TWMS J. App. Eng. Math. V.3, N.2, 2013, pp. 119-132.

EVOLUTION EQUATIONS IN WEIGHTED STEPANOV-LIKE PSEUDO ALMOST AUTOMORPHIC SPACES

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ABSTRACT. In this paper, we obtain new existence and uniqueness theorem of weighted pseudo almost automorphic solutions for non-autonomous neutral partial evolution equations by applying the theory of semigroups of operators to evolution families. An interesting example is presented to illustrate our abstract result.

Keywords: Evolution family, fixed point theorem, weighted pseudo almost automorphy, nonautonomous neutral evolution equations.

AMS Subject Classification: 43A60, 47D06, 34K40, 37B55.

1. Introduction

Let $(\mathbb{X}, \|.\|)$ be a Banach space. In [20], Xia *et al* introduced a new class of functions called weighted Stepanov-like pseudo almost automorphic functions as a natural generalization of the concept of weigthed pseudo almost automorphy as well as the one of weighted Stepanov-like pseudo almost periodicity, the authors studied the basic properties of this new concept and showed that under some suitable conditions there exists a unique weighted pseudo almost automorphic mild solutions to the autonomous abstract differential equation.

$$\frac{d}{dt}[u(t) + f(t, Bu(t))] = Au(t) + g(t, Cu(t)) \quad , t \in \mathbb{R},$$

where $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is weighted pseudo almost automorphic, $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is weighted Stepanov-like pseudo almost automorphic for $p > 1$ and B, C are bounded linear operators.

In this paper, we continue the investigation of [20]. That is, we aim to establish the existence and uniqueness theorem of weigthed pseudo almost automorphic mild solutions to a class of partial neutral functional differential evolution equations with weighted Stepanov-like pseudo almost automorphic term

$$\frac{d}{dt}[u(t) + f(t, u(t))] = A(t)u(t) + g(t, u(t)) \quad , t \in \mathbb{R}, \tag{1}$$

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§ Manuscript received March 12, 2013.

where $A(.)$ is a given family of densely defined closed linear operators on a domain D independant of t.

In recent years, the theory of almost automorphy and its various extensions have attracted a great deal of attention due to their significance and applications in areas such as physics, mathematical biology, control theory, and others. For more on the concepts of almost automorphy and related issues we refer the reader to [3, 4, 5, 6, 8, 11, 12, 18, 19, 21, 22, 23] and the references therein.

2. Preliminaries

In this section, we fix notation and collect some preliminary facts that will be used in the sequel. throughout this paper, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ stand for the sets of positive integer, real and complex numbers, $(\mathbb{X}, \|.\|)$, $(\mathbb{Y}, \|.\|_{\mathbb{Y}})$ stand for Banach spaces, $B(\mathbb{X}, \mathbb{Y})$ denotes the Banach space of all bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ equipped with its natural topology. If $\mathbb{Y} = \mathbb{X}$ it is simply denotes by $B(\mathbb{X})$. Let $C(\mathbb{R}, \mathbb{X})$ denote the collection of continuous functions from $\mathbb{R}$ into $\mathbb{X}$ and let $BC(\mathbb{R}, \mathbb{X})$ denote the Banach space of all $\mathbb{X}$-valued bounded continuous functions equipped with the sup norm $\|u\|_\infty := \sup_{t\in\mathbb{R}} \|u(t)\|$ for each $u \in BC(\mathbb{R}, \mathbb{X})$. Similarly, $C(\mathbb{R}\times\mathbb{Y}, \mathbb{X}))$ denotes the collection of all jointly continuous functions from $\mathbb{R}\times\mathbb{Y}$ into $\mathbb{X}$ and $BC(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$ denotes the collection of all jointly bounded continuous functions $F : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$.

2.1. **Weighted pseudo almost automorphy.** Let us first recall some properties of almost automorphic functions.

Definition 2.1. [2, 15] *A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_n$ there exists a subsequence $(s_n)_n$ such that*

$$\lim_{m\to\infty}\lim_{n\to\infty} f(t + s_n - s_m) = f(t) \text{ for each } t \in \mathbb{R}.$$

this limit means that

$$g(t) := \lim_{n\to\infty} f(t + s_n) \quad \text{is well defined for each } t \in \mathbb{R}$$

and

$$f(t) = \lim_{n\to\infty} g(t - s_n) \quad \text{for each } t \in \mathbb{R}.$$

The collection of all such functions will be denoted by $AA(\mathbb{R}, \mathbb{X})$.

Theorem 2.1. [15] *Assume $f, g : \mathbb{R} \to \mathbb{X}$ are almost automorphic and λ is any scalar. Then the following holds true:*

(a) *$f + g, \lambda f, f_\tau(t) := f(t + \tau)$ and $\widehat{f}(t) := f(-t)$ are almost automorphic.*
(b) *The range R_f of f is precompact, so f is bounded.*
(c) *If $\{f_n\}$ is a sequence of almost automorphic functions and $f_n \to f$ uniformly on $\mathbb{R}$, then f is almost automorphic.*

Definition 2.2. [10] *A function $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{X})$ is called bi-almost automorphic if for every sequence of real numbers $(s'_n)_n$ we can extract a subsequence $(s_n)_n$ such that*

$$\lim_{m\to\infty}\lim_{n\to\infty} f(t + s_n - s_m, s + s_n - s_m) = f(t, s) \text{ for each } t, s \in \mathbb{R}.$$

this limit means that

$$g(t, s) := \lim_{n\to\infty} f(t + s_n, t + s_n) \quad \text{is well defined for each } t, s \in \mathbb{R}$$

and

$$f(t, s) = \lim_{n\to\infty} g(t - s_n, t - s_n) \quad \text{for each } t, s \in \mathbb{R}.$$

The collection of all such functions will be denoted by $bAA(\mathbb{R}\times\mathbb{R},\mathbb{X})$.

Definition 2.3. [13] *A function* $f\in C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ *is said to be almost automorphic if* $f(t,u)$ *is almost automorphic in* $t\in\mathbb{R}$ *uniformly for all* $u\in\ K$*, where* K *is any bounded subset of* $\mathbb{Y}$*. The collection of all such functions will be denoted by* $AA(\mathbb{R}\times\mathbb{Y},\mathbb{X})$.

Let $\mathbb{U}$ denotes the collection of all functions (weights) $\rho:\mathbb{R}\to(0,\infty)$ which are locally integrable over $\mathbb{R}$ such that $\rho(x)>0$ for almost each $x\in\mathbb{R}$. From now on, if $\rho\in\mathbb{U}$ and for $T>0$, we then set

$$m(T,\rho):=\int_{-T}^{T}\rho(x)dx.$$

As in the particular case when $\rho(x)=1$ for each $x\in\mathbb{R}$, we are exclusively interested in those weights, $\rho(x)=1$, for which $\lim_{T\to+\infty}m(T,\rho)=\infty$. In fact, throughout the rest of the paper, the sets of weights $\mathbb{U}_\infty$ and $\mathbb{U}_B$ stand respectively for

$$\mathbb{U}_\infty:=\{\rho\in\mathbb{U}:\lim_{T\to+\infty}m(T,\rho)=\infty\}$$

$$\mathbb{U}_B:=\{\rho\in\mathbb{U}_\infty:\rho\text{ is bounded and }\inf_{x\in\mathbb{R}}\rho(x)>0\}$$

It is clear that $\mathbb{U}_B\subset\mathbb{U}_\infty\subset\mathbb{U}$.

