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PARTIAL CONE METRIC SPACE AND SOME FIXED POINT THEOREMS

D. DEY[1], M. SAHA[2] §

(Dedicated to the 71st birth anniversery of Professor A.P.Baisnab, F.N.A.Sc, India)

Abstract. In the present paper, we have proved some convergence properties of a sequence of elements in a partial cone metric space and thereby we have established some fixed point theorems on it.

Keywords: Partial metric space, partial cone metric space, fixed point.

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1. Introduction

Partial metric space was originally developed by Matthews ([7], [8]) to provide mechanism generalizing metric space theories. He introduced the notion of partial metric space as a part of the study of denotational semantics of dataflow networks in computer science. In particular, he established the precise relationship between partial metric spaces and the so called weightable quasi-metric space (see [10]). In partial metric spaces, the distance of a point in the self may not be zero. Introducing partial metric space, Matthews proved Banach fixed point theorem [1] in this space.

Definition 1.1. (Partial metric space) *A partial metric on a non-empty set X is a function $p : X \times X \to R^+$ such that for all $x, y, z \in X$:*
(p_1): $0 \leq p(x,x) \leq p(x,y)$,
(p_2): $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p_3): $p(x,y) = p(y,x)$,
(p_4): $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X, p) such that X is a non-empty set and p is a partial metric on X. It is clear that if $p(x,y) = 0$, then from (p_1) and (p_2) we obtain $x = y$. But if $x = y$, $p(x,y)$ may not be 0. A basic example of a partial metric space is the pair (R^+, p), where $p(x,y) = \max\{x,y\}$ for all $x, y \in R^+$. Various application on this space has been extensively investigated by many authors (see [5], [13] for details).
In 1980, Rzepecki [11] introduced a generalized metric d_E on a set X in a way that $d_E : X \times X \to P$, replacing the set of real numbers with a Banach space E in the metric function where P is a normal cone in E with a partial order $\leq$. Seven years later, Lin [6]

[1] Koshigram Union Institution, Koshigram-713150, Burdwan, West Bengal, India ,
e-mail: debashisdey@yahoo.com

[2] Department of Mathematics,The University of Burdwan, Burdwan-713104, West Bengal, India ,
e-mail: mantusaha@yahoo.com

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considered the notion of cone metric spaces by replacing real numbers with a cone P in the metric function in which it is called a K-metric. On the otherhand, L.-G. Huang and X. Zhang [3] replaced real numbers with an ordering Banach space to define cone metric space, which is the same as either the definition of Rzepecki or of Lin. In the same paper, they proved some important fixed point theorems including Banach's fixed point theorem in cone metric space.
Let E always be a real Banach space and P a subset of E. P is called a cone if
(C_1): P is closed, non-empty and $P \neq \{0\}$,
(C_2): $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,
(C_3): $P \cap (-P) = \{0\}$.
For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$, $\text{int}P$ denotes the interior of P. The cone P is called normal if there is a constant number $K > 0$ such that for all $x, y \in E$

$$\theta \leq x \leq y \text{ implies } \|x\| \leq K \|y\| .$$

The least positive number satisfying above is called the normal constant of P. The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that $x_1 \leq x_2 \leq ...x_n \leq ... \leq y$ for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.
In the following we always suppose E is a Banach space, P a cone in E with $\text{int}P \neq \Phi$ and $\leq$ is partial ordering with respect to P.

Definition 1.2. (Cone metric space) *Let X be a non-empty set. Suppose X the mapping $d : X \times X \to E$ satisfies such that :*
(CM_1): *$\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,*
(CM_2): *$d(x, y) = d(y, x)$ for all $x, y \in X$,*
(CM_3): *$d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.*

Then d is called a cone metric on X, and (X, d) is called a cone metric space.
Motivated by this beautiful generalization of metric space, recently A.Sönmez [12] introduced the notion of partial cone metric space and its topological characterization. He also developed some important fixed point theorems in this generalized setting. We now state the following definition of partial cone metric space due to A.Sönmez [12].

