TWMS J. App. Eng. Math. V.4, N.1, 2014, pp. 127-131.

SUFFICIENT CONDITIONS FOR GENERALIZED SAKAGUCHI TYPE FUNCTIONS OF ORDER β

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Abstract. In this paper, we obtain some sufficient conditions for generalized Sakaguchi type function of order β, defined on the open unit disk. Many interesting outcomes of our results are also calculated.
Keywords: Generalized Sakaguchi type function of order β, Univalent functions.

AMS Subject Classification: 30C45 , 30C50 , 30C80.

1. Introduction

Let A_n be the class of the form

$$f(z) = z + a_{n+1}z^{n+1} + ... \tag{1}$$

that are analytic in the unit disk $\Delta = \{z \in C : |z| < 1\}$ and let $A_1 = A$. An analytic function $f(z) \in A_n$ is said to be in the generalized Sakaguchi class $S_n(\beta, s, t)$ if it satisfies

$$Re\left\{\frac{(s-t)zf'(z)}{f(sz) - f(tz)}\right\} > \beta, \qquad z \in \Delta \tag{2}$$

for some $\beta(0 \leq \beta < 1)$, s and t are real parameters, $s > t$ and for all $z \in \Delta$.
For $n = 1$ the generalized Sakaguchi class $S_n(\beta, s, t)$ reduces to the subclass $S(\beta, s, t)$ studied by Frasin [[2], see also [6], [7]]. For $n = 1, s = 1$, this class is reduced to $S(\beta, t)$ studied by Owa et al. [9, 10], Goyal and Goswami [3] and Cho et al.[1]. The class $S(0, -1)$ was introduced by Sakaguchi [12]. Recently T. Mathur et al. [[6], [7]] have introduced and studied some properties of $S(\beta, s, t)$.
In this paper, we obtain some sufficient conditions for functions $f(z) \in S_n(\beta, s, t)$. To prove our results, we need the following:

Lemma 1.1 (8). *Let Ω be a set in the complex plane C and suppose that ϕ is a mapping from $C^2 \times \Delta$ to C which satisfies $\phi(ix, y; z) \notin \Omega$ for $z \in \Delta$, and for all real x, y such that $y \leq -n(1 + x^2)/2$. If the function $p(z) = 1 + c_nz^n + ...$ is analytic in Δ and $\phi(p(z), zp'(z); z) \in \Omega$ for all $z \in \Delta$, then $Re(p(z)) > 0$.*

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§ Manuscript received September 13, 2013; revised: November 21, 2013.
TWMS Journal of Applied and Engineering Mathematics Vol.4 No.2

2. Main Results

Theorem 2.1. *If $f(z) \in A_n$ satisfies*

$$Re\left[\frac{(s-t)^2 zf'(sz)}{f(sz)-f(tz)}\left\{\frac{\alpha szf''(sz)}{f'(sz)}+\frac{\alpha tzf'(tz)}{f(sz)-f(tz)}+1\right\}\right]$$

$$>\alpha\beta\left\{s\beta+\frac{n}{2}(s-t)-(s-t)\right\}+\left\{\beta-\frac{n\alpha}{2}\right\}(s-t) \tag{3}$$

for $(z\in\Delta, 0\le\alpha\le 1, 0\le\beta<1$ *and* $t<s)$, *then* $f(z)\in S_n(\beta,s,t)$.

Proof. Define $p(z)$ by

$$\left\{\frac{(s-t)zf'(sz)}{f(sz)-f(tz)}\right\}=(1-\beta)p(z)+\beta.$$

Then $p(z)=1+c_nz^n+.....$ and is analytic in Δ.
A computation shows that

$$\frac{szf''(sz)}{f'(sz)}+\frac{tzf'(tz)}{f(sz)-f(tz)}=\frac{(s-t)(1-\beta)zp'(z)+s[(1-\beta)p(z)+\beta]^2-(s-t)[(1-\beta)p(z)+\beta]}{(s-t)[(1-\beta)p(z)+\beta]}$$

and hence

$$\frac{(s-t)^2zf'(sz)}{f(sz)-f(tz)}\left[\frac{\alpha szf''(sz)}{f'(sz)}+\frac{\alpha tzf'(tz)}{f(sz)-f(tz)}+1\right]$$

$$=\alpha(s-t)(1-\beta)zp'(z)+\alpha s(1-\beta)^2p^2(z)+(1-\beta[2s\alpha\beta+(s-t)(1-\alpha)]p(z)+\beta[s\alpha\beta+(s-t)(1-\alpha)] \tag{4}$$

$$=\phi(p(z),zp'(z);z)\ \text{(say)}$$

where

$$\phi(u,v;z)=\alpha(s-t)(1-\beta)v+\alpha s(1-\beta)^2u^2+(1-\beta)[2s\alpha\beta+(s-t)(1-\alpha)]u+\beta[s\alpha\beta+(s-t)(1-\alpha)]$$