For $\rho\in\mathbb{U}_\infty$, define the classes of functions

$$PAA_0(\mathbb{R},\mathbb{X},\rho):=\{f\in BC(\mathbb{R},\mathbb{X}):\lim_{T\to\infty}\frac{1}{m(T,\rho)}\int_{-T}^{T}\rho(\sigma)\|f(\sigma)\|d\sigma=0\}$$

$$PAA_0(\mathbb{R}\times\mathbb{Y},\mathbb{X},\rho):=\{f\in BC(\mathbb{R}\times\mathbb{Y},\mathbb{X}):\lim_{T\to\infty}\frac{1}{m(T,\rho)}\int_{-T}^{T}\rho(\sigma)\|f(\sigma,u)\|d\sigma=0\\ \text{uniformly for u in any bounded subset of }\mathbb{Y}\}$$

We are now ready to introduce the concept of weighted pseudo almost automorphic functions:

Definition 2.4. [1] *Let* $\rho\in\mathbb{U}_\infty$*. A function* $f\in BC(\mathbb{R},\mathbb{X})$ *is said to be weighted pseudo almost automorphic if it can be decomposed as* $f=g+\varphi$ *where* $g\in AA(\mathbb{R},\mathbb{X})$ *and* $\varphi\in PAA_0(\mathbb{R},\mathbb{X},\rho)$*. The set of all such functions will be denoted by* $WPAA(\mathbb{R},\mathbb{X},\rho)$.

Definition 2.5. [1] *A function* $f\in C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ *is said to be weigthed pseudo almost automorphic if it can be decomposed as* $f=g+\varphi$ *where* $g\in AA(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ *and* $\varphi\in PAA_0(\mathbb{R}\times\mathbb{Y},\mathbb{X},\rho)$*. The set of all such functions will be denoted by* $WPAA(\mathbb{R}\times\mathbb{Y},\mathbb{X},\rho)$.

Theorem 2.2. [1] *Let* $\rho\in\mathbb{U}_B$*. If one equips* $WPAA(\mathbb{R},\mathbb{X},\rho)$ *with the sup norm, then* $WPAA(\mathbb{R},\mathbb{X},\rho)$ *turns out to be a Banach space.*

Theorem 2.3. [1] *Let* $\rho\in\mathbb{U}_\infty$*. Assume that* $f=g+\varphi\in WPAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho)$ *with* $g\in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $\varphi\in PAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho)$ *and there exists* $L_f>0$ *and* $L_g>0$ *such that*

$$\|f(t,u)-f(t,v)\|\le L_f\|u-v\|\ \text{and}\ \|g(t,u)-g(t,v)\|\le L_g\|u-v\|,\ \text{for } t\in\mathbb{R},u,v\in\mathbb{X}.$$

If $h(.)\in WPAA(\mathbb{R},\mathbb{X},\rho)$*, then* $F(.,h(.))\in WPAA(\mathbb{R},\mathbb{X},\rho)$.

2.2. **Weighted Stepanov-like pseudo almost automorphy.** Let $L^p(\mathbb{R},\mathbb{X})$ denote the space of all classes of equivalence (with respect to the equality almost everywhere on $\mathbb{R}$) of measurable functions $f:\mathbb{R}\to\mathbb{X}$ such that $\|f\|\in L^p(\mathbb{R})$. Let $L^p_{loc}(\mathbb{R},\mathbb{X})$ denote the space of all classes of equivalence of measurable functions $f:\mathbb{R}\to\mathbb{X}$ such that the restriction of every bounded subinterval of $\mathbb{R}$ is in $L^p(\mathbb{R},\mathbb{X})$.

Definition 2.6. [17] *The Bochner transform $f^b(t,s), t\in\mathbb{R}, s\in[0,1]$, of a function $f:\mathbb{R}\to\mathbb{X}$ is defined by $f^b(t,s):=f(t+s)$.*

Remark 2.1. [17] *A function $\varphi(t,s), t\in\mathbb{R}, s\in[0,1]$, is the Bochner transform of a certain function $f, \varphi(t,s)=f^b(t,s)$, if and only if $\varphi(t+\tau,s-\tau)=\varphi(s,t)$ for all $t\in\mathbb{R}, s\in[0,1]$ and $\tau\in[s-1,s]$.*

Definition 2.7. [17] *The Bochner transform $F^b(t,s,u), t\in\mathbb{R}, s\in[0,1], u\in\mathbb{X}$, of a function $F:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ is defined by $F^b(t,s,u):=F(t+s,u)$ for each $u\in\mathbb{X}$.*

Definition 2.8. [17] *Let $p\in[1,\infty)$. The space $BS^p(\mathbb{R},\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f:\mathbb{R}\to\mathbb{X}$ such that $f^b\in L^\infty(\mathbb{R},L^p(0,1;\mathbb{X}))$. This is a Banach space with the norm*

$$\|f\|_{S^p}=\|f^b\|_{L^\infty(\mathbb{R},L^p)}=\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\|f(\tau)\|^p d\tau\Big)^{\frac{1}{p}}$$

Definition 2.9. [16] *The space $S^pAA(\mathbb{R},\mathbb{X})$ of Stepanov-like almost automorphic functions (or $\mathbb{S}^p-$almost automorphic functions) , consists of all $f\in BS^p(\mathbb{R},\mathbb{X})$ such that $f^b\in AA(\mathbb{R},L^p(0,1;\mathbb{X}))$. That is, a function $f\in L^p_{loc}(\mathbb{R},\mathbb{X})$ is said to be Stepanov-like almost automorphic if its Bochner transform $f^b:\mathbb{R}\to L^p(0,1;\mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $(s'_n)_n$ there exists a subsequence $(s_n)_n$ such that*

$$\lim_{m\to\infty}\lim_{n\to\infty}\Big(\int_t^{t+1}\|f(s+s_n-s_m)-f(s)\|^p ds\Big)^{\frac{1}{p}} \text{ for each } t\in\mathbb{R}.$$

This limit means that there exists a function $g\in L^p_{loc}(\mathbb{R},\mathbb{X})$ such that

$$\Big(\int_0^1\|f(s+s_n)-g(s)\|^p ds\Big)^{\frac{1}{p}}\to 0 \quad \text{and} \ \Big(\int_0^1\|g(s-s_n)-f(s)\|^p ds\Big)^{\frac{1}{p}}\to 0$$

as $n\to\infty$ pointwise on $\mathbb{R}$.

Definition 2.10. [20] *Let $\rho\in\mathbb{U}_\infty$. A function $f\in BS^p(\mathbb{R},\mathbb{X})$ is said to be weighted Stepanov-like pseudo almost automorphic (or weighted $\mathbb{S}^p-$pseudo almost automorphic) if it can be decomposed as*

$$f=g+\varphi$$

where $g^b\in AA(\mathbb{R},L^p(0,1;\mathbb{X}))$ and $\varphi^b\in PAA_0(\mathbb{R},L^p(0,1;\mathbb{X}),\rho)$. The set of all such functions will be denoted by $S^pWPAA(\mathbb{R},\mathbb{X},\rho)$.