Definition 1.3. (Partial cone metric space) *A partial cone metric on a non-empty set X is a function $p : X \times X \to R^+$ such that for all $x, y, z \in X$:*
(PCM_1): *$\theta \leq p(x, x) \leq p(x, y)$,*
(PCM_2): *$x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,*
(PCM_3): *$p(x, y) = p(y, x)$,*
(PCM_4): *$p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$*

A partial cone metric space is a pair (X, p) such that X is a non-empty set and p is a partial cone metric on X. It is clear that, if $p(x, y) = \theta$, then from (PCM_1) and (PCM_2), $x = y$. But if $x = y$, $p(x, y)$ may not be equal to θ.
A cone metric space is a partial cone metric space, but there exists partial cone metric spaces which are not cone metric space. The following examples supports our contention.

Example 1.1. (A.Sönmez [12]) *Let $E = R^2$,*

$$P = \{(x, y) \in E : x, y \geq 0$$

and $X = R^+$ and $p : X \times X \to E$ defined by $p(x, y) = (\max\{x, y\}, \alpha \max\{x, y\})$ where $\alpha \geq 0$ is a constant. Then (X, p) is a partial cone metric space which is not a cone metric space.

Example 1.2. (A.Sönmez [12]) *Let $E = l_1$,*

$$P = \{\{x_n\} \in l_1 : x_n \geq 0\}$$

Also let $X = \{(x_n) : (x_n) \in (R^+)^w, \sum x_n < \infty\}$ where $(R^+)^w$ be the set of all infinite sequences over R^+, and $p : X \times X \to E$ defined by

$$p(x, y) = (x_1 \vee y_1, x_2 \vee y_2, ..., x_n \vee y_n, ...)$$

where the symbol $\vee$ denotes the maximum, i.e. $x \vee y = \max\{x, y\}$. Then (X, p) is a partial cone metric space which is not a cone metric space.

In the following sequel, (X, p) will denote a partial cone metric space. We now invite the following definitions and theorems due to A.Sönmez [12]

Theorem 1.1. (Theorem 1, [12]) Any partial cone metric space (X, p) is a topological space.

Theorem 1.2. (Theorem 2, [12]) Let (X, p) be a partial cone metric space and P be a normal cone with normal constant K, then (X, p) is T_0.

Definition 1.4. (Definition 3, [12]) *Let (X, p) be a partial cone metric space. Let (x_n) be a sequence in X and $x \in X$. If for every $c \in intP$, there is N such that for all $n > N$, $p(x_n, x) \ll c + p(x, x)$, then (x_n) is said to be convergent and (x_n) converges to x, and x is the limit of (x_n). We denote this by $\lim_{n\to\infty} x_n = x$ or, $x_n \to x (n \to \infty)$*

Theorem 1.3. (Theorem 3, [12]) *Let (X, p) be a partial cone metric space, P be a normal cone with normal constant K. Let (x_n) be a sequence in X. Then (x_n) converges to x if and only if $p(x_n, x) \to p(x, x)$ $(n \to \infty)$*

Sönmez [12] also noted that if (X, p) is a partial cone metric space, P be a normal cone with normal constant K and $p(x_n, x) \to p(x, x)$ $(n \to \infty)$, then $p(x_n, x_n) \to p(x, x)$ $(n \to \infty)$.

Lemma 1.1. (Lemma 1, [12]) Let (x_n) be a sequence in partial cone metric space (X, p). If a point x is the limit of (x_n) and $p(y, y) = p(y, x)$ then y is the limit point of (x_n).

