TWMS J. App. Eng. Math. V.3, N.2, 2013, pp. 214-222.

ON PARALLEL SURFACES IN MINKOWSKI 3-SPACE

YASİN ÜNLÜTÜRK[1], ERDAL ÖZÜSAĞLAM[2]

Abstract. In this paper, we study on some properties of parallel surfaces in Minkowski 3-space. The results given in this paper were given in Euclidean space by [7, 8]. By using these two former studies, we show these properties in Minkowski 3-space. Also we give the relation among the fundamental forms of parallel surfaces in Minkowski 3-space. Finally we show that how a curve which is geodesic on M become again a geodesic on parallel surface M^r by the normal map in Minkowski 3-space.

Keywords: parallel surfaces, fundamental forms, geodesic curve.

AMS Subject Classification: 53A05, 53B25, 53B30

1. Introduction

Parallel surfaces as a subject of differential geometry have been intriguing for mathematicians throughout history and so it has been a research field. In theory of surfaces, there are some special surfaces such as ruled surfaces, minimal surfaces and surfaces of constant curvature in which differential geometers are interested. Among these surfaces, parallel surfaces are also studied in many papers [1, 2, 3, 4, 5, 11, 13]. Craig had studied to find parallel of ellipsoid [2]. Eisenhart gave a chapter for parallel surfaces in his famous *A Treatise On the Differential Geometry of Curves and Surfaces* [3]. Nizamoğlu had stated parallel ruled surface as a curve depending on one-parameter and gave some geometric properties of such a surface [11].

We can explain a parallel surface something like that a surface M^r whose points are at a constant distance along the normal from another surface M is said to be parallel to M. So, there are infinite number of surfaces because we choose the constant distance along the normal arbitrarily. From the definition it follows that a parallel surface can be regarded as the locus of point which are on the normals to M at a non-zero constant distance r from M.

In this paper, we study on some properties of parallel surfaces in Minkowski 3-space. The results given in this paper were given in Euclidean space by [7, 8]. By using these two former studies, we show these properties in Minkowski 3-space. Also we give the relation among the fundamental forms of parallel surfaces in Minkowski 3-space. Finally we show that how a curve which is geodesic on M become again a geodesic on parallel surface M^r by the normal map in Minkowski 3-space.

[1] Department of Mathematics, Kırklareli University, 39060, Kırklareli, TURKEY.
e-mail: yasinunluturk@kirklareli.edu.tr
[2] Department of Mathematics, Aksaray University, 68100, Aksaray, TURKEY.
e-mail: materdalo@gmail.com
§ Manuscript received July 09, 2013.
TWMS Journal of Applied and Engineering Mathematics, Vol.3, No.2;

2. Preliminaries

Let $\mathbb{E}_1^3$ be the three-dimensional Minkowski space, that is, the three-dimensional real vector space $\mathbb{R}^3$ with the metric

$$< d\mathbf{x}, d\mathbf{x} >= dx_1^2 + dx_2^2 - dx_3^2$$

where (x_1, x_2, x_3) denotes the canonical coordinates in $\mathbb{R}^3$. An arbitrary vector $\mathbf{x}$ of $\mathbb{E}_1^3$ is said to be spacelike if $< \mathbf{x}, \mathbf{x} >> 0$ or $\mathbf{x} = \mathbf{0}$, timelike if $< \mathbf{x}, \mathbf{x} >< 0$ and lightlike or null if $< \mathbf{x}, \mathbf{x} >= 0$ and $\mathbf{x} \neq \mathbf{0}$. A timelike or light-like vector in $\mathbb{E}_1^3$ is said to be causal. For $\mathbf{x} \in \mathbb{E}_1^3$, the norm is defined by $\|\mathbf{x}\| = \sqrt{|< \mathbf{x}, \mathbf{x} >|}$, then the vector $\mathbf{x}$ is called a spacelike unit vector if $< \mathbf{x}, \mathbf{x} >= 1$ and a timelike unit vector if $< \mathbf{x}, \mathbf{x} >= -1$. Similarly, a regular curve in $\mathbb{E}_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [12]. For any two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ of $\mathbb{E}_1^3$, the inner product is the real number $< \mathbf{x}, \mathbf{y} >= x_1y_1 + x_2y_2 - x_3y_3$ and the vector product is defined by $\mathbf{x} \times \mathbf{y} = ((x_2y_3 - x_3y_2), (x_3y_1 - x_1y_3), -(x_1y_2 - x_2y_1))$ [10].

