^{-1}\begin{pmatrix}\varphi(x)\\ \psi(x)\end{pmatrix}=\begin{pmatrix}\varphi(x)\\ \psi(x)\end{pmatrix}-\lambda\int_0^l G(x,s;\lambda)\begin{pmatrix}\varphi(s)\\ \psi(s)\end{pmatrix}ds \tag{23}$$

$$=\begin{pmatrix}\left(1-\lambda\int_0^x G_{11}(x,s;\lambda)ds\right)\varphi(x)-\lambda\int_x^l G_{11}(x,s;\lambda)ds\varphi(l)\\ 0\end{pmatrix}$$

$$+\begin{pmatrix}-\lambda\int_0^x G_{22}(x,s;\lambda)ds\psi(0)+\left(1-\lambda\int_x^l G_{22}(x,s;\lambda)ds\right)\psi(x)\\ -\lambda\int_0^l G_{12}(x,s;\lambda)ds\psi(x)\end{pmatrix}$$

$$+\begin{pmatrix}\lambda\int_0^x G_{11}(x,s;\lambda)\left(\varphi(x)-\varphi(s)\right)ds+\lambda\int_x^l G_{11}(x,s;\lambda)\left(\varphi(l)-\varphi(s)\right)ds\\ 0\end{pmatrix}$$

$$+\begin{pmatrix}\lambda\int_0^l G_{12}(x,s;\lambda)\left(\psi(x)-\psi(s)\right)ds\\ \lambda\int_0^x G_{22}(x,s;\lambda)\left(\psi(0)-\psi(s)\right)ds+\lambda\int_x^l G_{22}(x,s;\lambda)\left(\psi(x)-\psi(s)\right)ds\end{pmatrix}.$$

Applying formula (23) and the following obvious equalities

$$1-\lambda\int_0^x G_{11}(x,s;\lambda)ds=1-Q\cdot\lambda\int_0^x\frac{1}{a(s)}\exp\left\{-(\delta+\lambda)\int_s^x\frac{dy}{a(y)}\right\}ds$$

$$=1-\frac{\lambda}{\delta+\lambda}Q\cdot\left(1-\exp\left\{-(\delta+\lambda)\int_0^x\frac{dy}{a(y)}\right\}\right)$$

$$=\frac{\delta}{\delta+\lambda}\cdot Q-\gamma\cdot Q\cdot\exp\left\{-(\delta+\lambda)\int_0^l\frac{dy}{a(y)}\right\}+\frac{\lambda}{\delta+\lambda}Q\cdot\exp\left\{-(\delta+\lambda)\int_0^x\frac{dy}{a(y)}\right\},$$

$$-\lambda\int_x^l G_{11}(x,s;\lambda)ds=-Q\cdot\lambda\int_x^l\frac{\gamma}{a(s)}\exp\left\{-(\delta+\lambda)\left(\int_0^x\frac{dy}{a(y)}+\int_s^l\frac{dy}{a(y)}\right)\right\}ds$$

$$=-\frac{\gamma\lambda}{\delta+\lambda}Q\cdot\left(\exp\left\{-(\delta+\lambda)\int_0^x\frac{dy}{a(y)}\right\}-\exp\left\{-(\delta+\lambda)\int_0^l\frac{dy}{a(y)}\right\}\right),$$

$$1-\lambda\int_x^l G_{22}(x,s;\lambda)ds=1-\lambda P\cdot\int_x^l\frac{1}{a(s)}\exp\left\{-(\delta+\lambda)\int_x^s\frac{dy}{a(y)}\right\}ds$$

$$=1-\frac{\lambda}{\delta+\lambda}P\cdot\left(1-\exp\left\{-(\delta+\lambda)\int_x^l\frac{dy}{a(y)}\right\}\right)$$

$$= \frac{\delta}{\delta+\lambda} \cdot P - \beta \cdot P \cdot \exp\left\{ - (\delta+\lambda) \int_0^l \frac{dy}{a(y)} \right\}$$

$$+ \frac{\lambda}{\delta+\lambda} P \cdot \exp\left\{ - (\delta+\lambda) \int_x^l \frac{dy}{a(y)} \right\},$$

$$-\lambda \int_0^x G_{22}(x,s;\lambda) ds = -\lambda P \cdot \int_0^x \frac{\beta}{a(s)} \exp\left\{ - (\delta+\lambda) \left(\int_x^l \frac{dy}{a(y)} + \int_0^s \frac{dy}{a(y)} \right) \right\} ds$$

$$= -\frac{\lambda\beta}{\delta+\lambda} P \cdot \left(\exp\left\{ - (\delta+\lambda) \int_x^l \frac{dy}{a(y)} \right\} - \exp\left\{ - (\delta+\lambda) \int_0^l \frac{dy}{a(y)} \right\} \right),$$

$$-\lambda \int_0^l G_{12}(x,s;\lambda) ds = -\lambda \int_0^l \delta \cdot \int_0^l G_{11}(x,p;\lambda) \cdot G_{22}(p,s;\lambda) dp ds$$

$$= -\lambda \int_0^x \delta \cdot \int_0^s G_{11}(x,p;\lambda) \cdot G_{22}(p,s;\lambda) dp ds$$

$$-\lambda \int_0^x \delta \cdot \int_s^x G_{11}(x,p;\lambda) \cdot G_{22}(p,s;\lambda) dp ds$$

$$-\lambda \int_0^x \delta \cdot \int_x^l G_{11}(x,p;\lambda) \cdot G_{22}(p,s;\lambda) dp ds$$

$$-\lambda \int_x^l \delta \cdot \int_0^x G_{11}(x,p;\lambda) \cdot G_{22}(p,s;\lambda) dp ds$$

$$-\lambda \int_x^l \delta \cdot \int_x^s G_{11}(x,p;\lambda) \cdot G_{22}(p,s;\lambda) dp ds$$

$$-\lambda \int_x^l \delta \cdot \int_s^l G_{11}(x,p;\lambda) \cdot G_{22}(p,s;\lambda) dp ds$$

$$= -\frac{\lambda\delta}{2(\delta+\lambda)^2} Q \cdot P \cdot \left(\left(1 - \exp\left\{ - (\delta+\lambda) \int_0^l \frac{dy}{a(y)} \right\} \right)^2 \right.$$

$$+ \beta \exp\left\{ - (\delta+\lambda) \int_x^l \frac{dy}{a(y)} \right\} \left(1 - \exp\left\{ - (\delta+\lambda) \int_0^x \frac{dy}{a(y)} \right\} \right)^2$$