Definition 1.5. (Definition 4, [12]) *Let (X, p) be a partial cone metric space. (x_n) be a sequence in X. (x_n) is Cauchy sequence if there is $a \in P$ such that for every $\epsilon > 0$ there is N such that for all $n, m > N$*

$$\|p(x_n, x_m) - a\| < \epsilon.$$

Definition 1.6. *A partial cone metric space (X, p) is said to be complete if every Cauchy sequence in (X, p) is convergent in (X, p).*

Theorem 1.4. (Theorem 4, [12]) *Let (X, p) be a partial cone metric space. If (x_n) is a Cauchy sequence in (X, p), then it is a Cauchy sequence in the cone metric space (X, d).*

Following the current literature, there is ample vicinity to explore and improve this new avenue of research area. Here we derive some important results of partial cone metric space and then prove some classical fixed point theorems, which are extensions of Banach's [1] fixed point principle and Kannan's [4] fixed point theorem in this new setting.

2. Some properties of partial cone metric space

We first prove an analogue of important lemma (Lemma 5, L-G Huang et al.[3]) in partial cone metric space.

Proposition 2.1. *Let (X, p) be a partial cone metric space, P be a normal cone with normal constant K. Let (x_n) and (y_n) be two sequences in X such that $x_n \to x$, $y_n \to x$ $(n \to \infty)$. Then $p(x_n, y_n) \to p(x, y)$ $(n \to \infty)$.*

Proof. For every real $\epsilon > 0$, choose $c \in \mathrm{int}P$ with $\|c\| < \frac{\epsilon}{4K+2}$. As $x_n \to x$, $y_n \to y$, there is N, for all $n > N$, $p(x_n, x) \ll c + p(x, x)$ and $p(y_n, y) \ll c + p(y, y)$.
Then we have for all $n > N$,

$$\begin{aligned} p(x_n, y_n) &\leq p(x_n, x) + p(x, y) + p(y_n, y) - p(x, x) - p(y, y) \\ &\leq p(x, y) + 2c \\ \text{and} & \\ p(x, y) &\leq p(x_n, x) + p(x_n, y_n) + p(y_n, y) - p(x_n, x_n) - p(y_n, y_n) \\ &\leq p(x_n, y_n) + 2c \end{aligned}$$

Hence for all $n > N$, $\theta \leq p(x, y) + 2c - p(x_n, y_n) \leq 4c$
and so for $n > N$,

$$\begin{aligned} \|p(x_n, y_n) - p(x, y)\| &\leq \|p(x, y) + 2c - p(x_n, y_n)\| + \|2c\| \\ &\leq (4K + 2)\|c\| < \epsilon \end{aligned}$$

Therefore $p(x_n, y_n) \to p(x, y)$ $(n \to \infty)$. □

Definition 2.1. *Let (X, p) be a partial cone metric space. If for any sequence (x_n) in X, there is a subsequence (x_{n_i}) of (x_n) such that (x_{n_i}) is convergent in X. Then X is called a sequentially compact partial cone metric space.*

Proposition 2.2. *Let (X, p) be a partial cone metric space, (x_n) be a sequence in X. If (x_n) converges to x, then (x_n) is a Cauchy sequence.*

Proof. For any $\epsilon > 0$, choose $c \in intP$ with $K\|c\| < \epsilon$, there is N such that for all $n, m > N$, $p(x_n, x) \ll \frac{c}{2} + p(x, x)$ and $p(x_m, x) \ll \frac{c}{2} + p(x, x)$. Then for $n, m > N$,

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x) + p(x, x_m) - p(x, x) \\ &\leq c + p(x, x) \end{aligned}$$

So $\|p(x_n, x_m) - p(x, x)\| \leq K\|c\| < \epsilon$. Therefore (x_n) is a Cauchy sequence. □

3. Fixed point theorems in partial cone metric space

Banach's fixed point theorem [1] in partial cone metric space has been proved by A.Sönmez [12]

Theorem 3.1. (Banach's fixed point theorem) *Let (X, p) be a complete partial cone metric space, P be a normal cone with constant K. Suppose that the mapping $T : X \to X$ satisfies the contractive condition*

$$p(Tx, Ty) \leq \alpha p(x, y) \text{ for all } x, y \in X$$

where $\alpha \in (0, 1)$, is a constant. Then T has a unique fixed point in X, and for any $x \in X$, the iterative sequence $(T^n x)$ converges to the fixed point.