Let $X = X(u, v)$ be a local parametrization. Let $\{X_u, X_v\}$ be a local base of the tangent plane at each point. Let us recall that the first fundamental form I is the metric on T_pM, i.e.,

$$\begin{aligned} I_p &= \langle , \rangle_p : T_pM \times T_pM \to \mathbb{R} \\ I_p(u, v) &= \langle u, v \rangle_p . \end{aligned} \tag{2.1}$$

Let $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ be the matricial expression of I with respect to $B = \{X_u, X_v\}$,

$$E = \langle X_u, X_u \rangle , \quad F = \langle X_u, X_v \rangle , \quad G = \langle X_v, X_v \rangle . \tag{2.2}$$

So the first fundamental form I is explained as

$$II = edu^2 + 2fdudv + gdv^2. \tag{2.3}$$

We take the normal vector field given by

$$\mathbf{N} = \frac{X_u \times X_v}{\|X_u \times X_v\|}. \tag{2.4}$$

The second fundamental form II at the point p is

$$\begin{aligned} II &: T_pM \times T_pM \to \mathbb{R} \\ II(u, v) &= \langle S(u), v \rangle \end{aligned} \tag{2.5}$$

where $S(u)$ is shape operator of the surface in Minkowski 3-space. Let $\begin{pmatrix} e & f \\ f & g \end{pmatrix}$ be the matricial expression of II with respect to B, that is,

$$\begin{aligned} e &= -\langle X_u, N_u \rangle = \langle N, X_{uu} \rangle \\ f &= -\langle X_u, N_v \rangle = -\langle X_v, N_u \rangle = \langle N, X_{uv} \rangle \\ g &= -\langle X_v, N_v \rangle = \langle N, X_{vv} \rangle . \end{aligned} \tag{2.6}$$

So, the second fundamental form II is written as follows:

$$II = edu^2 + 2fdudv + gdv^2 \tag{2.7}$$

[10]. The third fundamental form III of the surface is

$$III(u, v) = -\langle S^2(u), v \rangle = \langle S(u), S(v) \rangle \tag{2.8}$$

where $S(X)$ is shape operator of the surface in Minkowski 3-space [9].

Let a semi-Riemannian hypersurface of M be $\overline{M}$. Gauss equation for semi-Riemannian hypersurfaces for $\forall\ V,\ W \in \chi(\overline{M})$ is

$$D_V W = \overline{D}_V W + \varepsilon g(S(V), W)N \tag{2.9}$$

where D is the Levi-Civita connection of M and $\overline{D}$ is so closely related to the Levi-Civita connection of $\overline{M}$ we have used the same notation for both [12].

Let φ be the position vector of a point P on M and φ^r be the position vector of a point $f(P)$ on the parallel surface M^r. Then $f(P)$ is at a constant distance r from P along the normal to the surface M. Therefore the parametrization for M^r is given by

$$\varphi^r(u,v) = \varphi(u,v) + r\mathbf{N}(u,v) \tag{2.10}$$

where r is a constant scalar and $\mathbf{N}$ is the unit normal vector field on M [6].

Definition 2.1. Let M and M^r be two surfaces in Minkowski 3-space. The function

$$\begin{array}{llll} f: & M \longrightarrow & M^r \\ & p \longrightarrow & f(p) = p + r\mathbf{N}_p \end{array} \tag{2.11}$$

is called the parallellization function between M and M^r and furthermore M^r is called parallel surface to M in $\mathbb{E}_1^3$ where r is a given positive real number and $\mathbf{N}$ is the unit normal vector field on M [4].

Theorem 2.1. Let M be a surface and M^r be a parallel surface of M in Minkowski 3-space. Let $f: M \to M^r$ be the parallellization function. Then for $X \in \chi(M)$,

1) $f_*(X) = X + rS(X)$

2) $S^r(f_*(X)) = S(X)$

3) f preserves principal directions of curvature, that is

$$S^r(f_*(X)) = \frac{k}{1+rk} f_*(X) \tag{2.12}$$

where S^r is the shape operator on M^r, and k is a principal curvature of M at p in direction X [4].