Definition 2.11. [20] *Let $\rho\in\mathbb{U}_\infty$. A function $F:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ with $F(.,u)\in L^p_{loc}(\mathbb{R},\mathbb{X})$ for each $u\in\mathbb{Y}$, is said to be weighted Stepanov-like pseudo almost automorphic (or weighted $\mathbb{S}^p-$pseudo almost automorphic) if it can be decomposed as $F=G+\Phi$ where $G^b\in AA(\mathbb{R}\times\mathbb{Y},L^p(0,1;\mathbb{X}))$ and $\Phi^b\in PAA_0(\mathbb{R}\times\mathbb{Y},L^p(0,1;\mathbb{X}),\rho)$. The collection of such functions will be denoted by $S^pWPAA(\mathbb{R}\times\mathbb{Y},\mathbb{X},\rho)$.*

Theorem 2.4. [20] *$WPAA(\mathbb{R}\times\mathbb{Y},\mathbb{X},\rho)\subseteq S^pWPAA(\mathbb{R}\times\mathbb{Y},\mathbb{X},\rho)$ and $S^qWPAA(\mathbb{R}\times\mathbb{Y},\mathbb{X},\rho)\subseteq S^pWPAA(\mathbb{R}\times\mathbb{Y},\mathbb{X},\rho)$ for $1\le p<q<+\infty$.*

Theorem 2.5. [20] *Let $\rho \in \mathbb{U}_\infty$. Assume that $f, g \in S^pWPAA(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \rho)$. Then the following holds true:*

1. $f + g \in S^pWPAA(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \rho)$.
2. $\lambda f \in S^pWPAA(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \rho)$ *for any* $\lambda \in \mathbb{R}$.
3. *If*
$$\limsup_{t\to\infty} \frac{\rho(t+\tau)}{\rho(t)} \quad and \quad \limsup_{T\to\infty} \frac{m(T+|\tau|,\rho)}{m(T,\rho)}$$
are finite for any $\tau \in \mathbb{R}$, then $f(t-\tau) \in S^pWPAA(\mathbb{R}\times\mathbb{Y}, \mathbb{X}, \rho)$ and $S^pWPAA(\mathbb{R}\times \mathbb{Y}, \mathbb{X}, \rho)$ equipped with the norm $\|.\|_{S^p}$ is a Banach space.

Theorem 2.6. [20] *Let $\rho \in \mathbb{U}_\infty$. Assume that $f = g + \varphi \in S^pWPAA(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \rho)$ with $g^b \in AA(\mathbb{R} \times \mathbb{X}, L^p(0, 1; \mathbb{X}))$, $\varphi^b \in PAA_0(\mathbb{R} \times \mathbb{X}, L^p(0, 1; \mathbb{X}), \rho)$ and*

(a) *there exists $L_f > 0$ and $L_g > 0$ such that*
$$\|f(t, u) - f(t, v)\| \le L_f\|u - v\| \quad and \ \|g(t, u) - g(t, v)\| \le L_g\|u - v\|$$
for all $t \in \mathbb{R}$ and ll $u, v \in \mathbb{X}$.

(b) *$h = \alpha + \beta \in S^pWPAA(\mathbb{R}, \mathbb{X}, \rho)$ with $\alpha^b \in AA(\mathbb{R}, L^p(0, 1; \mathbb{X}))$, $\beta^b \in PAA_0(\mathbb{R}, L^p(0, 1; \mathbb{X}), \rho)$ and $K := \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact in $\mathbb{X}$.*

Then $f(., h(.)) \in S^pWPAA(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \rho)$.

3. Existence of pseudo almost automorphic mild solutions

Fix $\rho \in \mathbb{U}_\infty$. Throughout this paper, We assume that
$$\limsup_{t\to\infty} \frac{\rho(t+\tau)}{\rho(t)} \quad \text{and} \quad \limsup_{T\to\infty} \frac{m(T+|\tau|,\rho)}{m(T,\rho)}$$
are finite for any $\tau \in \mathbb{R}$,

This section is devoted to the search of a weighted pseudo almost automorphic solution to Eq.(1). For that , we suppose among others that there exists a Banach space $(\mathbb{Y}, \|.\|_{\mathbb{Y}})$ such that the embedding
$$(\mathbb{Y}, \|.\|_{\mathbb{Y}}) \hookrightarrow (\mathbb{X}, \|.\|)$$
is continuous. Let $C > 0$ be the bound of this embedding. In addition to the above, we assume that the following assumptions hold

(H_1) The system
$$u'(t) = A(t)u(t), \quad t \ge s, \quad u(s) = \varphi \in \mathbb{X}, \tag{2}$$
has an associated evolution family of operators $\{U(t, s) : t \ge s \ \text{ with } \ s, t \in \mathbb{R}\}$. In addition, we assume that the domain of the operators $A(t)$ are constant in t, that is, $D(A(t)) = D = \mathbb{Y}$ for all $t \in \mathbb{R}$ and that the evolution family $U(t, s)$ is asymptotically stable in the sense that there exist some constants $M, \delta > 0$ such that
$$\|U(t, s)\|_{B(\mathbb{X})} \le Me^{-\delta(t-s)}$$
for all $s, t \in \mathbb{R}$ with $t \ge s$.

(H_2) The function $s \to A(s)U(t, s)$ defined from $(-\infty, t)$ into $B(\mathbb{Y}, \mathbb{X})$ is strongly measurable and there exist a measurable function $H : (0, \infty) \to (0, \infty)$ with $H \in L^1(0, \infty)$ and a constant $\omega > 0$ such that
$$\|A(s)U(t, s)\|_{B(\mathbb{Y},\mathbb{X})} \le e^{-\omega(t-s)}H(t - s), \quad t, s \in \mathbb{R}, \quad t > s.$$
and the series $\sum_{k=1}^{\infty}[\int_{k-1}^{k} e^{-q\omega\sigma}H(\sigma)^q d\sigma]^{\frac{1}{q}}$ converges for $q > 1$.

(H_3) The function $(t,s) \to U(t,s)y$ belongs to $bAA(\mathbb{T},\mathbb{Y})$ uniformly for $y \in \mathbb{X}$, where $\mathbb{T} := \{(t,s) \in \mathbb{R} \times \mathbb{R} : t \geq s\}$.

(H_4) The function $(t,s) \to A(s)U(t,s)y \in bAA(\mathbb{T},\mathbb{X})$ uniformly for $y \in \mathbb{Y}$.

(H_5) The function $f = f_1 + f_2 \in WPAA(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \rho)$ with $f_1 \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$, $f_2 \in PAA_0(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \rho)$ and there exist $L, L_{f_1} > 0$ such that

$$\|f(t,u) - f(t,v)\|_{\mathbb{Y}} \leq L\|u-v\| \quad \text{and} \quad \|f_1(t,u) - f_1(t,v)\|_{\mathbb{Y}} \leq L_{f_1}\|u-v\|$$

for all $t \in \mathbb{R}$ and $u, v \in \mathbb{X}$.