$$\left. + \beta\gamma \left(1 - \exp\left\{ - (\delta+\lambda) \int_0^x \frac{dy}{a(y)} \right\} \right) \right.$$

$$\times \exp\left\{-(\delta+\lambda)\int_0^x \frac{dy}{a(y)}\right\}\left(1-\exp\left\{-2(\delta+\lambda)\int_x^l \frac{dy}{a(y)}\right\}\right)$$

$$+\left(1-\exp\left\{-(\delta+\lambda)\int_x^l \frac{dy}{a(y)}\right\}\right)\left(1-\exp\left\{-2(\delta+\lambda)\int_0^x \frac{dy}{a(y)}\right\}\right)$$

$$+\gamma\exp\left\{-(\delta+\lambda)\int_0^x \frac{dy}{a(y)}\right\}\left(1-\exp\left\{-(\delta+\lambda)\int_x^l \frac{dy}{a(y)}\right\}\right)^2$$

$$\left.+\beta\gamma\exp\left\{-2(\delta+\lambda)\int_0^x \frac{dy}{a(y)}\right\}\left(1-\exp\left\{-(\delta+\lambda)\int_x^l \frac{dy}{a(y)}\right\}\right)^2\right)$$

and using nonlocal boundary conditions

$$\varphi(0)=\gamma\varphi(l), \beta\psi(0)=\psi(l),$$

we get

$$A(A+\lambda)^{-1}\begin{pmatrix}\varphi(x)\\ \psi(x)\end{pmatrix}=\left(\begin{pmatrix}\left(\frac{\delta}{\delta+\lambda}\cdot Q-\gamma\cdot Q\cdot\exp\left\{-(\delta+\lambda)\int_0^l \frac{dy}{a(y)}\right\}\right)\varphi(x)\\ -\frac{\lambda\delta}{2(\delta+\lambda)^2}Q\cdot P\cdot\left(\left(1-\exp\left\{-(\delta+\lambda)\int_0^l \frac{dy}{a(y)}\right\}\right)^2\right.\end{pmatrix}\right.$$

$$+\begin{pmatrix}\frac{\lambda}{\delta+\lambda}Q\cdot\exp\left\{-(\delta+\lambda)\int_0^x \frac{dy}{a(y)}\right\}(\varphi(x)-\varphi(0))\\ \beta\exp\left\{-(\delta+\lambda)\int_x^l \frac{dy}{a(y)}\right\}\left(1-\exp\left\{-(\delta+\lambda)\int_0^x \frac{dy}{a(y)}\right\}\right)^2\end{pmatrix}$$

$$+\begin{pmatrix}\frac{\gamma\lambda}{\delta+\lambda}Q\exp\left\{-(\delta+\lambda)\int_0^l \frac{dy}{a(y)}\right\}\varphi(l)\\ \beta\gamma\left(1-\exp\left\{-(\delta+\lambda)\int_0^x \frac{dy}{a(y)}\right\}\right)\end{pmatrix}$$

$$+\begin{pmatrix}\left(\frac{\delta}{\delta+\lambda}\cdot P-\beta\cdot P\cdot\exp\left\{-(\delta+\lambda)\int_0^l \frac{dy}{a(y)}\right\}\right)(\psi(x)-\psi(0))\\ \times\exp\left\{-(\delta+\lambda)\int_0^x \frac{dy}{a(y)}\right\}\left(1-\exp\left\{-2(\delta+\lambda)\int_x^l \frac{dy}{a(y)}\right\}\right)\end{pmatrix}$$

$$+\begin{pmatrix}-\exp\left\{-(\delta+\lambda)\int_0^l \frac{dy}{a(y)}\right\}\psi(0)\\ \left(1-\exp\left\{-(\delta+\lambda)\int_x^l \frac{dy}{a(y)}\right\}\right)\left(1-\exp\left\{-2(\delta+\lambda)\int_0^x \frac{dy}{a(y)}\right\}\right)\end{pmatrix}$$

$$\left.+\begin{pmatrix}0\\ \beta\gamma\exp\left\{-2(\delta+\lambda)\int_0^x \frac{dy}{a(y)}\right\}\left(1-\exp\left\{-(\delta+\lambda)\int_x^l \frac{dy}{a(y)}\right\}\right)^2\end{pmatrix}\right)\right)$$

$$+\left(\begin{array}{c}\lambda\int\limits_0^x G_{11}(x,s;\lambda)\left(\varphi(x)-\varphi(s)\right)ds+\lambda\int\limits_x^l G_{11}(x,s;\lambda)\left(\varphi(l)-\varphi(s)\right)ds\\ \gamma\exp\left\{-(\delta+\lambda)\int\limits_0^x\frac{dy}{a(y)}\right\}\left(1-\exp\left\{-(\delta+\lambda)\int\limits_x^l\frac{dy}{a(y)}\right\}\right)^2\end{array}\right)$$

$$+\left(\begin{array}{c}\lambda\int\limits_0^l G_{12}(x,s;\lambda)\left(\psi(x)-\psi(s)\right)ds\\ \lambda\int\limits_0^x G_{22}(x,s;\lambda)\left(\psi(l)-\psi(s)\right)ds+\lambda\int\limits_x^l G_{22}(x,s;\lambda)\left(\psi(x)-\psi(s)\right)ds\end{array}\right).$$

Using this formula, the triangle inequality, estimates (11), (12), (13) and (14), the definition of spaces $E_\alpha(\mathbb{C}[0,l],A)$ and $\mathbb{C}^{2\alpha}[0,l]$, we get

$$\left\|\lambda^\alpha A(A+\lambda)^{-1}\left(\begin{array}{c}\varphi\\ \psi\end{array}\right)\right\|_{\mathbb{C}[0,l]}$$

$$\leq\left(\frac{\lambda^\alpha\delta}{\delta+\lambda}|Q|+|Q|\lambda^\alpha\exp\left\{-\frac{(\delta+\lambda)\,l}{a}\right\}\right)\max_{0\leq x\leq l}|\varphi(x)|$$

$$+\frac{\lambda^{1+\alpha}}{\delta+\lambda}|Q|\exp\left\{-\frac{(\delta+\lambda)\,x}{a}\right\}x^\alpha\sup_{0\leq x\leq l}\frac{|\varphi(x)-\varphi(0)|}{x^\alpha}$$