3. Parallel Surfaces in $\mathbb{E}_1^3$

Let M be a surface of $\mathbb{E}_1^3$ with unit normal $N = (a_1, a_2, a_3)$ where each a_i is a C^∞ function on M and $-a_1^2 + a_2^2 + a_3^2 = \pm 1$. For any constant r in $\mathbb{R}$, let $M^r = \{p + rN_p : p \in M\}$. Thus if $p = (p_1, p_2, p_3)$ is on M, then $f(p) = p + rN_p = (p_1 + ra_1(p), p_2 + ra_2(p), p_3 + ra_3(p))$ defines a new surface M^r. The map f is called the natural map on M into M^r, and if f is univalent, then M^r is a parallel surface of M with unit normal N, i.e., $N_{f(p)} = N_p$ for all p in M.

Let M be a surface and M^r be its parallel surface in $\mathbb{E}_1^3$. The fundamental forms I^r, II^r, III^r of the surfaces M^r given by (2.10) are as follows:

$$\begin{aligned} I^r &= \langle d\varphi^r, d\varphi^r \rangle \\ &= \left\langle \frac{\partial \varphi^r}{\partial u}, \frac{\partial \varphi^r}{\partial u} \right\rangle (du)^2 + 2\left\langle \frac{\partial \varphi^r}{\partial u}, \frac{\partial \varphi^r}{\partial v} \right\rangle dudv + \left\langle \frac{\partial \varphi^r}{\partial v}, \frac{\partial \varphi^r}{\partial v} \right\rangle (dv)^2 \\ &= E^r\,(du)^2 + 2F^r dudv + G^r\,(dv)^2 \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} II^r &= \langle -d\varphi^r, dN\rangle \\ &= -\left\langle \frac{\partial\varphi^r}{\partial u}, \frac{\partial N^r}{\partial u}\right\rangle (du)^2 - 2\left\langle \frac{\partial\varphi^r}{\partial u}, \frac{\partial N^r}{\partial v}\right\rangle dudv - \left\langle \frac{\partial\varphi^r}{\partial v}, \frac{\partial N^r}{\partial v}\right\rangle (dv)^2 \\ &= e^r (du)^2 + 2f^r dudv + g^r (dv)^2 \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} III^r &= \langle dN^r, dN^r\rangle \\ &= \left\langle \frac{\partial N^r}{\partial u}, \frac{\partial N^r}{\partial u}\right\rangle (du)^2 + 2\left\langle \frac{\partial N^r}{\partial u}, \frac{\partial N^r}{\partial v}\right\rangle dudv + \left\langle \frac{\partial N^r}{\partial v}, \frac{\partial N^r}{\partial v}\right\rangle (dv)^2 \\ &= \left\langle \frac{\partial N}{\partial u}, \frac{\partial N}{\partial u}\right\rangle (du)^2 + 2\left\langle \frac{\partial N}{\partial u}, \frac{\partial N}{\partial v}\right\rangle dudv + \left\langle \frac{\partial N}{\partial v}, \frac{\partial N}{\partial v}\right\rangle (dv)^2 \\ &= \langle dN, dN\rangle \\ &= III. \end{aligned} \tag{3.3}$$

The equation $III^r = III$ obtained in (3.3) means that the third fundamental form is preserved for parallel surfaces in Minkowski 3-space. Let's give the coefficients of the first and second fundamental forms of parallel surface M^r in terms of the coefficients of surface M. The coefficient E^r is found as follows:

$$\begin{aligned} E^r &= \left\langle \frac{\partial\varphi^r}{\partial u}, \frac{\partial\varphi^r}{\partial u}\right\rangle = \left\langle \frac{\partial(\varphi + rN)}{\partial u}, \frac{\partial(\varphi + rN)}{\partial u}\right\rangle = \langle \varphi_u + rN_u, \varphi_u + rN_u\rangle \\ &= \langle \varphi_u, \varphi_u\rangle + 2r\langle \varphi_u, N_u\rangle + r^2\langle N_u, N_u\rangle \end{aligned}$$