In the same paper [3], Kannan type mapping [4] is also considered to prove fixed point theorem in this setting.

Theorem 3.2. (Kannan's fixed point theorem) *Let (X, p) be a complete partial cone metric space, P be a normal cone with constant K. Supose that the mapping $T : X \to X$ satisfies the contractive condition*

$$p(Tx, Ty) \leq b\,[p(x, Tx) + p(y, Ty)] \text{ for all } x, y \in X$$

where $b \in (0, \frac{1}{2})$, is a constant. Then T has a unique fixed point in X, and for any $x \in X$, the iterative sequence $(T^n x)$ converges to the fixed point.

Following theorem is an important result of L.-G. Huang et al. (Corollary 1, [3]) proved here in partial cone metric space.

Theorem 3.3. *Let (X, p) be a complete partial cone metric space, P be a normal cone with constant K. For $c \in intP$ and $x_0 \in X$, set $B(x_0, c) = \{x \in X : p(x_0, x) \leq c\}$. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$p(Tx, Ty) \leq \alpha p(x, y) \text{ for all } x, y \in X$$

where $\alpha \in [0, 1)$, is a constant and $p(Tx_0, x_0) \leq (1 - \alpha)\, c$. Then T has a unique fixed point in $B(x_0, c)$.

Proof. As a consequence of Theorem 1 of A.Sönmez [12], we need only to prove that $B(x_0, c)$ is complete and $Tx \in B(x_0, c)$ for all $x \in B(x_0, c)$.
Suppose (x_n) is a Cauchy sequence in $B(x_0, c)$. Then (x_n) is also a Cauchy sequence in X. By the completeness of X, there is $x \in X$ such that $x_n \to x$ $(n \to \infty)$. Now

$$p(x_0, x) \leq p(x_n, x_0) + p(x_n, x) - p(x_n, x_n) \tag{1}$$

As

$$\begin{aligned} \left\| \lim_{n\to\infty} p(x_n, x) - \lim_{n\to\infty} p(x_n, x_n) \right\| &= \left\| \lim_{n\to\infty} [p(x_n, x) - p(x_n, x_n)] \right\| \\ &= \lim_{n\to\infty} \| p(x_n, x) - p(x_n, x_n) \| \\ &= 0 \end{aligned}$$

So $[p(x_n, x) - p(x_n, x_n)] \to \theta$ $(n \to \infty)$. Then by taking $n \to \infty$ in (1) we get $p(x_0, x) \leq c$ which implies that $x \in B(x_0, c)$. So $B(x_0, c)$ is closed and thus $B(x_0, c)$ is complete. Now for $x \in B(x_0, c)$,

$$\begin{aligned} p(x_0, Tx) &\leq p(x_0, Tx_0) + p(Tx_0, Tx) - p(Tx_0, Tx_0) \\ &\leq p(x_0, Tx_0) + \alpha p(x_0, x) \\ &\leq (1 - \alpha)c + \alpha c \\ &= c \end{aligned}$$

showing that $Tx \in B(x_0, c)$. □

Next result is also a consequence of Theorem 1 of A.Sönmez [12].

Theorem 3.4. *Let (X, p) be a complete partial cone metric space, P be a normal cone with constant K. Suppose a mapping $T : X \to X$ satisfies for some positive integer n,*

$$p(T^n x, T^n y) \leq \alpha p(x, y) \text{ for all } x, y \in X \tag{2}$$

where $\alpha \in (0, 1)$, is a constant. Then T has a unique fixed point in X.

Proof. Let $S = T^n$, then from (2)

$$p(Sx, Sy) \leq \alpha p(x, y) \text{ for all } x, y \in X$$

So by Theorem 3.1, S i.e. T^n has a unique fixed point x_0. But $T^n(Tx_0) = T(T^n x_0) = Tx_0$. So Tx_0 is also a fixed point of T^n. Hence $Tx_0 = x_0$ i.e. x_0 is a fixed point of T. Since the fixed point of T is also fixed point of T^n, the fixed point of T is unique. □

Theorem 3.5. *Let (X, p) be a sequentially compact partial cone metric space, P be a regular cone. Suppose the mapping $T : X \to X$ satisfies*

$$p(Tx, Ty) < p(x, y) \textit{ for all } x, y \in X,\ x \neq y$$

Then T has a unique fixed point in X.

