The Modified Reductive Perturbation Method as Applied to the Boussinesq Equation

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In this work, we extended the application of “the modified reductive perturbation method” to long water waves and obtained the governing equations of Korteweg – de Vries (KdV) hierarchy. Seeking localized travelling wave solutions to these evolution equations we have determined the scale parameter g_1 so as to remove the possible secularities that might occur. To indicate the effectiveness and the elegance of the present method, we studied the problem of the “dressed solitary wave method” and obtained exactly the same result. The present method seems to be fairly simple and practical as compared to the renormalization method and the multiple scale expansion method existing in the current literature.

Key words: Modified Reductive Perturbation Method; Korteweg – de Vries Hierarchy; Solitary Waves.

1. Introduction

In collisionless cold plasma, in fluid-filled elastic tubes and in shallow-water waves, due to nonlinearity of the governing equations, for the weakly dispersive case one obtains the Korteweg – de Vries (KdV) equation for the lowest-order term in the perturbation expansion, the solution of which may be described by solitons [1]. To study the higher-order terms in the perturbation expansion, the reductive perturbation method has been introduced by use of the stretched time and space variables [2]. However, in such an approach some secular terms appear in the solution and they can be eliminated by introducing some slow scale variables [3] or by a renormalization procedure of the velocity of the KdV soliton [4]). Nevertheless, this approach remains somewhat artificial, since there is no reasonable connection between the smallness parameters of the stretched variables and the one used in the perturbation expansion of the field variables. The choice of the former parameter is based on the linear wave analysis of the concerned problem, and the wave number or the frequency is taken as the perturbation parameter [5]. On the other hand, at the lowest order, the amplitude and the width of the wave are expressed in terms of the unknown perturbed velocity, which is also used as the smallness parameter. This causes some ambiguity over the correction terms. Another attempt to remove such secularities was made by Kraenkel and Manna [6] for long water waves by use of the multiple time scale expansion but they could not obtain explicitly the correction terms to the speed of the wave. In order to remove some possible secularities, He [7, 8] presented the “homotopy perturbation method” and “modified Lindstedt-Poincare method” and applied them successfully to certain one-dimensional problems. For other perturbation methods the readers are referred to [9, 10].

In order to remove these ambiguities, Malfliet and Wieers [11] presented a dressed solitary wave approach, which is based on the assumption that the field variables admit localized travelling wave solutions. Then, for the longwave limit, they expanded the field variables and the wave speed into a power series of the wave number, which is assumed to be the only smallness parameter, and obtained the explicit solution for various order terms in the expansion. However, this approach can only be used when one studies progressive wave solutions to the original nonlinear equations, and it does not give any idea about the form of evolution equations governing the various order terms in the perturbation expansion. In our previous paper [12], we have presented the so-called “modified reductive perturbation method” to examine the contributions of higher-order terms in the perturbation expansion and applied it to weakly dispersive ion-acoustic plasma waves and solitary waves in a fluid-filled elastic tube [13]. In these works, we have shown that the lowest-order term in the perturbation expansion is governed by the nonlinear Korteweg – de Vries equation, whereas the higher-order terms in the expansion are governed by the degenerate Korteweg – De Vries equation with the nonhomogeneous term. By employing the hyperbolic tangent method a progressive wave type of solution was sought and the possible secularities were removed by selecting the scaling parameter in a special way. The basic idea in this method was the inclusion of higher-order dispersive effects through the introduction of the scaling parameter g, to balance the higher-order nonlinearities with dispersion. The negligence of higher-order dispersive effects in the classical reductive perturbation method leads to the imbalance between the nonlinearity and dispersion, which resulted in some secular terms in the solution of evolution equations. As a matter of fact, the renormalization method presented by Kodama and Taniuti [4] is different but rather involved the formulation of the same idea.

In the present work, we extended the application of "the modified reductive perturbation method" to long water waves and obtained the equations of KdV hierarchy. Seeking for localized progressive wave solutions to these equations we determined the scale parameter g_1 so as to remove the possible secularities that might occur. To indicate the power and the elegance of the present method, we studied the same problem by use of the "dressed solitary wave method" and obtained exactly the same result. The present method appears to be fairly simple and practical as compared to the renormalization method of Kodama and Taniuti [4] and the multiple scale expansion method of Kraenkel and Manna [6].