$$+\frac{\lambda^{1+\alpha}}{\delta+\lambda}|Q|\exp\left\{-\frac{(\delta+\lambda)\,l}{a}\right\}|\varphi(l)|+\lambda^\alpha\exp\left\{-\frac{(\delta+\lambda)\,l}{a}\right\}|\psi(0)|$$

$$+\left(\frac{\lambda^\alpha\delta}{\delta+\lambda}|P|+|P|\lambda^\alpha\exp\left\{-\frac{(\delta+\lambda)\,x}{a}\right\}\right)x^\alpha\sup_{0\leq x\leq l}\frac{|\psi(x)-\psi(0)|}{x^\alpha}$$

$$+\frac{\lambda^{1+\alpha}\delta}{2(\delta+\lambda)^2}|Q|\,|P|\left(\left(1-\exp\left\{-\frac{(\delta+\lambda)\,l}{a}\right\}\right)^2\right.$$

$$+\exp\left\{-\frac{(\delta+\lambda)(l-x)}{a}\right\}\left(1-\exp\left\{-\frac{(\delta+\lambda)\,l}{a}\right\}\right)^2$$

$$+\left(1-\exp\left\{-\frac{(\delta+\lambda)\,x}{a}\right\}\right)\exp\left\{-\frac{(\delta+\lambda)\,x}{a}\right\}\left(1-\exp\left\{-\frac{2(\delta+\lambda)(l-x)}{a}\right\}\right)$$

$$+\left(1-\exp\left\{-\frac{(\delta+\lambda)(l-x)}{a}\right\}\right)\left(1-\exp\left\{-\frac{2(\delta+\lambda)\,x}{a}\right\}\right)$$

$$+\exp\left\{-\frac{2(\delta+\lambda)\,x}{a}\right\}\left(1-\exp\left\{-\frac{(\delta+\lambda)(l-x)}{a}\right\}\right)^2$$

$$\left.+\exp\left\{-\frac{2(\delta+\lambda)\,x}{a}\right\}\left(1-\exp\left\{-\frac{(\delta+\lambda)(l-x)}{a}\right\}\right)^2\right)\max_{0\leq x\leq l}|\psi(x)|$$

$$+\frac{\lambda^{1+\alpha}}{a\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)}\int\limits_0^x(x-s)^\alpha\exp\left\{-\frac{(\delta+\lambda)(x-s)}{a}\right\}ds$$

$$\times\sup_{0\leq s\leq x}\frac{|\varphi(x)-\varphi(s)|}{(x-s)^\alpha}+\frac{\lambda^{1+\alpha}}{a\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)}\sup_{0\leq s\leq l}\frac{|\varphi(l)-\varphi(s)|}{(l-s)^\alpha}$$

$$\times\int\limits_x^l(l-s)^\alpha\exp\left\{-\frac{(\delta+\lambda)(l+x-s)}{a}\right\}ds$$

$$+\frac{\lambda^{1+\alpha}}{a\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)}\int_x^l (s-x)^\alpha \exp\left\{-\frac{(\delta+\lambda)(s-x)}{a}\right\}ds$$

$$\times\sup_{x\le s\le l}\frac{|\psi(s)-\psi(x)|}{(s-x)^\alpha}+\frac{\lambda^{1+\alpha}}{a\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)}\sup_{0\le s\le l}\frac{|\psi(l)-\psi(s)|}{(l-s)^\alpha}$$

$$\times\int_0^x (l-s)^\alpha \exp\left\{-\frac{(\delta+\lambda)(l+x-s)}{a}\right\}ds$$

$$+\frac{\lambda^{1+\alpha}}{a\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)}\sup_{0\le s\le l}\frac{|\psi(x)-\psi(s)|}{|x-s|^\alpha}$$

$$\times\int_0^l |x-s|^\alpha \exp\left\{-\frac{(\delta+\lambda)|x-s|}{a}\right\}ds$$

$$\le J\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|_{\dot{\mathbb{C}}^{\alpha}[0,l]}.$$

Here

$$J=\left(\frac{\lambda^\alpha\delta}{\delta+\lambda}|Q|+|Q|\lambda^\alpha\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)+\frac{\lambda^{1+\alpha}}{\delta+\lambda}|Q|\exp\left\{-\frac{(\delta+\lambda)x}{a}\right\}x^\alpha$$

$$+\frac{\lambda^{1+\alpha}}{\delta+\lambda}|Q|\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}+\lambda^\alpha\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}$$

$$+\left(\frac{\lambda^\alpha\delta}{\delta+\lambda}|P|+|P|\lambda^\alpha\exp\left\{-\frac{(\delta+\lambda)x}{a}\right\}\right)x^\alpha$$

$$+\frac{\lambda^{1+\alpha}\delta}{2(\delta+\lambda)^2}|Q||P|\left(\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)^2\right.$$

$$+\exp\left\{-\frac{(\delta+\lambda)(l-x)}{a}\right\}\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)^2$$

$$+\left(1-\exp\left\{-\frac{(\delta+\lambda)x}{a}\right\}\right)\exp\left\{-\frac{(\delta+\lambda)x}{a}\right\}\left(1-\exp\left\{-\frac{2(\delta+\lambda)(l-x)}{a}\right\}\right)$$

$$+\left(1-\exp\left\{-\frac{(\delta+\lambda)(l-x)}{a}\right\}\right)\left(1-\exp\left\{-\frac{2(\delta+\lambda)x}{a}\right\}\right)$$

$$+\exp\left\{-\frac{2(\delta+\lambda)x}{a}\right\}\left(1-\exp\left\{-\frac{(\delta+\lambda)(l-x)}{a}\right\}\right)^2$$

$$\left.+\exp\left\{-\frac{2(\delta+\lambda)x}{a}\right\}\left(1-\exp\left\{-\frac{(\delta+\lambda)(l-x)}{a}\right\}\right)^2\right)$$

$$+\frac{\lambda^{1+\alpha}}{a\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)}\int_0^x (x-s)^\alpha \exp\left\{-\frac{(\delta+\lambda)(x-s)}{a}\right\}ds$$

$$+\frac{\lambda^{1+\alpha}}{a\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)}\int_x^l (l-s)^\alpha \exp\left\{-\frac{(\delta+\lambda)(l+x-s)}{a}\right\}ds$$

$$+\frac{\lambda^{1+\alpha}}{a\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)}\int\limits_{x}^{l}(s-x)^{\alpha}\exp\left\{-\frac{(\delta+\lambda)(s-x)}{a}\right\}ds$$

$$+\frac{\lambda^{1+\alpha}}{a\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)}\int\limits_{0}^{x}(l-s)^{\alpha}\exp\left\{-\frac{(\delta+\lambda)(l+x-s)}{a}\right\}ds$$

$$+\frac{\lambda^{1+\alpha}}{a\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)}\int\limits_{0}^{l}|x-s|^{\alpha}\exp\left\{-\frac{(\delta+\lambda)|x-s|}{a}\right\}ds.$$