or

$$E^r = E - 2re + r^2\langle N_u, N_u\rangle . \tag{3.4}$$

The coefficient F^r is obtained as follows:

$$\begin{aligned} F^r &= \left\langle \frac{\partial\varphi^r}{\partial u}, \frac{\partial\varphi^r}{\partial v}\right\rangle = \left\langle \frac{\partial(\varphi + rN)}{\partial u}, \frac{\partial(\varphi + rN)}{\partial v}\right\rangle = \langle \varphi_u + rN_u, \varphi_v + rN_v\rangle \\ &= \langle \varphi_u, \varphi_v\rangle + r\langle \varphi_u, N_v\rangle + r\langle N_u, \varphi_v\rangle + r^2\langle N_u, N_v\rangle \\ &= F - 2rf + r^2\langle N_u, N_v\rangle . \end{aligned} \tag{3.5}$$

The coefficient G^r is

$$\begin{aligned} G^r &= \left\langle \frac{\partial\varphi^r}{\partial v}, \frac{\partial\varphi^r}{\partial v}\right\rangle = \left\langle \frac{\partial(\varphi + rN)}{\partial v}, \frac{\partial(\varphi + rN)}{\partial v}\right\rangle = \langle \varphi_v + rN_v, \varphi_v + rN_v\rangle \\ &= \langle \varphi_v, \varphi_v\rangle + 2r\langle \varphi_v, N_v\rangle + r^2\langle N_v, N_v\rangle \\ &= G - 2rg + r^2\langle N_v, N_v\rangle . \end{aligned} \tag{3.6}$$

The coefficient e^r is

$$e^r = -\left\langle \frac{\partial\varphi^r}{\partial u}, \frac{\partial N}{\partial u}\right\rangle = -\langle \varphi_u + rN_u, N_u\rangle = -\langle \varphi_u, N_u\rangle - \langle N_u, N_u\rangle = e - r\langle N_u, N_u\rangle . \tag{3.7}$$

The coefficient f^r is

$$f^r = -\left\langle \frac{\partial \varphi^r}{\partial u}, \frac{\partial N}{\partial v} \right\rangle = -\langle \varphi_u + rN_u, N_v\rangle = -\langle \varphi_u, N_v\rangle - \langle N_u, N_v\rangle = f - r\langle N_u, N_v\rangle . \quad (3.8)$$

The coefficient g^r is

$$g^r = -\left\langle \frac{\partial \varphi^r}{\partial v}, \frac{\partial N}{\partial v} \right\rangle = -\langle \varphi_v + rN_v, N_v\rangle = -\langle \varphi_v, N_v\rangle - \langle N_v, N_v\rangle = g - r\langle N_v, N_v\rangle . \quad (3.9)$$

Theorem 3.1. Let M be a surface in $\mathbb{E}_1^3$. The fundamental forms of M are, respectively, denoted by I, II, III and Gaussian and mean curvatures of M denoted by K and H, respectively. Hence

$$III - 2HII + \varepsilon KI = 0 \quad (3.10)$$

where $\langle N, N\rangle = \varepsilon$.

Proof. $\dim T_pM = 2$ since $\dim M = 2$ for $n = 3$. Therefore the shape operator

$$S = T_M(P) \to T_M(P)$$

has characteristic polynomial of the second order. Furthermore, since the principal curvatures k_1 and k_2 are zeros of the polynomial, the characteristic polynomial of S is

$$P_S(\lambda) = \det(S - \lambda I_2) = \lambda^2 - (k_1 + k_2)\lambda + k_1.k_2$$