2. Modified Reductive Perturbation Formalism for the Boussinesq Equation

The one-dimensional form of the Boussinesq equation may be given by

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} - 3\frac{\partial^2}{\partial x^2}(u^2) = 0. \tag{1}$$

The dispersion relation of the linearized equation of (1) reads

$$\omega = k(1+k^2)^{1/2}. \tag{2}$$

For the longwave limit, the frequency may be expressed as follows:

$$\omega = k\left(1 + \frac{1}{2}k^2 - \frac{1}{8}k^4 + \frac{1}{24}k^6 - \dots\right). \tag{3}$$

Motivated with this dispersion relation, it is convenient to introduce the stretched coordinates

$$\xi = \epsilon^{1/2}(x-t), \quad \tau = \epsilon^{3/2} g t, \tag{4}$$

where $\epsilon \sim k^2$ is the smallness parameter characterizing the longwave length and g is a scale parameter to be determined from the solution.

Inserting the transformation (4) into the field equation (1) we have

$$-\epsilon\left(2g\frac{\partial^2 u}{\partial \xi \partial \tau}\right) + \epsilon^2 g^2 \frac{\partial^2 u}{\partial \tau^2} + \epsilon \frac{\partial^4 u}{\partial \xi^4} - 3\frac{\partial^2}{\partial \xi^2}(u^2) = 0. \tag{5}$$

We shall further assume that the field variable u and the scale parameter g can be expanded into asymptotic series as

$$\begin{aligned} u &= \epsilon u_0 + \epsilon^2 u_1 + \epsilon^3 u_2 + \dots, \\ g &= 1 + \epsilon g_1 + \epsilon^2 g_2 + \dots \ . \end{aligned} \tag{6}$$

Introducing the expansion (6) into (5) and setting the coefficients of similar powers ϵ equal to zero we obtain the following set of differential equations:

$O(\epsilon)$ equation:

$$\frac{\partial}{\partial \xi}\left(\frac{\partial u_0}{\partial \tau} + 3u_0\frac{\partial u_0}{\partial \xi} - \frac{1}{2}\frac{\partial^3 u_0}{\partial \xi^3}\right) = 0; \tag{7}$$

$O(\epsilon^2)$ equation:

$$\begin{aligned} &\frac{\partial}{\partial \xi}\left[g_1\frac{\partial u_0}{\partial \tau} + \frac{\partial u_1}{\partial \tau} + 3\frac{\partial}{\partial \xi}(u_0 u_1) - \frac{1}{2}\frac{\partial^3 u_1}{\partial \xi^3}\right] \\ &-\frac{1}{2}\frac{\partial^2 u_0}{\partial \tau^2} = 0; \end{aligned} \tag{8}$$

$O(\epsilon^3)$ equation:

$$\begin{aligned} &\frac{\partial}{\partial \xi}\left[\frac{\partial u_2}{\partial \tau} + g_1\frac{\partial u_1}{\partial \tau} + g_2\frac{\partial u_0}{\partial \tau}\right. \\ &\qquad \left. - \frac{1}{2}\frac{\partial^3 u_2}{\partial \xi^3} + \frac{3}{2}\frac{\partial}{\partial \xi}(u_1^2 + 2u_0 u_2)\right] \\ &-\frac{1}{2}\frac{\partial^2 u_1}{\partial \tau^2} - g_1\frac{\partial^2 u_0}{\partial \tau^2} = 0. \end{aligned} \tag{9}$$

2.1. $O(\epsilon)$ Equation

From the integration of (7) we get

$$\frac{\partial u_0}{\partial \tau} + 3u_0 \frac{\partial u_0}{\partial \xi} - \frac{1}{2}\frac{\partial^3 u_0}{\partial \xi^3} = f(\tau), \tag{10}$$

where $f(\tau)$ is an arbitrary function of its argument and can be chosen to be zero. Then, it follows the conventional Korteweg – de Vries equation

$$\frac{\partial u_0}{\partial \tau} + 3u_0 \frac{\partial u_0}{\partial \xi} - \frac{1}{2}\frac{\partial^3 u_0}{\partial \xi^3} = 0. \tag{11}$$

We shall seek a travelling wave solution to (11) of the form

$$u_0 = F(\zeta), \quad \zeta = \alpha(\xi - \beta\tau), \tag{12}$$

where α and β are two constants to be determined from the solution and $F(\zeta)$ is an arbitrary function of its argument. Introducing the proposed solution (12) into (11) we have

$$-\beta F' + 3FF' - \frac{\alpha^2}{2}F''' = 0, \tag{13}$$

where a prime denotes the differentiation of the corresponding quantity with respect to ζ.