Proof. Choose $x_0 \in X$. Set $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0$, . . . , $x_{n+1} = Tx_n = T^{n+1}x_0$, ... If for some n, $x_{n+1} = x_n$, then x_n is a fixed point of T, the proof is complete. So we assume that for all n, $x_{n+1} \neq x_n$. Set $d_n = p(x_n, x_{n+1})$, then

$$d_{n+1} = p(x_{n+1}, x_{n+2}) = p(Tx_n, Tx_{n+1}) < p(x_n, x_{n+1}) = d_n.$$

Therefore (d_n) is a decreasing sequence bounded below by 0. Since P is regular, there is $d^* \in E$ such that $d_n \to d^*$ $(n \to \infty)$. From the sequence compactness of X, there are subsequence (x_{n_i}) of (x_n) and $x^* \in X$ such that $x_{n_i} \to x^*$ $(i \to \infty)$. We have

$$p(Tx_{n_i}, Tx^*) \leq p(x_{n_i}, x^*),\ i = 1, 2, ..$$

So

$$\|p(Tx_{n_i}, Tx^*)\| \leq \|p(x_{n_i}, x^*)\| \to 0\ (n \to \infty)$$

where K is the normal constant of E. Hence $Tx_{n_i} \to Tx^*$ $(i \to \infty)$. Similarly, $T^2x_{n_i} \to T^2x^*$ $(i \to \infty)$. By using Proposition 2.1, we have $p(Tx_{n_i}, x_{n_i}) \to p(Tx^*, x^*)$ $(i \to \infty)$ and $p(T^2x_{n_i}, Tx_{n_i}) \to p(T^2x^*, Tx^*)$ $(i \to \infty)$. It is obvious that $p(Tx_{n_i}, x_{n_i}) = d_{n_i} \to d^* = p(Tx^*, x^*)$ $(i \to \infty)$. Now we shall prove that $Tx^* = x^*$. If $Tx^* \neq x^*$, then $d^* \neq 0$. We have

$$d^* = p(Tx^*, x^*) > p(T^2x^*, Tx^*) = \lim_{i \to \infty} p(T^2x_{n_i}, Tx_{n_i}) = \lim d_{n_i+1} = d^*$$

which is a contradiction. So $Tx^* = x^*$. That is x^* is a fixed point of T . The uniqueness of fixed point is obvious. □

We now prove the fixed point theorem due to Chatterjea [2] in in partial cone metric space.

Theorem 3.6. (Chatterjea fixed point theorem) *Let (X, p) be a complete partial cone metric space, P be a normal cone with constant K. Supose that the mapping $T : X \to X$ satisfies the contractive condition*

$$p(Tx, Ty) \leq b\,[p(Tx, y) + p(Ty, x)] \textit{ for all } x, y \in X$$

where $b \in (0, \frac{1}{2})$, is a constant. Then T has a unique fixed point in X, and for any $x \in X$, the iterative sequence $(T^n x)$ converges to the fixed point.

Proof. Choose $x_0 \in X$. Define the sequence (x_n) as $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, ..., x_{n+1} = Tx_n = T^{n+1}x_0$,... We have

$$\begin{aligned} p(x_{n+1}, x_n) &= p(Tx_n, Tx_{n-1}) \\ &\leq b\,[p(Tx_n, x_{n-1}) + p(Tx_{n-1}, x_n)] \\ &\leq b\,[p(x_{n+1}, x_{n-1}) + p(x_n, x_n)] \end{aligned}$$