(H_6) The function $g = g_1 + g_2 \in SWPAA(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \rho)$ with $g_1^b \in AA(\mathbb{R}, L^p(0,1;\mathbb{X}))$, $g_2^b \in PAA_0(\mathbb{R}, L^p(0,1;\mathbb{X}), \rho)$ and there exist $L, L_{g_1} > 0$ such that

$$\|g(t,u) - g(t,v)\| \leq L\|u-v\| \quad \text{and} \quad \|g_1(t,u) - g_1(t,v)\| \leq L_{g_1}\|u-v\|,$$

for all $t \in \mathbb{R}$ and $u, v \in \mathbb{X}$.

Definition 3.1. [14, 9] *A family of bounded linear operators $(U(t,s))_{t\geq s}$ on a Banach space X is called an evolution family for Eq.(2) if*

(1) *$U(t,s) = U(t,r)U(r,s)$ and $U(t,t) = I$ for all $t, r, s \in \mathbb{R}$ such that $t \geq r \geq s$.*

(2) *the map $(t,s) \mapsto U(t,s)x$ is continuous and $U(t,s) \in B(\mathbb{X},\mathbb{Y})$ for all $x \in \mathbb{X}$ and $t, s \in \mathbb{R}$ such that $t > s$.*

(3) *the map $(s,t] \to B(\mathbb{X}), t \mapsto U(t,s)$ is differentiable with*

$$\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s).$$

Definition 3.2. *A continuous function $u : \mathbb{R} \to \mathbb{X}$ is said to be a mild solution to Eq.(1) provided that the function $s \mapsto A(s)U(t,s)f(s,u(s))$ is integrable on (s,t) and*

$$\begin{aligned} u(t) &= -f(t,u(t)) + U(t,s)[u(s) + f(s,u(s))] \\ &\quad - \int_s^t A(\sigma)U(t,\sigma)f(\sigma,u(\sigma))d\sigma + \int_s^t U(t,\sigma)g(\sigma,u(\sigma))d\sigma, \end{aligned} \tag{3}$$

for all $t \geq s$ and all $s \in \mathbb{R}$.

Now, we state our main result

Theorem 3.1. *Assume that $(H_1) - (H_6)$ hold. If L is small enough, then there exists a unique mild solution $u \in WPAA(\mathbb{R},\mathbb{X})$ of Eq.(1) such that*

$$u(t) = -f(t,u(t)) + \int_{-\infty}^t U(t,\sigma)g(\sigma,u(\sigma))d\sigma - \int_{-\infty}^t A(\sigma)U(t,\sigma)f(\sigma,u(\sigma))d\sigma, \quad t \in \mathbb{R}. \tag{4}$$

Lemma 3.1. *Suppose that $(H_1), (H_3)$ and (H_6) hold. Then the nonlinear integral operator Λ defined by*

$$(\Lambda u)(t) = \int_{-\infty}^t U(t,\sigma)g(\sigma,u(\sigma))d\sigma$$

maps $WPAA(\mathbb{R},\mathbb{X},\rho)$ into itself.

Proof of lemma. From previous assumptions one can easily see that Λ is well defined. Moreover, let $u \in WPAA(\mathbb{R},\mathbb{X})$, Using (H_6) and composition theorem on weighted Stepanov-like pseudo almost automorphic functions (Theorem 2.6), we deduce that $G(t) := g(t,u(t)) \in S^pWPAA(\mathbb{R},\mathbb{X})$. Now let $G = \phi + \psi$, where $\phi^b \in AA(\mathbb{R}, L^p(0,1;\mathbb{X}))$ and $\psi^b \in PAA_0(\mathbb{R}, L^p(0,1;\mathbb{X}), \rho)$, then Λ can be decomposed as

$$(\Lambda u)(t) = \Phi(t) + \Psi(t)$$

where

$$\Phi(t) = \int_{-\infty}^{t} U(t,\sigma)\phi(\sigma)d\sigma \quad \text{and} \quad \Psi(t) = \int_{-\infty}^{t} U(t,\sigma)\psi(\sigma)d\sigma.$$

Next, we show that $\Phi \in AA(\mathbb{R},\mathbb{X})$ and $\Psi \in PAA_0(\mathbb{R},\mathbb{X},\rho)$.

To prove that $\Phi \in AA(\mathbb{R},\mathbb{X})$, we consider

$$\Phi_k(t) := \int_{k-1}^{k} U(t,t-\sigma)\phi(t-\sigma)d\sigma = \int_{t-k}^{t-k+1} U(t,\sigma)\phi(\sigma)d\sigma,$$

for each $t \in \mathbb{R}$ and $k \in \mathbb{N}$.

Let $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, where $p > 1$. From (H_1) it follows that the function $s \mapsto U(t,s)\phi(s)$ is integrable over $(-\infty, t)$ for each $t \in \mathbb{R}$. Using the Hölder's inequality, it follows that

$$\begin{aligned}
\|\Phi_k(t)\| &\leq M \int_{t-k}^{t-k+1} e^{-\delta(t-\sigma)}\|\phi(\sigma)\|d\sigma \\
&\leq M \left(\int_{t-k}^{t-k+1} e^{-q\delta(t-\sigma)}d\sigma\right)^{\frac{1}{q}} \left(\int_{t-k}^{t-k+1} \|\phi(\sigma)\|^p d\sigma\right)^{\frac{1}{p}} \\
&\leq \frac{M}{\sqrt[q]{q\delta}} \left(e^{-q(k-1)\delta} - e^{-qk\delta}\right)^{\frac{1}{q}} \|\phi\|_{S^p} \\
&\leq M e^{-k\delta} \sqrt[q]{\frac{1+e^{q\delta}}{q\delta}} \|\phi\|_{S^p}.
\end{aligned}$$

Since $M \sqrt[q]{\frac{1+e^{q\delta}}{q\delta}} \sum_{k=1}^{\infty} e^{-k\delta} < \infty$, we deduce from the well-known Weierstrass test that the series $\sum_{k=1}^{\infty} \Phi_k(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore

$$\Phi(t) = \int_{-\infty}^{t} U(t,\sigma)\phi(\sigma)d\sigma = \sum_{k=1}^{\infty} \Phi_k(t),$$

$\Phi \in C(\mathbb{R},\mathbb{X})$, and $\|\Phi(t)\| \leq \sum_{k=1}^{\infty} \|\Phi_k(t)\| \leq M \sqrt[q]{\frac{1+e^{q\delta}}{q\delta}} \sum_{k=1}^{\infty} e^{-k\delta} \|\phi\|_{S^p}$.