Since

$$\frac{\lambda^{\alpha}\delta^{1-\alpha}}{\delta+\lambda}\leq 1,\frac{\lambda^{1+\alpha}\delta^{1-\alpha}}{(\delta+\lambda)^{2}}\leq 1,$$

$$\lambda^{\alpha}\exp\left\{-\frac{(\delta+\lambda)x}{a}\right\}\leq\frac{1}{x^{\alpha}(\delta+\lambda)^{\alpha}},|Q|\,|P|\left(1-\exp\left\{-\frac{(\delta+\lambda)l}{a}\right\}\right)^{2}\leq 1,$$

$$\lambda^{1+\alpha}\int\limits_{0}^{x}(x-s)^{\alpha}\exp\left\{-\frac{(\delta+\lambda)(x-s)}{a}\right\}ds\leq a^{1+a}\int\limits_{0}^{\infty}y^{\alpha}\exp\{-y\}\,dy,$$

$$\lambda^{1+\alpha}\int\limits_{x}^{l}(l-s)^{\alpha}\exp\left\{-\frac{(\delta+\lambda)(l+x-s)}{a}\right\}ds\leq a^{1+a}\int\limits_{0}^{\infty}y^{\alpha}\exp\{-y\}\,dy,$$

$$\lambda^{1+\alpha}\int\limits_{x}^{l}(s-x)^{\alpha}\exp\left\{-\frac{(\delta+\lambda)(s-x)}{a}\right\}ds\leq a^{1+a}\int\limits_{0}^{\infty}y^{\alpha}\exp\{-y\}\,dy,$$

$$\lambda^{1+\alpha}\int\limits_{0}^{x}(l-s)^{\alpha}\exp\left\{-\frac{(\delta+\lambda)(l+x-s)}{a}\right\}ds\leq a^{1+a}\int\limits_{0}^{\infty}y^{\alpha}\exp\{-y\}\,dy,$$

$$\lambda^{1+\alpha}\int\limits_{0}^{l}|x-s|^{\alpha}\exp\left\{-\frac{(\delta+\lambda)|x-s|}{a}\right\}ds\leq 2a^{1+a}\int\limits_{0}^{\infty}y^{\alpha}\exp\{-y\}\,dy$$

we have that

$$J\leq M(a,\delta).$$

Then

$$\left\|\lambda^{\alpha}A(A+\lambda)^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|_{\mathbb{C}[0,l]}\leq M(a,\delta)\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|_{\dot{\mathbb{C}}^{\alpha}[0,l]}$$

for any $\lambda\geq 0$. This means that

$$\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|_{E_{\alpha}(\mathbb{C}[0,l],A)}\leq M(a,\delta)\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|_{\dot{\mathbb{C}}^{\alpha}[0,l]}.$$

Let us prove the opposite inequality. For any positive operator A we can write

$$f=\int_{0}^{\infty}A(\lambda+A)^{-2}fd\lambda.$$

From this relation and formula (7) it follows that

$$\begin{pmatrix}\varphi(x)\\ \psi(x)\end{pmatrix}=\int_{0}^{\infty}(\lambda+A)^{-1}A(\lambda+A)^{-1}\begin{pmatrix}\varphi(x)\\ \psi(x)\end{pmatrix}d\lambda$$

$$= \int_0^\infty \int_0^l G(x,s;\lambda) A(\lambda + A)^{-1} \begin{pmatrix} \varphi(s) \\ \psi(s) \end{pmatrix} ds d\lambda.$$

Consequently,

$$\begin{pmatrix} \varphi(x+\tau) \\ \psi(x+\tau) \end{pmatrix} - \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix}$$

$$= \int_0^\infty \int_0^l \left(G(x+\tau,s;\lambda) - G(x,s;\lambda)\right) A(\lambda + A)^{-1} \begin{pmatrix} \varphi(s) \\ \psi(s) \end{pmatrix} ds d\lambda,$$

whence

$$|\varphi(x+\tau) - \varphi(x)| \le \left(\int_0^\infty \int_0^l \lambda^{-\alpha} \left| G_{11}(x+\tau,s;\lambda) - G_{11}(x,s;\lambda) \right| ds d\lambda \right.$$

$$\left. + \int_0^\infty \int_0^l \lambda^{-\alpha} \left| G_{12}(x+\tau,s;\lambda) - G_{12}(x,s;\lambda) \right| ds d\lambda \right) \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|_{E_\alpha(\mathbb{C}[0,l],A)},$$

$$|\psi(x+\tau) - \psi(x)| \le \int_0^\infty \int_0^l \lambda^{-\alpha} \left| G_{22}(x+\tau,s;\lambda) - G_{22}(x,s;\lambda) \right| ds d\lambda$$

$$\times \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|_{E_\alpha(\mathbb{C}[0,l],A)}.$$

Let

$$P = |\tau|^{-\alpha} \left(\int_0^\infty \lambda^{-\alpha} \int_0^l \left| G_{11}(x+\tau,s;\lambda) - G_{11}(x,s;\lambda) \right| ds d\lambda \right.$$

$$+ \int_0^\infty \lambda^{-\alpha} \int_0^l \left| G_{12}(x+h,s;\lambda) - G_{12}(x,s;\lambda) \right| ds d\lambda$$

$$\left. + \int_0^\infty \lambda^{-\alpha} \int_0^l \left| G_{22}(x+\tau,s;\lambda) - G_{22}(x,s;\lambda) \right| ds d\lambda \right).$$

Then for any $x+\tau, x \in [0,l]$ we have that

$$|\tau|^{-\alpha} |\varphi(x+\tau) - \varphi(x)| + |\tau|^{-\alpha} |\psi(x+\tau) - \psi(x)| \le P \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|_{E_\alpha(\mathbb{C}[0,l],A)}.$$