By Hamilton-Cayley theorem, the shape operator S is zero of the polynomial as follows:

$$S^2 - (k_1 + k_2)S + k_1.k_2 I_2 = 0. \quad (3.11)$$

On the other hand,

$$\begin{aligned} 0 &= [S^2 - (k_1 + k_2)S + k_1.k_2 I_2](X_P) \\ &= S^2(X_P) - (k_1 + k_2)S(X_P) + (k_1.k_2)X_P \end{aligned}$$

for $\forall\ X_P \in T_M(P)$. Also for $\forall\ Y_P \in T_M(P)$, we get the followings

$$\begin{aligned} 0 &= \left\langle S^2(X_P) - (k_1 + k_2)S(X_P) + (k_1.k_2)X_P, Y_P \right\rangle \\ &= \left\langle S^2(X_P), Y_P \right\rangle - (k_1 + k_2)\langle S(X_P), Y_P\rangle + k_1.k_2\langle X_P, Y_P\rangle . \end{aligned}$$

From the definitions, we have

$$III - 2HII + \varepsilon KI = 0.$$

Theorem 3.2. Let M be a surface and M^r be its parallel surface in $\mathbb{E}_1^3$. The fundamental forms of M^r are, respectively, denoted by I^r, II^r, III^r and Gaussian and mean curvatures of M^r denoted by K^r and H^r, respectively. Hence

$$III^r - 2H^r II^r + \varepsilon K^r I^r = 0 \quad (3.12)$$

where $\langle N, N\rangle = \varepsilon$.

Proof. $\dim T_{M^r}(f(P)) = 2$ since $\dim M^r = 2$ for $n = 3$. Therefore the shape operator

$$S^r = T_{M^r}(f(P)) \to T_{M^r}(f(P))$$

has characteristic polynomial of the second order. Furthermore, since the principal curvatures $\dfrac{k_1}{1 + rk_1}$ and $\dfrac{k_2}{1 + rk_2}$ are zeros of the polynomial, the characteristic polynomial of S^r is

$$\begin{aligned} P_S(\lambda) &= \det(S^r - \lambda I_2) \\ &= \lambda^2 - \left(\frac{k_1}{1 + rk_1} + \frac{k_2}{1 + rk_2} \right)\lambda + \frac{k_1}{1 + rk_1}\frac{k_2}{1 + rk_2} \end{aligned} \quad (3.13)$$

By Hamilton-Cayley theorem, the shape operator S^r is zero of the polynomial as follows:

$$S^{r^2} - \left(\frac{k_1}{1+rk_1} + \frac{k_2}{1+rk_2}\right) S^r + \left(\frac{k_1}{1+rk_1}\right)\left(\frac{k_2}{1+rk_2}\right) I_2 = 0. \tag{3.14}$$

On the other hand,

$$\begin{aligned} 0 &= \left[S^{r^2} - \left(\frac{k_1}{1+rk_1} + \frac{k_2}{1+rk_2}\right) S^r + \frac{k_1}{1+rk_1}\frac{k_2}{1+rk_2} I_2\right](f_*(X_p)) \\ &= S^{r^2}(f_*(X_p)) - \left(\frac{k_1}{1+rk_1} + \frac{k_2}{1+rk_2}\right) S^r(f_*(X_p)) + \left(\frac{k_1}{1+rk_1}\frac{k_2}{1+rk_2}\right) f_*(X_p) \end{aligned} \tag{3.15}$$

for $\forall\ f_*(X_p) \in T_{M^r}(f(p))$.

$$\begin{aligned} 0 &= \langle S^{r^2}(f_*(X_p)) - \left(\frac{k_1}{1+rk_1} + \frac{k_2}{1+rk_2}\right) S^r(f_*(X_p)) + \left(\frac{k_1}{1+rk_1}\frac{k_2}{1+rk_2}\right) f_*(X_p), f_*(Y_p)\rangle \\ &= \left\langle S^{r^2}(f_*(X_p)), f_*(Y_p)\right\rangle - \left(\frac{k_1}{1+rk_1} + \frac{k_2}{1+rk_2}\right)\langle S^r(f_*(X_p)), f_*(Y_p)\rangle \\ &+ \left(\frac{k_1}{1+rk_1}\right)\left(\frac{k_2}{1+rk_2}\right)\langle f_*(X_p), f_*(Y_p)\rangle \\ &= \left\langle S^{r^2}(f_*(X_p)), f_*(Y_p)\right\rangle - TrS^r\langle S^r(f_*(X_p)), f_*(Y_p)\rangle + \det S^r\langle f_*(X_p), f_*(Y_p)\rangle \end{aligned} \tag{3.16}$$