Fix $k \in \mathbb{N}$, let us take a sequence $(s'_n)_n$ and show that there exists a subsequence $(s_n)_n$ of $(s'_n)_n$ such that

$$\lim_{m\to\infty}\lim_{n\to\infty} \|\Phi_k(t+s_n-s_m) - \Phi_k(t)\| = 0 \quad \text{for each } t \in \mathbb{R}. \tag{5}$$

Since $U(t,s)x \in bAA(\mathbb{R}\times\mathbb{R},\mathbb{Y})$ uniformly for $x \in \mathbb{X}$ and $\phi \in S^pAA(\mathbb{R},\mathbb{X})$, there exists a subsequence $(s_n)_n$ of $(s'_n)_n$ such that, for each $t \in \mathbb{R}, x \in \mathbb{X}$

$$\lim_{m\to\infty}\lim_{n\to\infty} \|U(t+s_n-s_m, t+s_n-s_m-\sigma)x - U(t,t-\sigma)x\| = 0 \tag{6}$$

$$\lim_{m\to\infty}\lim_{n\to\infty} \left(\int_{k-1}^{k} \|\phi(\sigma+s_n-s_m) - \phi(\sigma)\|^p d\sigma\right)^{\frac{1}{p}} = 0. \tag{7}$$

On the other hand, using triangle inequality, (H_1) and Hölder's inequality, we obtain that

$$\begin{aligned}
\|\Phi_k(t+s_n-s_m)-\Phi_k(t)\| &\leq M\int_{k-1}^{k} e^{-\delta\sigma}\|\phi(t+s_n-s_m-\sigma)-\phi(t-\sigma)\|d\sigma \\
&+\int_{k-1}^{k}\Big\|\left(U(t+s_n-s_m,t+s_n-s_m-\sigma)-U(t-\sigma)\right)\phi(t-\sigma)\Big\|d\sigma \\
&\leq M\left(\int_{k-1}^{k} e^{-q\delta\sigma}\right)^{\frac{1}{q}}\left(\int_{k-1}^{k}\|\phi(t+s_n-s_m-\sigma)-\phi(t-\sigma)\|^p d\sigma\right)^{\frac{1}{p}} \\
&+\int_{k-1}^{k}\|\left(U(t+s_n-s_m,t+s_n-s_m-\sigma)-U(t-\sigma)\right)\phi(t-\sigma)\|d\sigma \\
&\leq Me^{-k\delta}\sqrt[q]{\frac{1+e^{q\delta}}{q\delta}}\left(\int_{k-1}^{k}\|\phi(t+s_n-s_m-\sigma)-\phi(t-\sigma)\|^p d\sigma\right)^{\frac{1}{p}} \\
&+\int_{k-1}^{k}\|(U(t+s_n-s_m,t+s_n-s_m-\sigma)-U(t-\sigma))\,\phi(t-\sigma)\|\,d\sigma.
\end{aligned}$$

Thus, the Lebesgue dominated convergence theorem and Eq.(6)-(7) lead to Eq.(5), therefore, to $\Phi_k \in AA(\mathbb{R},\mathbb{X})$. Applying (Theorem 2.1), we deduce that the uniform limit

$$\Phi(.)=\sum_{k=1}^{\infty}\Phi_k(.)\in AA(\mathbb{R},\mathbb{X}).$$

Now, we prove that $\Psi \in PAA_0(\mathbb{R},\mathbb{X},\rho)$. For this, we set

$$\Psi_k(t):=\int_{k-1}^{k}U(t,t-\sigma)\psi(t-\sigma)d\sigma=\int_{t-k}^{t-k+1}U(t,\sigma)\psi(\sigma)d\sigma,$$

for each $t\in\mathbb{R}$ and $k\in\mathbb{N}$.
By carrying similar arguments as above, one has

$$\begin{aligned}
\|\Psi_k(t)\| &\leq M\int_{t-k}^{t-k+1} e^{-\delta(t-\sigma)}\|\psi(\sigma)\|d\sigma \\
&\leq M\left(\int_{t-k}^{t-k+1} e^{-q\delta(t-\sigma)}d\sigma\right)^{\frac{1}{q}}\left(\int_{t-k}^{t-k+1}\|\psi(\sigma)\|^p d\sigma\right)^{\frac{1}{p}} \\
&\leq Me^{-k\delta}\sqrt[q]{\frac{1+e^{q\delta}}{q\delta}}\left(\int_{t-k}^{t-k+1}\|\psi(\sigma)\|^p d\sigma\right)^{\frac{1}{p}} \\
&\leq C(M,q,\delta)\|\psi\|_{S^p}.
\end{aligned} \tag{8}$$

So, we deduce that $\Psi_k(t)\in BC(\mathbb{R},\mathbb{X})$, $\sum_{k=1}^{\infty}\Psi_k(t)$ is uniformly convergent on $\mathbb{R}$ and

$$\Psi(t):=\sum_{k=1}^{\infty}\Psi_k(t)=\int_{-\infty}^{t}U(t,\sigma)\psi(\sigma)d\sigma\in BC(\mathbb{R},\mathbb{X}).$$

To complete the proof, it remains to show that

$$\lim_{T\to\infty}\frac{1}{m(T,\rho)}\int_{-T}^{T}\rho(t)\|\Psi(t)\|dt=0.$$

In fact, one has

$$\frac{1}{m(T,\rho)}\int_{-T}^{T}\rho(t)\|\Psi_k(t)\|dt \leq \frac{C(M,q,\delta)}{m(T,\rho)}\int_{-T}^{T}\rho(t)\left(\int_{t-k}^{t-k+1}\|\psi(\sigma)\|^p d\sigma\right)^{\frac{1}{p}}dt.$$

Since $\psi^b \in PAA_0(\mathbb{R}, L^p(0,1;\mathbb{X}),\rho)$, the above inequality leads to $\Psi_k \in PAA_0(\mathbb{R},\mathbb{X},\rho)$. Using the following inequality

$$\frac{1}{m(T,\rho)}\int_{-T}^{T}\rho(t)\|\Psi(t)\|dt \leq \frac{1}{m(T,\rho)}\int_{-T}^{T}\rho(t)\|\Psi(t)-\sum_{k=1}^{\infty}\Psi_k(t)\|dt + \sum_{k=1}^{\infty}\frac{1}{m(T,\rho)}\int_{-T}^{T}\rho(t)\|\Psi_k(t)\|dt$$

we deduce that the uniform limit $\Psi(.) = \sum_{k=1}^{\infty}\Psi_k(.) \in PAA_0(\mathbb{R},\mathbb{X},\rho)$, which ends the proof. □

Lemma 3.2. *Suppose that $(H_1),(H_2),(H_4)$ and (H_5) hold. Then the nonlinear integral operator Γ defined by*

$$(\Gamma u)(t) = \int_{-\infty}^{t} A(\sigma)U(t,\sigma)f(\sigma,u(\sigma))d\sigma$$

maps $WPAA(\mathbb{R},\mathbb{X},\rho)$ into itself.

Proof of lemma. Let $u \in WPAA(\mathbb{R},\mathbb{X})$, Using (H_5) and composition theorem on weighted pseudo almost automorphic functions (Theorem 2.3), we deduce that $F(t) := f(t,u(t)) \in WPAA(\mathbb{R},\mathbb{Y}) \subset S^pWPAA(\mathbb{R},\mathbb{Y}) \subset S^pWPAA(\mathbb{R},\mathbb{X})$. The proof is, up to some slight modifications, similar to the proof of Lemma 3.1. Indeed, write $F = \phi + \psi$, where $\phi^b \in AA(\mathbb{R}, L^p(0,1;\mathbb{X}))$ and $\psi^b \in PAA_0(\mathbb{R}, L^p(0,1;\mathbb{X}),\rho)$, then Γ can be decomposed as

$$(\Gamma u)(t) = \Phi(t) + \Psi(t)$$

where

$$\Phi(t) = \int_{-\infty}^{t} A(\sigma)U(t,\sigma)\phi(\sigma)d\sigma \quad \text{and} \quad \Psi(t) = \int_{-\infty}^{t} A(\sigma)U(t,\sigma)\psi(\sigma)d\sigma.$$

Next, we show that $\Phi \in AA(\mathbb{R},\mathbb{X})$ and $\Psi \in PAA_0(\mathbb{R},\mathbb{X},\rho)$.