So

$$\begin{aligned} p(x_{n+1}, x_n) &\leq b\,[p(x_{n+1}, x_n) + p(x_n, x_{n-1}) - p(x_n, x_n)] + bp(x_n, x_n) \\ &\leq b\,[p(x_{n+1}, x_n) + p(x_n, x_{n-1})] \end{aligned}$$

which imlies that

$$p(x_{n+1}, x_n) \leq \frac{b}{1-b} p(x_n, x_{n-1}) = hp(x_n, x_{n-1})$$

where $h = \frac{b}{1-b} < 1$.
For $m > n$,

$$\begin{aligned} p(x_m, x_n) &\leq p(x_m, x_{m-1}) + p(x_{m-1}, x_{m-2}) + ... + p(x_{n+1}, x_n) - \sum_{r=1}^{m-n-1} p(x_{m-r}, x_{m-r}) \\ &\leq p(x_m, x_{m-1}) + p(x_{m-1}, x_{m-2}) + ... + p(x_{n+1}, x_n) \\ &\leq (h^{m-1} + h^{m-2} + ... + h^n)p(x_1, x_0) \\ &\leq \frac{h^n}{1-h} p(x_1, x_0) \end{aligned}$$

and

$$\|p(x_m, x_n)\| \leq \frac{h^n}{1-h} K \|p(x_1, x_0)\|$$

This implies that $p(x_m, x_n) \to 0$ $(n, m \to \infty)$. Hence (x_n) is a Cauchy sequence. By completeness of X, there is $u \in X$ such that $x_n \to u$ $(n \to \infty)$. Therefore

$$\lim_{n\to\infty} p(x_n, u) = p(u, u) = \lim_{n\to\infty} p(x_n, x_n) = 0.$$

Since

$$\begin{aligned} p(Tu, u) &\leq p(Tx_n, Tu) + p(Tx_n, u) - p(Tx_n, Tx_n) \\ &\leq p(Tx_n, Tu) + p(Tx_n, u) \\ &\leq b\,[p(Tx_n, u) + p(Tu, x_n)] + p(x_{n+1}, u) \\ &\leq b\,[p(x_{n+1}, u) + p(Tu, u) + p(u, x_n) - p(u, u)] + p(x_{n+1}, u) \\ &\leq b\,[p(x_{n+1}, u) + p(Tu, u) + p(u, x_n)] + p(x_{n+1}, u) \end{aligned}$$

gives

$$p(Tu, u) \leq \frac{1}{1-b}\,[b(p(x_{n+1}, u) + p(u, x_n)) + p(x_{n+1}, u)]$$

and

$$\|p(Tu, u)\| \leq b. \frac{1}{1-b}\,[b(\|p(x_{n+1}, u)\| + \|p(u, x_n)\|) + \|p(x_{n+1}, u)\|] \to 0.$$

So $\|p(Tu, u)\| = 0$, which implies that $Tu = u$ and so u is a fixed point of T.
Let v be another fixed point of T, then $p(u, v) = p(Tu, Tv) \leq b\,[p(Tu, v) + p(Tv, u)] = 2bp(u, v) \Rightarrow (1 - 2b)p(u, v) \leq 0$. Since $b \in [0, \frac{1}{2})$, $p(u, v) = 0$ i.e. $u = v$. So the fixed point is unique. □

Following is the Reich's [9] type contraction mapping considered here to prove an another fixed point theorem in partial cone metric space.

Theorem 3.7. *Let (X, p) be a complete partial cone metric space, P be a normal cone with constant K. Supose that the mapping $T : X \to X$ satisfies the contractive condition*

$$p(Tx, Ty) \leq \alpha p(Tx, x) + \beta p(Ty, y) + \gamma p(x, y) \text{ for all } x, y \in X$$

where $0 \leq \alpha + \beta + \gamma < 1$ and α, β, γ are non-negative. Then T has a unique fixed point in X, and for any $x \in X$, the iterative sequence $(T^n x)$ converges to the fixed point.