From Definitions,

$$\left\langle S^{r^2}(f_*(X_p)), f_*(Y_p)\right\rangle - \varepsilon H^r\langle S^r(f_*(X_p)), f_*(Y_p)\rangle + \varepsilon K^r\langle f_*(X_p), Y_P\rangle = 0$$

$$III^r - 2H^r II^r + \varepsilon K^r I^r = 0$$

Lemma 3.1. Let M be a surface and M^r be its parallel surface in $\mathbb{E}^3_1$. The fundamental forms of M and M^r are, respectively, denoted by I, II and I^r, II^r, and also Gaussian and mean curvatures of M denoted by K and H, respectively. Hence

$$I^r = (1-\varepsilon r^2 K)I - 2r(1-rH)II \quad \text{and} \quad II^r = \varepsilon rKI + (1-2rH)II.$$

Proof. Let the parallel surface M^r be given with the following parametrization:

$$\varphi^r(u,v) = \varphi(u,v) + rN(u,v).$$

The first fundamental form of M^r is obtained as follows:

$$\begin{aligned} I^r &= \langle d\varphi^r, d\varphi^r\rangle \\ &= \langle d(\varphi + rN), d(\varphi + rN)\rangle \\ &= \langle \varphi_u du + rN_u du + \varphi_v dv + rN_v dv, \varphi_u du + rN_u du + \varphi_v dv + rN_v dv\rangle \\ &= \langle\varphi_u,\varphi_u\rangle du^2 + 2r\langle\varphi_u, N_u\rangle du^2 + 2\langle\varphi_u,\varphi_v\rangle dudv + 2r\langle\varphi_u, N_v\rangle dudv \\ &+ 2r\langle\varphi_v, N_u\rangle dudv + r^2\langle N_u, N_u\rangle du^2 + 2r^2\langle N_u, N_v\rangle dudv + \langle\varphi_v,\varphi_v\rangle dv^2 \\ &+ 2r\langle\varphi_v, N_v\rangle dv^2 + r^2\langle N_v, N_v\rangle dv^2. \end{aligned} \tag{3.17}$$

By using (2.6) and (2.7) in (3.18) with keeping the expression $III = \langle dN, dN\rangle$ and Theorem 3.1 in mind, we get

$$\begin{aligned} I^r &= I - 2rII + r^2 III \\ &= I - 2rII + r^2(2HII - \varepsilon KI) \\ &= (1 - r^2\varepsilon K)I - 2r(1 - rH)II. \end{aligned}$$

Let's look at the second fundamental form of the parallel surfaces, that is,

$$\begin{aligned} & II^r = -\langle d\varphi^r, dN\rangle \\ & = -\langle \varphi_u du + rN_u du + \varphi_v dv + rN_v dv, N_u du + N_v dv\rangle \\ & = -\langle \varphi_u, N_u\rangle du^2 - \langle \varphi_u, N_v\rangle dudv - r\langle N_u, N_u\rangle du^2 - r\langle N_u, N_v\rangle dudv \\ & -\langle \varphi_v, N_u\rangle dudv\text{-} \langle \varphi_v, N_v\rangle dv^2 - r\langle N_u, N_v\rangle dudv - r\langle N_v, N_v\rangle dv^2. \end{aligned} \tag{3.18}$$

By using (2.6) and (2.7) in (3.18) with keeping the expression $III = \langle dN, dN\rangle$ and Theorem 3.1 in mind, we get

$$\begin{aligned} II^r &= II - rIII \\ &= II - r(2HII - \varepsilon KI) \\ &= \varepsilon rKI + (1 - 2rH)II. \end{aligned}$$

Theorem 3.3. Let M be a surface and M^r be a parallel surface of M in Minkowski 3-space. Let $f : M \to M^r$ be the parallellization function. Then for $X \in \chi(M)$,

1) f preserves the third fundamental form

2) f preserves umbilical point

3) There is the relation

$$I^r(f_*(X_p), f_*(Y_p)) = I(X_p, Y_p) + II(X_p, Y_p) + III(X_p, Y_p)$$

for $\forall\ X, Y \in \chi(M)$, $\forall\ p \in M$.