To prove that $\Phi \in AA(\mathbb{R},\mathbb{X})$, we consider

$$\Phi_k(t) := \int_{k-1}^{k} A(t-\sigma)U(t,t-\sigma)\phi(t-\sigma)d\sigma = \int_{t-k}^{t-k+1} A(\sigma)U(t,\sigma)\phi(\sigma)d\sigma,$$

for each $t \in \mathbb{R}$ and $k \in \mathbb{N}$.

Let $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, where $p > 1$. From (H_2) it follows that the function $s \mapsto A(s)U(t,s)\phi(s)$ is integrable over $(-\infty,t)$ for each $t \in \mathbb{R}$. Using the Hölder's inequality, it follows that

$$\begin{aligned}\|\Phi_k(t)\| &\leq \int_{t-k}^{t-k+1} e^{-\omega(t-\sigma)}H(t-\sigma)\|\phi(\sigma)\|_{\mathbb{Y}}d\sigma \\ &\leq \left(\int_{t-k}^{t-k+1} e^{-q\omega(t-\sigma)}H(t-\sigma)^q d\sigma\right)^{\frac{1}{q}}\left(\int_{t-k}^{t-k+1}\|\phi(\sigma)\|_{\mathbb{Y}}^p d\sigma\right)^{\frac{1}{p}} \\ &\leq C\left(\int_{k-1}^{k} e^{-q\omega\sigma}H(\sigma)^q d\sigma\right)^{\frac{1}{q}}\|\phi\|_{S^p}.\end{aligned}$$

Since the series $\sum_{k=1}^{\infty}\left(\int_{k-1}^{k} e^{-q\omega\sigma}H(\sigma)^q d\sigma\right)^{\frac{1}{q}}$ is convergent, we deduce from the well-known Weierstrass test that the series $\sum_{k=1}^{\infty}\Phi_k(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore

$$\Phi(t)=\int_{-\infty}^{t} A(\sigma)U(t,\sigma)\phi(\sigma)d\sigma=\sum_{k=1}^{\infty}\Phi_k(t),$$

$\Phi\in C(\mathbb{R},\mathbb{X})$, and $\|\Phi(t)\|\leq\sum_{k=1}^{\infty}\|\Phi_k(t)\|\leq K_1\|\phi\|_{S^p}$. Where $K_1>0$ is a constant.

Fix $k\in\mathbb{N}$, let us take a sequence $(s'_n)_n$ of real numbers. Since $A(s)U(t,s)x\in bAA(\mathbb{R}\times\mathbb{R},\mathbb{X})$ uniformly for $x\in\mathbb{X}$ and $\phi\in S^pAA(\mathbb{R},\mathbb{Y})\subset S^pAA(\mathbb{R},\mathbb{X})$, then for every sequence $(s'_n)_n$ there exists a subsequence $(s_n)_n$ and functions θ,ϕ_1 such that

$$\lim_{n\to\infty}A(s+s_n)U(t+s_n,s+s_n)x=\theta(t,s)x \quad \text{for each} \quad t,s\in\mathbb{R},x\in\mathbb{X}. \tag{9}$$

$$\lim_{n\to\infty}\theta(t-s_n,s-s_n)x=A(s)U(t,s)x \quad \text{for each} \quad t,s\in\mathbb{R},x\in\mathbb{X}. \tag{10}$$

$$\lim_{n\to\infty}\|\phi(t+s_n+.)-\phi_1(t+.)\|_{S^p}=0, \quad \text{for each} \quad t\in\mathbb{R}. \tag{11}$$

$$\lim_{n\to\infty}\|\phi_1(t-s_n+.)-\phi(t+.)\|_{S^p}=0 \quad \text{for each} \quad t\in\mathbb{R}. \tag{12}$$

We set

$$\Phi_k^1(t):=\int_{k-1}^{k}\theta(t,t-\sigma)\phi_1(t-\sigma)d\sigma$$

Using triangle inequality, (H_2) and Hölder's inequality, we obtain that

$$\begin{aligned}
I &:= \|\Phi_k(t+s_n)-\Phi_k^1(t)\|\\
&\leq \Big\|\int_{k-1}^{k}A(t+s_n-\sigma)U(t+s_n,t+s_n-\sigma)(\phi(t+s_n-\sigma)-\phi_1(t-\sigma))d\sigma\Big\|\\
&+\Big\|\int_{k-1}^{k}\left(A(t+s_n-\sigma)U(t+s_n,t+s_n-\sigma)-\theta(t,t-\sigma)\right)\phi_1(t-\sigma)d\sigma\Big\|\\
&\leq \left(\int_{k-1}^{k}e^{-q\omega\sigma}H(\sigma)^q d\sigma\right)^{\frac{1}{q}}\left(\int_{k-1}^{k}\|\phi(t+s_n-\sigma)-\phi_1(t-\sigma)\|_{\mathbb{Y}}^p d\sigma\right)^{\frac{1}{p}}\\
&+\int_{k-1}^{k}\|\left(A(t+s_n-\sigma)U(t+s_n,t+s_n-\sigma)-\theta(t,t-\sigma)\right)\phi_1(t-\sigma)\|d\sigma\\
&:= I_1+I_2.
\end{aligned}$$

By Eq.(11), $\lim_{n\to\infty}I_1=0$. By using the Lebesgue dominated convergence theorem and Eq.(9), one can get $\lim_{n\to\infty}I_2=0$. Thus,

$$\lim_{n\to\infty}\Phi_k(t+s_n)=\int_{k-1}^{k}\theta(t,t-\sigma)\phi_1(t-\sigma)d\sigma, \quad \text{for each} \quad t\in\mathbb{R}.$$

Analogously, one can prove that

$$\lim_{n\to\infty}\int_{k-1}^{k}\theta(t-s_n,t-s_n-\sigma)\phi_1(t-s_n-\sigma)d\sigma=\Phi_k(t), \quad \text{for each} \quad t\in\mathbb{R}.$$

Therefore, $\Phi_k\in AA(\mathbb{R},\mathbb{X})$. Applying (Theorem 2.1), we deduce that the uniform limit

$$\Phi(.)=\sum_{k=1}^{\infty}\Phi_k(.)\in AA(\mathbb{R},\mathbb{X}).$$

Now, we prove that $\Psi \in PAA_0(\mathbb{R},\mathbb{X},\rho)$. For this, we set

$$\Psi_k(t) := \int_{k-1}^{k} A(t-\sigma)U(t,t-\sigma)\psi(t-\sigma)d\sigma = \int_{t-k}^{t-k+1} A(\sigma)U(t,\sigma)\psi(\sigma)d\sigma,$$

for each $t \in \mathbb{R}, k \in \mathbb{N}$. note that

$$\begin{aligned} \|\Psi_k(t)\| &\leq \int_{t-k}^{t-k+1} e^{-\omega(t-\sigma)}H(t-\sigma)\|\psi(\sigma)\|_{\mathbb{Y}}d\sigma \\ &\leq \left(\int_{t-k}^{t-k+1} e^{-q\omega(t-\sigma)}H(t-\sigma)^q d\sigma\right)^{\frac{1}{q}} \left(\int_{t-k}^{t-k+1} \|\psi(\sigma)\|_{\mathbb{Y}}^p d\sigma\right)^{\frac{1}{p}} \\ &\leq C\left(\int_{k-1}^{k} e^{-q\omega\sigma}H(\sigma)^q d\sigma\right)^{\frac{1}{q}} \left(\int_{t-k}^{t-k+1} \|\psi(\sigma)\|^p d\sigma\right)^{\frac{1}{p}} \\ &:= K_2\left(\int_{t-k}^{t-k+1} \|\psi(\sigma)\|^p d\sigma\right)^{\frac{1}{p}}. \end{aligned} \tag{13}$$