Now let us estimate $P = P_1 + P_2 + P_3$, where

$$P_1 = |\tau|^{-\alpha} \int_0^\infty \lambda^{-\alpha} \int_0^l \left| G_{11}(x+\tau,s;\lambda) - G_{11}(x,s;\lambda) \right| ds d\lambda,$$

$$P_2 = |\tau|^{-\alpha} \int_0^\infty \lambda^{-\alpha} \int_0^l \left| G_{12}(x+h,s;\lambda) - G_{12}(x,s;\lambda) \right| ds d\lambda,$$

$$P_3 = |\tau|^{-\alpha} \int_0^\infty \lambda^{-\alpha} \int\limits_0^l |G_{22}(x+\tau,s;\lambda) - G_{22}(x,s;\lambda)|\, dsd\lambda.$$

Note that it suffices to consider the case when $0 \le \tau \le \frac{1}{2}$. Let us estimate the expression

$$P_1 = \tau^{-\alpha} \int_0^\infty \lambda^{-\alpha} \int\limits_0^x |G_{11}(x+\tau,s;\lambda) - G_{11}(x,s;\lambda)|\, dsd\lambda$$

$$+\tau^{-\alpha} \int_0^\infty \lambda^{-\alpha} \int\limits_x^{x+\tau} |G_{11}(x+\tau,s;\lambda) - G_{11}(x,s;\lambda)|\, dsd\lambda$$

$$+\tau^{-\alpha} \int_0^\infty \lambda^{-\alpha} \int\limits_{x+\tau}^{l} |G_{11}(x+\tau,s;\lambda) - G_{11}(x,s;\lambda)|\, dsd\lambda = P_{11} + P_{12} + P_{13}.$$

Using formula (8) and estimates (11), (12), we get

$$P_{11} \le \tau^{-\alpha} \int_0^\infty \frac{1}{a} \lambda^{-\alpha}$$

$$\times \int\limits_0^x \left| \exp\left\{ -\frac{(\delta+\lambda)(x+\tau-s)}{a} \right\} - \exp\left\{ -\frac{(\delta+\lambda)(x-s)}{a} \right\} \right| dsd\lambda$$

$$\le \tau^{-\alpha} \int_0^\delta \frac{1}{a} \lambda^{-\alpha} \int\limits_0^x M(a) \left((\delta+\lambda)\tau \right)^\alpha dsd\lambda$$

$$+2^\alpha \tau^{-\alpha} \int_\delta^\infty \frac{1}{a} (\delta+\lambda)^{-\alpha} \left(1 - \exp\left\{ -\frac{(\delta+\lambda)\tau}{a} \right\} \right)$$

$$\times \int\limits_0^x \exp\left\{ -\frac{(\delta+\lambda)(x-s)}{a} \right\} dsd\lambda$$

$$\le M(a,\delta) + M_1(a,\delta)\tau^{-\alpha} \int_\delta^\infty (\delta+\lambda)^{-\alpha-1} \left(1 - \exp\left\{ -\frac{(\delta+\lambda)\tau}{a} \right\} \right) d\lambda.$$

Making the substitution $z = \frac{(\delta+\lambda)\tau}{a}$, we get

$$\tau^{-\alpha} \int_\delta^\infty (\delta+\lambda)^{-\alpha-1} \left(1 - \exp\left\{ -\frac{(\delta+\lambda)\tau}{a} \right\} \right) d\lambda$$

$$\le a^{-\alpha} \tau^{-\alpha} \int_\delta^\infty \left(\frac{az}{\tau} \right)^{-\alpha-1} \left(1 - \exp\{-z\} \right) \tau dz.$$

Then

$$P_{11} \le M(a,\delta) + M_2(a,\delta) \int_0^\infty \frac{1-\exp\{-z\}}{z^{\alpha+1}} dz.$$

Since

$$\int_0^\infty \frac{1-\exp\{-z\}}{z^{\alpha+1}} dz = \int_0^\infty \frac{d\tau}{z^{1+\alpha}} \int_0^z e^{-s} ds = \int_0^\infty e^{-s} \int_s^\infty \frac{dz}{z^{1+\alpha}} ds \tag{24}$$

$$= \frac{1}{\alpha} \int_0^\infty e^{-s} s^{-\alpha} ds = \frac{1}{\alpha} G(1-\alpha) \le \frac{M_1}{\alpha(1-\alpha)},$$

we have that

$$P_{11} \leq \frac{M_3(a,\delta)}{\alpha(1-\alpha)}.$$

Using formula (8) and estimates (11), (12), we get

$$P_{12} \leq \tau^{-\alpha} \int_0^\infty \frac{1}{a}\lambda^{-\alpha} \int\limits_x^{x+\tau} \exp\left\{-\frac{\lambda\,(x+\tau-s)}{a}\right\} dsd\lambda$$

$$= \tau^{-\alpha} \int_0^\infty \lambda^{-1-\alpha}\left(1-\exp\left\{-\frac{\lambda\tau}{a}\right\}\right) d\lambda.$$

Making the substitution $\lambda = \frac{a}{\tau}z$, we obtain

$$P_{12} \leq \tau^{-\alpha} \int_0^\infty \left(\frac{az}{\tau}\right)^{-(1+\alpha)} (1-\exp\{-z\})\,\frac{adz}{\tau}$$

$$= a^{-\alpha} \int_0^\infty \frac{1-\exp\{-z\}}{z^{\alpha+1}} dz.$$

Using formula (24), we get

$$P_{12} \leq \frac{M_4(a,\delta)}{\alpha(1-\alpha)}.$$

Using formula (8) and estimates (11), (12), we get

$$P_{13} \leq \tau^{-\alpha} \int_0^\infty \frac{1}{a}\lambda^{-\alpha}$$

$$\times \int\limits_{x+\tau}^{l} \left(\exp\left\{-\frac{(\delta+\lambda)\,(l+x-s)}{a}\right\} - \exp\left\{-\frac{(\delta+\lambda)\,(l+x+\tau-s)}{a}\right\}\right) dsd\lambda$$

$$\leq \tau^{-\alpha} \int_0^\delta \frac{1}{a}\lambda^{-\alpha} \int\limits_{x+\tau}^{l} M(a)\,((\delta+\lambda)\,\tau)^\alpha\, dsd\lambda$$

$$+2^\alpha \tau^{-\alpha} \int_\delta^\infty \frac{1}{a}\,(\delta+\lambda)^{-\alpha}\left(1-\exp\left\{-\frac{(\delta+\lambda)\,\tau}{a}\right\}\right)$$

$$\times \int\limits_{x+\tau}^{l} \exp\left\{-\frac{(\delta+\lambda)\,(l+x-s)}{a}\right\} dsd\lambda$$

$$\leq M(a,\delta) + M_1(a,\delta)\tau^{-\alpha} \int_\delta^\infty (\delta+\lambda)^{-\alpha-1}\left(1-\exp\left\{-\frac{(\delta+\lambda)\,\tau}{a}\right\}\right) d\lambda.$$