Proof. 1) The expression (3.1) is the proof. But we can give another proof as follows: Let the third fundamental forms of M and M^r be III and III^r, then

$$III^r(f_*(X_p), f_*(Y_p)) = \langle S^r(f_*(X_p)), S^r(f_*(Y_p))\rangle = \langle S(X), S(Y)\rangle_p = III(X_p, Y_p).$$

2) Let $p \in M$ be an umbilical point of M, then

$$S(X_p) = \lambda X_p$$

for $\forall\ X_p \in T_pM$ and only one $\lambda \in \mathbb{R}$. On the other hand, we have

$$f_*(X_p) = X + rS(X) = (1 + r\lambda)X.$$

Accordingly, we obtain

$$S^r(f_*(X_p)) = S(X) = \lambda X = \frac{\lambda}{1 + r\lambda} f_*(X_p) \tag{3.19}$$

for $\forall\ f_*(X_p) \in T_{f(p)}M^r$. The expression means that S^r is one time of the identity mapping of M^r at the point $f(p) \in M^r$. Thus if $p \in M$ be an umbilical point of M, then the point $f(p) \in M^r$ is also an umbilical point of M^r.

3)

$$\begin{aligned}\langle f_*(X), f_*(Y)\rangle_{f(p)} &= \langle X + rS(X), Y + rS(Y)\rangle_{f(p)} \\ &= \langle X, Y\rangle_p + 2r\langle S(X), Y\rangle_p + r^2\langle S(X), S(Y)\rangle_p \\ &= I(X_p, Y_p) + 2rII(X_p, Y_p) + r^2 III(X_p, Y_p)\end{aligned}$$

for $\forall\ X, Y \in \chi(M)$, $\forall\ p \in M$.

Theorem 3.4. If f preserves the second fundamental form, then M is a Lorentzian plane.

Proof. From the definition of the second fundamental form, we can write

$$\begin{aligned}&II^r(f_*(X), f_*(Y)) = \langle S^r(f_*(X)), f_*(Y)\rangle \\ &= \langle S(X), Y + rS(Y)\rangle \\ &= \langle S(X), Y\rangle + r\langle S(X), s(Y)\rangle\,;\end{aligned} \tag{3.20}$$

thus $\langle S(X), rS(Y)\rangle = \left\langle X, rS^2(Y)\right\rangle = 0$ for all X and Y, and hence

$$S^2(Y) = 0.$$

Thus the principal curvatures are zero, $S = 0$, and M is a Lorentzian plane.

Theorem 3.5. Let α be a geodesic curve on M, then $f \circ \alpha$ is also a geodesic curve on M^r if and only if $\widetilde{\nabla}_{\alpha'(u)} S(\alpha'(u)) = 0$, where $\widetilde{\nabla}$ is a Riemannian connection on M.

Proof. Let the parametrization of M^r be given as follows:

$$\varphi^r(u, v) = \varphi(u, v) + r\mathbf{N}(u, v)$$

where $\varphi(u, v)$ is the parametrization of M and N is differentiable normal vector field of M. Thus we can write

$$\varphi^r_u = \alpha'(u) + rS(\alpha'(u)) \tag{3.21}$$

where α is a curve such that $u \to \alpha'(u) = \varphi(u, v)$, $v = const.$

Let Riemannian connection on M be denoted by $\widetilde{\nabla}$, the tangent component of φ^r_{uu} be denoted by φ^{rT}_{uu}, and the normal component be denoted by $\widetilde{\varphi^r_{uu}}$, then Gauss equation can be written as follows:

$$\begin{aligned}&\widetilde{\varphi^r_{uu}} + \varphi^{rT}_{uu} = \widetilde{\nabla}_{\alpha'(u)}\alpha'(u) + r\widetilde{\nabla}_{\alpha'(u)}S(\alpha'(u)) + \varepsilon\langle S(\alpha'(u)), \alpha'(u)\rangle N \\ &+ r\varepsilon\left\langle S^2(\alpha'(u)), \alpha'(u)\right\rangle N.\end{aligned} \tag{3.22}$$