By carrying similar arguments as above, we deduce that $\Psi_k(t) \in BC(\mathbb{R},\mathbb{X})$, $\sum_{k=1}^{\infty}\Psi_k(t)$ is uniformly convergent on $\mathbb{R}$ and

$$\Psi(t) := \sum_{k=1}^{\infty}\Psi_k(t) = \int_{-\infty}^{t} U(t,\sigma)\psi(\sigma)d\sigma \in BC(\mathbb{R},\mathbb{X})$$

To complete the proof, it remains to show that

$$\lim_{T\to\infty}\frac{1}{m(T,\rho)}\int_{-T}^{T}\rho(t)\|\Psi(t)\|dt = 0.$$

In fact, one has

$$\frac{1}{m(T,\rho)}\int_{-T}^{T}\rho(t)\|\Psi_k(t)\|dt \leq \frac{K_2}{m(T,\rho)}\int_{-T}^{T}\rho(t)\left(\int_{t-k}^{t-k+1}\|\psi(\sigma)\|^p d\sigma\right)^{\frac{1}{p}} dt.$$

Since $\psi^b \in PAA_0(\mathbb{R}, L^p(0,1;\mathbb{X}),\rho)$, the above inequality leads to $\Psi_k \in PAA_0(\mathbb{R},\mathbb{X},\rho)$. and from the following inequality

$$\begin{aligned} \frac{1}{m(T,\rho)}\int_{-T}^{T}\rho(t)\|\Psi(t)\|dt &\leq \frac{1}{m(T,\rho)}\int_{-T}^{T}\rho(t)\|\Psi(t) - \sum_{k=1}^{\infty}\Psi_k(t)\|dt \\ &+ \sum_{k=1}^{\infty}\frac{1}{m(T,\rho)}\int_{-T}^{T}\rho(t)\|\Psi_k(t)\|dt \end{aligned}$$

we deduce that the uniform limit $\Psi(.) = \sum_{k=1}^{\infty}\Psi_k(.) \in PAA_0(\mathbb{R},\mathbb{X},\rho)$, which ends the proof. □

Proof of theorem. first of all, define the nonlinear operator Ξ on $WPAA(\mathbb{R},\mathbb{X},\rho)$ by setting

$$(\Xi u)(t) = -f(t,u(t)) + \int_{-\infty}^{t} U(t,\sigma)g(\sigma,u(\sigma))d\sigma - \int_{-\infty}^{t} A(\sigma)U(t,\sigma)f(\sigma,u(\sigma))d\sigma, \quad t \in \mathbb{R}.$$

Let $u \in WPAA(\mathbb{R},\mathbb{X})$. Using (H_5) and composition theorem on weighted pseudo almost automorphic functions (Theorem 2.3), we deduce that $f(t,u(t)) \in WPAA(\mathbb{R},\mathbb{Y}) \subset WPAA(\mathbb{R},\mathbb{X})$, thanks to Lemma 3.1 and Lemma 3.2 we deduce that the operator Ξ is well defined and maps $WPAA(\mathbb{R},\mathbb{X},\rho)$ into itself.

Secondly, we will show that $\Xi : WPAA(\mathbb{R},\mathbb{X},\rho) \mapsto WPAA(\mathbb{R},\mathbb{X},\rho)$ has a unique fixed point.
For $u, v \in WPAA(\mathbb{R},\mathbb{X},\rho)$, we get

$$\|\Xi(u)(t) - \Xi(v)(t)\|_\infty \le L\left[M\delta^{-1} + C(1 + \int_0^\infty e^{-\omega s}H(s)ds)\right]\|u - v\|_\infty.$$

Clearly, if $L < \left[M\delta^{-1} + C(1 + \int_0^\infty e^{-\omega s}H(s)ds)\right]^{-1}$, then the operator Ξ becomes a strict contraction on $WPAA(\mathbb{R},\mathbb{X},\rho)$, and by the Banach fixed-point principle, Ξ has a unique fixed point in $WPAA(\mathbb{R},\mathbb{X},\rho)$.

Finally, we will prove that u satisfies Eq.(4) for all $t \ge s$, all $s \in \mathbb{R}$. For this, we let

$$u(s) = -f(s, u(s)) + \int_{-\infty}^{s} U(s,\sigma)g(\sigma, u(\sigma))d\sigma - \int_{-\infty}^{s} A(\sigma)U(s,\sigma)f(\sigma, u(\sigma))d\sigma. \tag{14}$$

Multiply both sides of Eq.(14) by $U(t, s)$ for all $t \ge s$, then

$$\begin{aligned}
U(t,s)u(s) &= -U(t,s)f(s,u(s)) + \int_{-\infty}^{s} U(t,\sigma)g(\sigma,u(\sigma))d\sigma - \int_{-\infty}^{s} A(\sigma)U(t,\sigma)f(\sigma,u(\sigma))d\sigma \\
&= -U(t,s)f(s,u(s)) + \int_{-\infty}^{t} U(t,\sigma)g(\sigma,u(\sigma))d\sigma - \int_{s}^{t} U(t,\sigma)g(\sigma,u(\sigma))d\sigma \\
&\quad - \int_{-\infty}^{t} A(\sigma)U(t,\sigma)f(\sigma,u(\sigma))d\sigma + \int_{s}^{t} A(\sigma)U(t,\sigma)f(\sigma,u(\sigma))d\sigma \\
&= -U(t,s)f(s,u(s)) + u(t) + f(t,u(t)) + \int_{s}^{t} A(\sigma)U(t,\sigma)f(\sigma,u(\sigma))d\sigma \\
&\quad - \int_{s}^{t} U(t,\sigma)g(\sigma,u(\sigma))d\sigma.
\end{aligned}$$

Hence u is a mild solution to Eq.(1). The proof is complete. □

4. Example

Consider the following heat equation with Dirichlet boundary conditions:

$$\begin{cases}
\dfrac{\partial}{\partial t}\left[u(t,x) + f(t,u(t,x))\right] = \dfrac{\partial^2}{\partial x^2}u(t,x) + (-2 + \sin(t) + \sin(\pi t))u(t,x) + g(t,u(t,x)), \\
u(t,x) = 0, \text{ for } x = 0,\ \pi \text{ and } t \in \mathbb{R}, \\
f(t,u(t,x)) = 0, \text{ for } x = 0,\ \pi \text{ and } t \in \mathbb{R}.
\end{cases} \tag{15}$$

Let $\mathbb{X} := L^2([0,\pi])$ equipped with its natural topology and let A be the operator given by

$$A\psi(\xi) := \psi''(\xi), \qquad \forall \xi \in [0,\pi], \psi \in D(A),$$

where, $\quad D(A) = \left\{\psi \in L^2([0,\pi]) : \psi'' \in L^2([0,\pi]), \psi(0) = \psi(\pi) = 0\right\}.$

It is well known [10, 7, 14] that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\ge 0}$ on $\mathbb{X}$. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2, n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by

$$\psi_n(\xi) := \sqrt{\frac{2}{\pi}}\sin(n\xi).$$

In addition to the above, the following properties hold

(a) $T(t)\psi = \sum_{n=1}^{\infty} e^{-n^2 t}\langle \psi, \psi_n \rangle \psi_n$ for each $\psi \in D(A)$.
(b) $\|T(t)\| \leq e^{-t}$, for $t \geq 0$.