Making the substitution $z = \frac{(\delta+\lambda)\tau}{a}$, we get

$$P_{13} \leq M(a,\delta) + M_2(a,\delta) \int_0^\infty \frac{1-\exp\{-z\}}{z^{\alpha+1}} dz.$$

Using formula (24), we get

$$P_{13} \leq \frac{M_5(a,\delta)}{\alpha(1-\alpha)}.$$

Combining the estimates of expressions P_{11}, P_{12}, P_{13}, we get

$$P_1 \leq \frac{M_6(a,\delta)}{\alpha(1-\alpha)}. \tag{25}$$

Now, let us estimate the expression

$$P_3 = \tau^{-\alpha} \int_0^\infty \lambda^{-\alpha} \int\limits_0^x |G_{22}(x+\tau, s; \lambda) - G_{22}(x, s; \lambda)| \, ds d\lambda$$

$$+\tau^{-\alpha} \int_0^\infty \lambda^{-\alpha} \int\limits_x^{x+\tau} |G_{22}(x+\tau, s; \lambda) - G_{22}(x, s; \lambda)| \, ds d\lambda$$

$$+\tau^{-\alpha} \int_0^\infty \lambda^{-\alpha} \int\limits_{x+\tau}^{l} |G_{22}(x+\tau, s; \lambda) - G_{22}(x, s; \lambda)| \, ds d\lambda = P_{31} + P_{32} + P_{33}.$$

Using formula (9) and estimates (11), (13), we get

$$P_{31} \leq \tau^{-\alpha} \int_0^\infty \frac{1}{a} \lambda^{-\alpha}$$

$$\times \int\limits_0^x \left| \exp\left\{ -\frac{(\delta+\lambda)(l-x-\tau+s)}{a} \right\} - \exp\left\{ -\frac{(\delta+\lambda)(l-x+s)}{a} \right\} \right| ds d\lambda$$

$$\leq \tau^{-\alpha} \int_0^\delta \frac{1}{a} \lambda^{-\alpha} \int\limits_0^x M(a) \left((\delta+\lambda)\tau \right)^\alpha ds d\lambda$$

$$+2^\alpha \tau^{-\alpha} \int_\delta^\infty \frac{1}{a} (\delta+\lambda)^{-\alpha} \left(1 - \exp\left\{ -\frac{(\delta+\lambda)\tau}{a} \right\} \right)$$

$$\times \int\limits_0^x \exp\left\{ -\frac{(\delta+\lambda)(l-x-\tau+s)}{a} \right\} ds d\lambda$$

$$\leq M(a,\delta) + M_1(a,\delta) \tau^{-\alpha} \int_\delta^\infty (\delta+\lambda)^{-\alpha-1} \left(1 - \exp\left\{ -\frac{(\delta+\lambda)\tau}{a} \right\} \right) d\lambda.$$

Making the substitution $z = \frac{(\delta+\lambda)\tau}{a}$, we get

$$P_{31} \leq M(a,\delta) + M_2(a,\delta) \int_0^\infty \frac{1-\exp\{-z\}}{z^{\alpha+1}} dz.$$

Using formula (24), we get

$$P_{31} \leq \frac{M_4(a,\delta)}{\alpha(1-\alpha)}.$$

Using formula (9) and estimates (11), (13), we get

$$P_{32} \leq \tau^{-\alpha} \int_0^\infty \frac{1}{a} \lambda^{-\alpha} \int\limits_x^{x+\tau} \exp\left\{ -\frac{\lambda(l-x-\tau+s)}{a} \right\} ds d\lambda$$

$$\leq \tau^{-\alpha} \int_0^\infty \lambda^{-1-\alpha} \left(1 - \exp\left\{ -\frac{\lambda\tau}{a} \right\} \right) d\lambda.$$

Making the substitution $\lambda = \frac{a}{\tau} z$, we obtain

$$P_{12} \leq \tau^{-\alpha} \int_0^\infty \left(\frac{az}{\tau} \right)^{-(1+\alpha)} (1 - \exp\{-z\}) \frac{adz}{\tau}$$

$$= a^{-\alpha} \int_0^\infty \frac{1-\exp\{-z\}}{z^{\alpha+1}} dz.$$

Using formula (24), we get

$$P_{32} \leq \frac{M_4(a,\delta)}{\alpha(1-\alpha)}.$$

Using formula (9) and estimates (11), (13), we get

$$P_{33} \leq \tau^{-\alpha} \int_0^{\infty} \frac{1}{a} \lambda^{-\alpha}$$

$$\times \int\limits_{x+\tau}^{l} \left(\exp\left\{ -\frac{(\delta+\lambda)(s-x-\tau)}{a} \right\} - \exp\left\{ -\frac{(\delta+\lambda)(s-x)}{a} \right\} \right) dsd\lambda$$

$$\leq \tau^{-\alpha} \int_0^{\delta} \frac{1}{a} \lambda^{-\alpha} \int\limits_{x+\tau}^{l} M(a) \left((\delta+\lambda)\tau \right)^{\alpha} dsd\lambda$$

$$+2^{\alpha}\tau^{-\alpha} \int_{\delta}^{\infty} \frac{1}{a} (\delta+\lambda)^{-\alpha} \left(1 - \exp\left\{ -\frac{(\delta+\lambda)\tau}{a} \right\} \right)$$

$$\times \int\limits_{x+\tau}^{l} \exp\left\{ -\frac{(\delta+\lambda)(s-x-\tau)}{a} \right\} dsd\lambda$$

$$\leq M(a,\delta) + M_1(a,\delta)\tau^{-\alpha} \int_{\delta}^{\infty} (\delta+\lambda)^{-\alpha-1} \left(1 - \exp\left\{ -\frac{(\delta+\lambda)\tau}{a} \right\} \right) d\lambda.$$

Making the substitution $z = \frac{(\delta+\lambda)\tau}{a}$, we get

$$P_{33} \leq M(a,\delta) + M_2(a,\delta) \int_0^{\infty} \frac{1-\exp\{-z\}}{z^{\alpha+1}} dz.$$