Thus, we get

$$\varphi^{rT}_{uu} = 0 \Leftrightarrow \widetilde{\nabla}_{\alpha'(u)}\alpha'(u) + r\widetilde{\nabla}_{\alpha'(u)}S(\alpha'(u)) = 0. \tag{3.23}$$

While the curve α is geodesic on M, if the curve $\alpha(u) + rN_u$ is also geodesic on M^r, then

$$\widetilde{\nabla}_{\alpha'(u)}S(\alpha'(u)) = 0.$$

Conversely, let's take $\widetilde{\nabla}_{\alpha'(u)}S(\alpha'(u))$=0, then

$$\widetilde{\nabla}_{\alpha'(u)}\alpha'(u) = 0$$

since

$$\varphi^{rT}_{uu} = \widetilde{\nabla}_{\alpha'(u)}\alpha'(u) + r\widetilde{\nabla}_{\alpha'(u)}S(\alpha'(u)) = 0$$

and α geodesic on M. In that case, φ^{rT}_{uu}=0, that is, the curve $\alpha(u) + rN_u$ is also geodesic on M^r.

References

[1] Çöken, A. Ç., Çiftci, Ü. and Ekici, C., 2008, On parallel timelike ruled surfaces with timelike rulings, Kuwait Journal of Science & Engineering, 35, 21–31.

[2] Craig, T., Note on parallel surfaces, Journal für die Reine und Angewandte Mathematik (Crelle's journal), 94, 162–170., 1883.

[3] Eisenhart, L.P., 1909, A treatise on the differential geometry of curves and surfaces, Ginn and Company, Boston: New York.

[4] Görgülü, A. and Çöken, C., The dupin indicatrix for parallel pseudo-Euclidean hypersurfaces in pseudo-Euclidean space in semi-Euclidean space R_1^n. Journ. Inst. Math. and Comp. Sci. (Math Series), 7(3), 221-225., 1994.

[5] A. Görgülü and A. C. Çöken, The Euler theorem for parallel pseudo-Euclidean hypersurfaces in pseudo-Euclidean space $\mathbb{E}_1^{n+1}$, Journ. Inst. Math. and Comp. Sci. (Math. Series) Vol:6, No.2 (1993), 161-165.

[6] Gray, A., Modern differential geometry of curves and surfaces, CRC Press, Inc., 1993.

[7] Hacısalihoğlu, H. H., 2000, Diferensiyel geometri, Cilt I-II, Ankara Üniversitesi, Fen Fakültesi Yayınları.

[8] Hicks, Noel J., Notes on Differential Geometry, Van Nostrant Reinhold Company, London, 1974.

[9] Kühnel, W., 2002, Differential geometry, curves-surfaces-manifolds, American Mathematical Society.

[10] Lopez, R., Differential geometry of curves and surfaces in Lorentz-Minkowski space, Mini-Course taught at the Instituto de Matematica e Estatistica (IME-USP), University of Sao Paulo, Brasil, 2008

[11] Nizamoglu, Ş., 1986, Surfaces réglées parallèles, Ege Üniv. Fen Fak. Derg., 9 (Ser. A), 37-48.

[12] O'Neill, B., Semi Riemannian geometry with applications to relativity, Academic Press, Inc. New York, 1983.

[13] Patriciu, A. M., 2010, On some ${}^{1,3}H_3$-helicoidal surfaces and their parallel surfaces at a certain distance in 3-dimensional Minkowski space, Annals of the University of Craiova, Maths. and Compt.Science Series, V 37 (4), 93-98.

Erdal Özüsağlam is an Assistant Professor at the Department of Mathematics in Aksaray University, Aksaray, Turkey. He received his MS degree Master of Science in 2002. Then started to work doctorate program in Eskisehir Osmangazi University Science Institute and he received degree of PhD, Doctor of Philosophy in 2007. His major research interests include ruled surfaces, Lorentz Geometry and Contact Manifolds.

Yasin Ünlütürk is an Assistant Professor at the Department of Mathematics in Kirklareli University, Kirklareli, Turkey. He received his PhD degree in Differential Geometry in 2011 from Eskisehir Osmangazi University, Eskisehir, Turkey. His major research interests include theory of curves and surfaces in semi-Riemannian geometry and particularly Weingarten surfaces, Parallel surfaces.