For all real x and y satisfying $y\le -n(1+x^2)/2$, we have

$$Re[\phi(ix,y;z)]\le\alpha(s-t)(1-\beta)y-\alpha s(1-\beta)^2x^2+\beta[s\alpha\beta+(s-t)(1-\alpha)]$$

$$\le\alpha(s-t)(1-\beta)\left\{\frac{-(1+x^2)}{2}\right\}-\alpha s(1-\beta)^2x^2+\beta[s\alpha\beta+(s-t)(1-\alpha)]$$

$$=\frac{-\alpha n}{2}(s-t)(1-\beta)-\left\{\frac{\alpha n}{2}(s-t)(1-\beta)+\alpha\beta(1-\beta)^2\right\}x^2+\beta[s\alpha\beta+(1-\alpha)(s-t)]$$

$$\le\frac{-\alpha n}{2}(s-t)(1-\beta)+\beta[s\alpha\beta+(1-\alpha)(s-t)]$$

$$=\alpha\beta\left\{s\beta+\frac{n}{2}(s-t)-(s-t)\right\}+\left\{\beta-\frac{n\alpha}{2}\right\}(s-t)$$

Let $\Omega=\left\{w; Re(w)>\alpha\beta\left\{\beta+\frac{n}{2}(s-t)-(s-t)\right\}+\left\{\beta-\frac{n\alpha}{2}\right\}(s-t)\right\}$
Then $\phi(p(z),zp'(z);z)\in\Omega$ and $\phi(ix,y;z)\notin\Omega$ for all real x and $y\le -n(1+x^2)/2,\ z\in\Delta$.
By an application of Lemma 1.1, the result follows. □

Remark 2.1. *On putting $s=1$, in Theorem 2.1, we get the known results due to Goyal et al.*[9]

Theorem 2.2. *Let* $0 \leq \beta < 1, t < s$ *with* $-1 \leq \frac{t}{s} + \beta < 1$,

$$\lambda = (1-\beta)^2 \left\{ \frac{n}{2}(s-t) + s(1-\beta \right\}^2, \quad \mu = \left\{ \frac{n}{2}|(s-t)|(1-\beta) + \beta|(s-t-s\beta)| \right\}^2,$$

$$\nu = \left\{ s(1-\beta)^2 - \beta(s-t-s\beta) \right\}^2 \quad and \quad \sigma = \{(1-\beta)(2s\beta - t - s)\}^2 \tag{5}$$

satisfy$(\lambda + \mu - \nu + \sigma)\beta^2 < (1-2\beta)\mu$.
Also suppose that r_0 *be the positive real root of the equation*

$$2\lambda(1-\beta)^2 r^3 + \left\{(1-\beta)^2(2\lambda + \mu - \nu + \sigma) + 3\lambda\beta^2\right\} r^2 + 2\beta^2(2\lambda + \mu - \nu + \sigma)r$$
$$+(\lambda + 2\mu - \nu + \sigma)\beta^2 - (1-\beta)^2\mu = 0 \tag{6}$$

and

$$\rho^2 = \frac{(1-\beta)^2(1+r_0)}{(s-t)^2\left\{(1-\beta)^2 r_0 + \beta^2\right\}}[\lambda r_0^2 + (\lambda + \mu - \nu + \sigma)r_0 + \mu] \tag{7}$$

Now if $f(z) \in A_n$ *satisfies*

$$\left| \left(\frac{s-t)zf'(sz)}{f(sz) - f(tz)} - 1 \right) \left(\frac{szf''(sz)}{f'(sz)} + \frac{tzf'(tz)}{f(sz) - f(tz)} \right) \right| \leq \rho \quad (z \in \Delta)$$

then $f(z) \in S_n(\beta, s, t)$.