Using formula (24), we get

$$P_{33} \leq \frac{M_5(a,\delta)}{\alpha(1-\alpha)}.$$

Combining the estimates of expressions P_{31}, P_{32}, P_{33}, we get

$$P_3 \leq \frac{M_6(a,\delta)}{\alpha(1-\alpha)}. \tag{26}$$

Now, let us estimate the expression

$$P_2 = \tau^{-\alpha} \int_0^{\infty} \lambda^{-\alpha} \int\limits_0^{x} \left| G_{12}(x+\tau,s;\lambda) - G_{12}(x,s;\lambda) \right| dsd\lambda$$

$$+\tau^{-\alpha} \int_0^{\infty} \lambda^{-\alpha} \int\limits_x^{x+\tau} \left| G_{12}(x+\tau,s;\lambda) - G_{12}(x,s;\lambda) \right| dsd\lambda$$

$$+\tau^{-\alpha} \int_0^{\infty} \lambda^{-\alpha} \int\limits_{x+\tau}^{l} \left| G_{12}(x+\tau,s;\lambda) - G_{12}(x,s;\lambda) \right| dsd\lambda = P_{21} + P_{22} + P_{23}.$$

Using formula (10) and estimates (11), (14), we get

$$P_{21} \leq \tau^{-\alpha} \int_0^{\infty} \frac{1}{a} \lambda^{-\alpha}$$

$$\times \int_0^x \left| \exp\left\{ -\frac{(\delta+\lambda)(x+\tau-s)}{a} \right\} - \exp\left\{ -\frac{(\delta+\lambda)(x-s)}{a} \right\} \right| dsd\lambda$$

$$\leq \tau^{-\alpha} \int_0^{\delta} \frac{1}{a} \lambda^{-\alpha} \int_0^x M(a) \left((\delta+\lambda) \tau \right)^{\alpha} dsd\lambda$$

$$+2^{\alpha}\tau^{-\alpha} \int_{\delta}^{\infty} \frac{1}{a} (\delta+\lambda)^{-\alpha} \left(1 - \exp\left\{ -\frac{(\delta+\lambda)\tau}{a} \right\} \right)$$

$$\times \int_0^x \exp\left\{ -\frac{(\delta+\lambda)(x-s)}{a} \right\} dsd\lambda$$

$$\leq M(a,\delta) + M_1(a,\delta)\tau^{-\alpha} \int_{\delta}^{\infty} (\delta+\lambda)^{-\alpha-1} \left(1 - \exp\left\{ -\frac{(\delta+\lambda)\tau}{a} \right\} \right) d\lambda.$$

Making the substitution $z = \frac{(\delta+\lambda)\tau}{a}$, we get

$$P_{21} \leq M(a,\delta) + M_2(a,\delta) \int_0^{\infty} \frac{1-\exp\{-z\}}{z^{\alpha+1}} dz.$$

Using formula (24), we get

$$P_{21} \leq \frac{M_4(a,\delta)}{\alpha(1-\alpha)}.$$

Using formula (10) and estimates (11), (14), we get

$$P_{22} \leq \tau^{-\alpha} \int_0^{\infty} \frac{1}{a} \lambda^{-\alpha} \int_x^{x+\tau} \exp\left\{ -\frac{\lambda(x-s)}{a} \right\} dsd\lambda$$

$$\leq \tau^{-\alpha} \int_0^{\infty} \lambda^{-1-\alpha} \left(1 - \exp\left\{ -\frac{\lambda\tau}{a} \right\} \right) d\lambda.$$

Making the substitution $\lambda = \frac{a}{\tau}z$, we obtain

$$P_{22} \leq \tau^{-\alpha} \int_0^{\infty} \left(\frac{az}{\tau} \right)^{-(1+\alpha)} (1-\exp\{-z\}) \frac{adz}{\tau}$$

$$= a^{-\alpha} \int_0^{\infty} \frac{1-\exp\{-z\}}{z^{\alpha+1}} dz.$$

Using formula (24), we get

$$P_{22} \leq \frac{M_4(a,\delta)}{\alpha(1-\alpha)}.$$

Using formula (10) and estimates (11), (14), we get

$$P_{23} \leq \tau^{-\alpha} \int_0^{\infty} \frac{1}{a} \lambda^{-\alpha}$$

$$\times \int_{x+\tau}^{l} \left(\exp\left\{ -\frac{(\delta+\lambda)(s-x-\tau)}{a} \right\} - \exp\left\{ -\frac{(\delta+\lambda)(s-x)}{a} \right\} \right) dsd\lambda$$

$$\leq \tau^{-\alpha} \int_0^{\delta} \frac{1}{a} \lambda^{-\alpha} \int_{x+\tau}^{l} M(a) \left((\delta+\lambda)\tau \right)^{\alpha} dsd\lambda$$

$$+2^{\alpha}\tau^{-\alpha}\int_{\delta}^{\infty}\frac{1}{a}(\delta+\lambda)^{-\alpha}\left(1-\exp\left\{-\frac{(\delta+\lambda)\tau}{a}\right\}\right)$$

$$\times\int\limits_{x+\tau}^{l}\exp\left\{-\frac{(\delta+\lambda)(s-x-\tau)}{a}\right\}dsd\lambda$$

$$\leq M(a,\delta)+M_1(a,\delta)\tau^{-\alpha}\int_{\delta}^{\infty}(\delta+\lambda)^{-\alpha-1}\left(1-\exp\left\{-\frac{(\delta+\lambda)\tau}{a}\right\}\right)d\lambda.$$

Making the substitution $z=\frac{(\delta+\lambda)\tau}{a}$, we get

$$P_{33}\leq M(a,\delta)+M_2(a,\delta)\int_0^{\infty}\frac{1-\exp\{-z\}}{z^{\alpha+1}}dz.$$

Using formula (24), we get

$$P_{23}\leq\frac{M_5(a,\delta)}{\alpha(1-\alpha)}.$$

Combining the estimates of expressions P_{21},P_{22},P_{23}, we get

$$P_2\leq\frac{M_6(a,\delta)}{\alpha(1-\alpha)}. \tag{27}$$

Applying the triangle inequality and estimates (25), (26), (27), we get

$$P\leq\frac{M_7(a,\delta)}{\alpha(1-\alpha)}.$$

Thus, for any $x+\tau,x\in[0,l]$ we have that

$$|\tau|^{-\alpha}|\varphi(x+\tau)-\varphi(x)|+|\tau|^{-\alpha}|\psi(x+\tau)-\psi(x)|\leq\frac{M_8(a,\delta)}{\alpha(1-\alpha)}\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|_{E_\alpha(\mathbb{C}[0,l],A)}.$$