Proof. Choose $x_0 \in X$. Define the sequence (x_n) as $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, ..., x_{n+1} = Tx_n = T^{n+1}x_0$,... We have

$$\begin{aligned} p(x_{n+1}, x_n) &= p(Tx_n, Tx_{n-1}) \\ &\leq \alpha p(Tx_n, x_n) + \beta p(Tx_{n-1}, x_{n-1}) + \gamma p(x_n, x_{n-1}) \\ &\leq \alpha p(x_{n+1}, x_n) + \beta p(x_n, x_{n-1}) + \gamma p(x_n, x_{n-1}) \end{aligned}$$

which implies that

$$p(x_{n+1}, x_n) \leq \frac{\beta + \gamma}{1 - \alpha} p(x_n, x_{n-1}) = hp(x_n, x_{n-1})$$

where $h = \frac{\beta+\gamma}{1-\alpha} < 1$. Proceeding as previous theorem, we see that (x_n) is a Cauchy sequence. By completeness of X, there is $u \in X$ such that $x_n \to u$ $(n \to \infty)$. Therefore

$$\lim_{n\to\infty} p(x_n, u) = p(u, u) = \lim_{n\to\infty} p(x_n, x_n) = 0.$$

Now

$$\begin{aligned} p(Tu, u) &\leq p(Tx_n, Tu) + p(Tx_n, u) - p(Tx_n, Tx_n) \\ &\leq p(Tx_n, Tu) + p(Tx_n, u) \\ &\leq \alpha p(Tx_n, x_n) + \beta p(Tu, u) + \gamma p(x_n, u) + p(x_{n+1}, u) \\ &\leq \alpha p(x_{n+1}, x_n) + \beta p(Tu, u) + \gamma p(x_n, u) + p(x_{n+1}, u) \end{aligned}$$

gives

$$p(Tu, u) \leq \frac{1}{1 - \beta} (\alpha p(x_{n+1}, x_n) + \gamma p(x_n, u) + p(x_{n+1}, u))$$

and

$$\|p(Tu, u)\| \leq K \frac{1}{1 - \beta} (\alpha \|p(x_{n+1}, x_n)\| + \gamma \|p(x_n, u)\| + \|p(x_{n+1}, u)\|) \to 0$$

So $\|p(Tu, u)\| = 0$, which implies that $Tu = u$ and so u is a fixed point of T.
Let v be another fixed point of T, then $p(u, v) = p(Tu, Tv) \leq \alpha p(Tu, u) + \beta p(Tv, v) + \gamma p(u, v) = \gamma p(u, v)$ gives $u = v$. So the fixed point is unique. □

4. Conclusion

Some additional properties of partial cone metric space have been established in this paper. We have generalized some more fixed point theorems due to Chatterjea [2] and Reich [9] in partial cone metric space apart from A.Sönmez's [12] work. However, these results have vast potential in solving various nonlinear problems in functional analysis, differential and integral equation and of course, in computer science and engineering.

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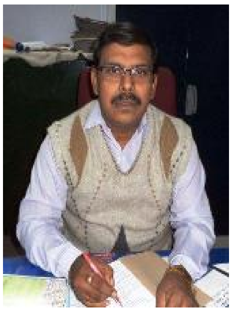
Debashis Dey received his M.Sc and Ph.D degree from The University of Burdwan, Burdwan-713104, West Bengal, India. His major research interests include fixed point theory in different topological spaces, fuzzy metric space and random fixed point theory. Dr. Dey has published several research papers on fixed point theory in various international journals of repute.



Mantu Saha awarded with Gold medal for securing position first in first class in M.Sc degree in Mathematics and he obtained Ph.D degree from The University of Burdwan, Burdwan-713104, West Bengal, India. Presently, Dr. Saha is working as an Associate Professor at the Department of Mathematics in The University of Burdwan. The area of his major research interest is fixed point theory in different topological spaces including stochastic fixed point theory. Dr. Saha also finds interest in the area of general topology and bitopological spaces. He had published several research papers on fixed point theory in various international journals of repute. He has been serving as reviewers in different international journals in the area of his expertise.