Proof. Define $p(z)$ by

$$\left\{ \frac{(s-t)zf'(sz)}{f(sz) - f(tz)} \right\} = (1-\beta)p(z) + \beta.$$

Then $p(z) = 1 + c_n z^n + ...$ and is analytic in Δ.
A computation shows that

$$\frac{szf''(sz)}{f'(sz)} + \frac{tzf'(tz)}{f(sz) - f(tz)} = \frac{(s-t)(1-\beta)zp'(z) + s[(1-\beta)p(z) + \beta]^2 - (s-t)[(1-\beta)p(z) + \beta]}{(s-t)[(1-\beta)p(z) + \beta]}$$

and hence

$$\left(\frac{(s-t)zf'(sz)}{f(sz) - f(tz)} - 1 \right) \left(\frac{szf''(sz)}{f'(sz)} + \frac{tzf'(tz)}{f(sz) - f(tz)} \right)$$
$$= \frac{(1-\beta)(p(z) - 1)}{(s-t)[(1-\beta)p(z) + \beta]} \left\{ (s-t)(1-\beta)zp'(z) + s[(1-\beta)p(z) + \beta]^2 - (s-t)[(1-\beta)p(z) + \beta] \right\}$$
$$= \phi(p(z), zp'(z); z)$$

Then for all real x and y satisfying $y \leq -n(1+x^2)/2$, we have

$$|\phi(ix, y; z)|^2 = \frac{(1-\beta)^2(1+x^2)}{(s-t)^2[(1-\beta)^2 x^2 + \beta^2]}$$
$$\times \left[(s-t)(1-\beta)y - s(1-\beta)^2 x^2 - \beta(s-t-s\beta) \right]^2 + (1-\beta)^2[2s\beta - (s-t)]^2 x^2$$
$$= \frac{(1-\beta)^2(1+r)}{(s-t)^2[(1-\beta)^2 r + \beta^2]}$$
$$\times \left[(s-t)(1-\beta)y - s(1-\beta)^2 r - \beta(s-t-s\beta) \right]^2 + (1-\beta)^2[2s\beta - (s-t)]^2 r$$
$$= g(r, y)$$

where $r = x^2 > 0$ and $y \leq -n(1+x^2)/2$
Since

$$\frac{\partial g}{\partial y} = \frac{2(1-\beta)^3(1+r)}{(s-t)[(1-\beta)^2 r + \beta^2]} \left\{ (s-t)(1-\beta)y - \beta(s-t-s\beta) - s(1-\beta)^2 r \right\} < 0$$

therefore we have

$$h(r) = g\left[r, -n(1+r)/2\right] \leq g(r, y),$$

where

$$h(r)=\frac{(1-\beta)^2(1+r)}{(s-t)^2[(1-\beta)^2r+\beta^2]}[\lambda r^2+(\lambda+\mu-\nu+\sigma)r+\mu] \tag{8}$$

where λ,μ,ν, and σ are given in (5).
Now differentiating (8) and using $h'(r)=0$, we get

$$2\lambda(1-\beta)^2r^3+\left\{(1-\beta)^2(2\lambda+\mu-\nu+\sigma)+3\lambda\beta^2\right\}r^2$$
$$+2\beta^2(2\lambda+\mu-\nu+\sigma)r+(\lambda+2\mu-\nu+\sigma)\beta^2-(1-\beta)^2\mu=0$$

which is a cubic equation in r. Since r_0 is the positive real root of this equation we have $h(r)\geq h(r_0)$ and hence

$$|\phi(ix,y;z)|^2\geq h(r_0)=\rho^2.$$

Define $\Omega=\{w;|w|<\rho\}$, then $\phi(p(z),zp'(z);z)\in\Omega$ for all real x and $y\leq -n(1+x^2)/2,\ z\in\Delta$. Therefore by an application of Lemma 1.1 the result follows. □

Remark 2.2. *By taking $s=1$ in Theorem 2.2 we get the known results of Goyal et al.*[4] *For $s=1$ and $t=0$ in Theorem 2.2 gives the known results due to Ravichandran et al.*[11] *and for $n=1,\beta=0,t=0$, our Theorem 2.2 reduces to another known result of Li and Owa.*[5]

ACKNOWLEDGMENT

Authors are thankful to Prof. S.P. Goyal, Emeritus Scientist(CSIR), University of Rajasthan, Jaipur, India, for his kind help and valuable suggestions during the preparation of this paper.

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