Now, define a family of linear operators $A(t)$ by

$$\begin{cases} D(A(t)) = D(A), \quad \text{for} \quad t \in \mathbb{R} \\ A(t)\psi(\xi) := (A - 2 + \sin(t) + \sin(\pi t))\psi(\xi), \qquad \forall \xi \in [0,\pi], \psi \in D(A), \end{cases} \tag{16}$$

Then, $A(t)$ generates an evolution family $(U(t,s))_{t\geq s}$ such that

$$U(t,s)\psi(\xi) = T(t-s)e^{\int_s^t (-2+\sin(\sigma)+\sin(\pi\sigma))d\sigma}\psi(\xi)$$

Hence, $\|U(t,s)\| \leq e^{-(t-s)}, \quad \text{for} \quad t \geq s.$

Let $\mathbb{Y} := D(A(t))$ denote the space $D(A(t))$ endowed with the graph norm, then $\mathbb{Y} \hookrightarrow \mathbb{X}$ is continuously embedded with bound $C = 1$.

In view of the above, Eq.(15) can be transformed into abstract form Eq.(1) and assumptions $(H_1) - (H_4)$ are satisfied with $M = 1, \delta = 1$.

Now the following proposition is an immediate consequence of Theorem 3.1.

Proposition 4.1. *Assume that (H_5) and (H_6) hold. If L is small enought, then Eq.(15) admits a unique weighted pseudo almost automorphic mild solution.*

5. Conclusion

In this manuscript, the authors studied the existence and uniqueness of weighted pseudo almost automorphic mild solutions to the abstract differential equations Eq.(1) with weighted stepanov-like pseudo almost automorphic coefficients. The authors consider an equation similar to that in [10], however consider an operator A(t) more general. The results presented are interesting and will be useful for other people working in the area. Such a result (Theorem 3.1) generalizes most of the results encountered in the literature on the existence and uniqueness of weighted pseudo-almost automorphic solutions to Eq.(1).

References

[1] J. Blot, G.M. Mophou, G.M. N'Guérékata, (2009), Weighted pseudo-almost automorphic functions and applications, Nonlinear Anal., 71, 903-909.
[2] S. Bochner, (1962), A new approach to almost periodicity, Proc. Nat. Acad. Sci. U.S.A., 48 , 2039-2043.
[3] H.S. Ding, J. Liang, T.J. Xiao, (2009), Some prorerties of Stepanov-like almost automorphic functions and applications to abstract evolutions equations, Appl. Anal., 88 (7), 1079-1091.
[4] H.S. Ding, J. Liang, T.J. Xiao, (2010), Almost automorphic mild solutions to nonautonomous semilinear evolution equations in Banach spaces, Nonlinear Anal. TMA., 73, 1426-1438.
[5] T. Diagana, (2008), Stepanov-like pseudo-almost periodicity and its applications to some nonautonomous differential equations, Nonlinear Anal. TMA., 69, 4277-4285.
[6] T. Diagana, (2007), Stepanov-like pseudo-almost periodic functions and their applications to differential equations, Commun. Math. Anal., 3 (1), 9-18.
[7] T. Diagana, (2009), Existence of pseudo-almost automorphic solutions to some abstract differential equations with S^p-pseudo-almost automorphic coefficients, Nonlinear Anal. TMA., 70, 3781-3790.
[8] T. Diagana, (2012), Evolution equations in generalized Stepanov-like pseudo almost automorphic spaces, EJDE. 49, 1-19.
[9] D. henry, (1981), Geometry Theory of Semilinear Parabolic Equations, Springer-verlag.
[10] Z. Hu, Z. jin, (2009), Stepanov-like pseudo almost automorphic mild solutions to nonautonomous evolution equations, Nonlinear Anal. TMA., 71, 2349-2360.
[11] Z. Hu, Z. jin, (2009), Stepanov-like pseudo almost periodic mild solutions to perturbed nonautonomous evolution equations with infinite delay, Nonlinear Anal. TMA., 71, 5381-5391.

[12] Z. Hu, Z. jin, (2012), Stepanov-like pseudo almost periodic mild solutions to nonautonomous neutral partial evolution equations, Nonlinear Anal. TMA., 75, 244-252.
[13] J. Liang, J. Zhang, T.J. Xiao, (2008), Composition of pseudo-almost automorphic and asymptotically almost automorphic functions, J. Math. Anal. Appl., 340, 1493-1499.
[14] K.J. Negel, (2000), One-Parameter Semigroups for Linear Evolution Equatons, Springer-Verlag., Vol. 194.
[15] G.M. N'Guérékata, (2005), Topics in almost automorphy, Springer-Verlag, New York.
[16] G.M. N'Guérékata, A. Pankov, (2008), Stepanov-like almost automorphic functions and monotone evolution equations, Nonlinear Anal. TMA., 68, 2658-2667.
[17] A. Pankov, (1990), Bounded and Almost Periodic Solutions of Nonlinear Operator Differential Equations, Kluwer, Dordrecht.
[18] T.J. Xiao, J. Liang, J. Zhang, (2008), Pseudo-almost automorphic solutions to semilinear differential equations in Banach space, Semigroup Forum., 76, 518-524.
[19] T.J. Xiao, X.X. ZHU, J. Liang, (2009), Pseudo-almost automorphic mild solutions to nonautonomous differential equations and applications, Nonlinear Anal. TMA., 70, 4079-4085.
[20] Z. Xia, M. Fan, (2012), weighted Stepanov-like pseudo almost automorphy and applications, Nonlinear Anal. TMA., 75, 2378-2397.
[21] M. Zitane, C. Bensouda, (2013), Generalized Stepanov-like pseudo almost automorphic solutions to some classes of nonautonomous evolution equations, J. Math. Comput. Sci., 3 (1), 278-303.
[22] M. Zitane, C. Bensouda, (2012), Stepanov-like pseudo almost automorphic solutions to nonautonomous neutral partial evolution equations, Journal of Applied Mathematics & Bioinformatics., 2 (3), 193-211.
[23] M. Zitane, C. Bensouda, (2012), Weighted pseudo-almost automorphic solutions to a neutral delay integral equation of advanced type, Applied Mathemathical sciences ., 6 (122), 6087-6095.

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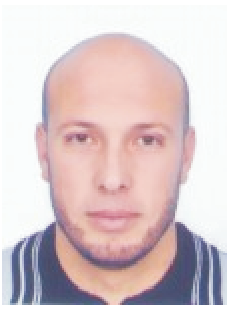

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