This means that the following inequality holds:

$$\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|_{\overset{\circ}{\mathbb{C}}^{\alpha}[0,l]}\leq\frac{M_8(a,\delta)}{\alpha(1-\alpha)}\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|_{E_\alpha(\mathbb{C}[0,l],A)}.$$

Theorem 3.1. is proved. $\square$

Since the A is a positive operator in the fractional spaces $E_\alpha(\mathbb{C}[0,l],A)$, from the result of Theorem 3.1 it follows also it is positive operator in the Hölder space $\overset{\circ}{\mathbb{C}}^{\alpha}[0,l]$. Namely,

Theorem 3.2. *The operator $(\lambda I+A)$ has a bounded in $\overset{\circ}{\mathbb{C}}^{\alpha}[0,l]$ inverse for any $\lambda\geq 0$ and the following estimate holds:*

$$\|(\lambda+A)^{-1}\|_{\overset{\circ}{\mathbb{C}}^{\alpha}[0,l]\to\overset{\circ}{\mathbb{C}}^{\alpha}[0,l]}\leq\frac{M_8(a,\delta)}{\alpha(1-\alpha)}\frac{M_1}{1+\lambda}.$$

4. Applications

In this section we consider the application of results of sections 2 and 3. For A a positive operator in E the following result was established in papers [17]-[18].

Theorem 4.1. *Let A be a positive operator in E. Then obeys the following estimate*

$$\|\mathbf{R}_{q,q-1}^{k}(\tau A)\|_{E\to E}\leq M,1\leq k\leq N,N\tau=T, \tag{28}$$

where M does not depend on τ and k. Here $\mathbf{R}_{q,q-1}^{k}(z)$ is the Pade approximation of $exp(-z)$ near $z=0$.

Putting $t_k = k\tau$ and passing to limit when $\tau \to 0$, we get $t_k \to t$ and

$$\| \exp\{-tA\}\|_{E\to E} \le M, 0 \le t \le T. \tag{29}$$

We introduce the Banach space $\mathbb{C}([0,T],E)$ of all continuous abstract vector functions $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ defined on $[0,T]$ with values in E, equipped with the norm

$$\|u\|_{\mathbb{C}([0,T],E)} = \max_{0\le t\le T} \|u_1(t)\|_E + \max_{0\le t\le T} \|u_2(t)\|_E .$$

Note that the problem (1) can be written in the form as the abstract Cauchy problem

$$\frac{d}{dt}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + A\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, \tag{30}$$

$$0 < t < T, \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

in a Banach space $E = \mathbb{C}([0,l],R) = C([0,l],R) \times C([0,l],R)$ with a positive operator A defined by (4). Here $\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} f_1(t,x) \\ f_2(t,x) \end{pmatrix}$ is the given abstract vector function defined on $[0,T]$ with values in E, $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix}$ is the element of $D(A)$.

It is well known that (see, for example [3]) the following formula

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \exp\{-tA\}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t \exp\{-(t-s)A\}\begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} ds \tag{31}$$

gives a solution of problem (30) in $\mathbb{C}([0,T],E)$ for continuously differentiable on $[0,T]$ vector function $\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$ and smooth given element $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$.

Theorem 4.2. *For the solution of problem* (30) *the stability inequality holds:*

$$\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\|_{\mathbb{C}([0,T],E)} \le M\left[\left\|\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}\right\|_E + \left\|\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right\|_{\mathbb{C}([0,T],E)}\right].$$

The proof of Theorem 4.2 is based on the positivity of operator A, the formula (31) and estimate (29).

Applying results of Theorem 2.1 and Theorem 4.2, we get the following theorem.

Theorem 4.3. *The solution of problem*(1) *satisfy the following estimate*

$$\max_{t\in[0,T]}\max_{x\in[0,l]} |u(t,x)| + \max_{t\in[0,T]}\max_{x\in[0,l]} |v(t,x)|$$

$$\le M\left[\max_{x\in[0,l]} |u_0(x)| + \max_{x\in[0,l]} |v_0(x)| + \max_{t\in[0,T]}\max_{x\in[0,l]} |f_1(t,x)| + \max_{t\in[0,T]}\max_{x\in[0,l]} |f_2(t,x)|\right].$$

Applying results of Theorem 3.2, Theorem 4.1 and Theorem 4.2, we get the following theorem.

Theorem 4.4. *Assume that*

$$f_1(t,0) = \gamma f_1(t,l), 0 \le \gamma \le 1, \quad \beta f_2(t,0) = f_2(t,l), 0 \le \beta \le 1, t \in [0,T].$$

Then the solution of problem(1) *satisfy the following estimate*

$$\max_{t\in[0,T]}\left(\max_{x\in[0,l]}|u(t,x)|+\max_{\substack{x,x+\tau\in[0,l]\\ \tau\neq 0}}\frac{|u(t,x+\tau)-u(t,x)|}{|\tau|^{\alpha}}\right)$$
$$+\max_{t\in[0,T]}\left(\max_{x\in[0,l]}|v(t,x)|+\max_{\substack{x,x+\tau\in[0,l]\\ \tau\neq 0}}\frac{|v(t,x+\tau)-v(t,x)|}{|\tau|^{\alpha}}\right)$$
$$\leq M\left[\max_{x\in[0,l]}|u_0(x)|+\max_{\substack{x,x+\tau\in[0,l]\\ \tau\neq 0}}\frac{|u_0(x+\tau)-u_0(x)|}{|\tau|^{\alpha}}\right.$$
$$+\max_{x\in[0,l]}|v_0(x)|+\max_{\substack{x,x+\tau\in[0,l]\\ \tau\neq 0}}\frac{|v_0(x+\tau)-v_0(x)|}{|\tau|^{\alpha}}$$
$$+\max_{t\in[0,T]}\left(\max_{x\in[0,l]}|f_1(t,x)|+\max_{\substack{x,x+\tau\in[0,l]\\ \tau\neq 0}}\frac{|f_1(t,x+\tau)-f_1(t,x)|}{|\tau|^{\alpha}}\right)$$
$$\left.+\max_{t\in[0,T]}\left(\max_{x\in[0,l]}|f_2(t,x)|+\max_{\substack{x,x+\tau\in[0,l]\\ \tau\neq 0}}\frac{|f_2(t,x+\tau)-f_2(t,x)|}{|\tau|^{\alpha}}\right)\right].$$

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