In the present work we shall be concerned with a localized travelling wave solution, i. e. F and its various order derivatives vanish as $\zeta \to \mp\infty$. Integrating (13) with respect to ζ and utilizing the localization condition we obtain

$$-\beta F + \frac{3}{2}F^2 - \frac{\alpha^2}{2}F'' = 0. \tag{14}$$

This nonlinear ordinary differential equation admits a solution of the form

$$F = -a\,\mathrm{sech}^2\zeta, \tag{15}$$

where a is the wave amplitude. Introducing (15) into (14) and setting the coefficients powers of $\mathrm{sech}\zeta$ equal to zero we obtain

$$\alpha = \left(\frac{a}{2}\right)^{1/2}, \quad \beta = -a. \tag{16}$$

2.2. $O(\epsilon^2)$ Equation

To obtain the solution for this order equation we need the expression of $\partial^2 u_0/\partial\tau^2$. Using (11) we obtain the following identity:

$$\frac{\partial^2 u_0}{\partial \tau^2} = \frac{\partial^2}{\partial \xi^2}\Big[3u_0^3 - \frac{3}{4}\left(\frac{\partial u_0}{\partial \xi}\right)^2 - 3u_0\frac{\partial^2 u_0}{\partial \xi^2} + \frac{1}{4}\frac{\partial^4 u_0}{\partial \xi^4}\Big]. \tag{17}$$

Introducing the expression (17) into (8) we have

$$\frac{\partial}{\partial \xi}\left[g_1\frac{\partial u_0}{\partial \tau} + \frac{\partial u_1}{\partial \tau} + 3\frac{\partial}{\partial \xi}(u_0u_1) - \frac{1}{2}\frac{\partial^3 u_1}{\partial \xi^3}\right] \tag{18}$$
$$= \frac{\partial^2}{\partial \xi^2}\left[\frac{3}{2}u_0^3 - \frac{3}{8}\left(\frac{\partial u_0}{\partial \xi}\right)^2 - \frac{3}{2}u_0\frac{\partial^2 u_0}{\partial \xi^2} + \frac{1}{8}\frac{\partial^4 u_0}{\partial \xi^4}\right].$$

Integrating (18) with respect to ξ we obtain

$$g_1\frac{\partial u_0}{\partial \tau} + \frac{\partial u_1}{\partial \tau} + 3\frac{\partial}{\partial \xi}(u_0u_1) - \frac{1}{2}\frac{\partial^3 u_1}{\partial \xi^3} = \frac{\partial}{\partial \xi}\left[\frac{3}{2}u_0^3 - \frac{3}{8}\left(\frac{\partial u_0}{\partial \xi}\right)^2 - \frac{3}{2}u_0\frac{\partial^2 u_0}{\partial \xi^2} + \frac{1}{8}\frac{\partial^4 u_0}{\partial \xi^4}\right] + g(\tau), \tag{19}$$

where $g(\tau)$ is an arbitrary function of τ. Without loosing the generality of the problem we may take $g(\tau)$ to be equal to zero. Eliminating $\partial u_0/\partial\tau$ from the expression of (19), by use of (11), we get the evolution equation

$$\frac{\partial u_1}{\partial \tau} + 3\frac{\partial}{\partial \xi}(u_0u_1) - \frac{1}{2}\frac{\partial^3 u_1}{\partial \xi^3} = \frac{\partial}{\partial \xi}S_0(u_0), \tag{20}$$

where the function $S_0(u_0)$ is defined by

$$S_0(u_0) = \frac{3}{2}u_0^3 - \frac{3}{8}\left(\frac{\partial u_0}{\partial \xi}\right)^2 - \frac{3}{2}u_0\frac{\partial^2 u_0}{\partial \xi^2} + \frac{1}{8}\frac{\partial^4 u_0}{\partial \xi^4} + \frac{3}{2}g_1u_0^2 - \frac{g_1}{2}\frac{\partial^2 u_0}{\partial \xi^2}. \tag{21}$$

Here we notice that (20) is the degenerate Korteweg – de Vries equation with a nonhomogeneous term.

Now, we shall seek a travelling wave solution to (20) of the following form:

$$u = G(\zeta). \tag{22}$$

Introducing (15) and (22) into (20) and (21) we obtain

$$-\beta G' + 3(FG)' - \frac{\alpha^2}{2}G''' = S_0'(F). \tag{23}$$

Integrating (23) and utilizing the localization condition we have

$$G'' + (12\text{sech}^2\zeta - 4)G = -\frac{4}{a}S_0(F), \tag{24}$$

where $S_0(F)$ is given by

$$S_0(F) = a^2\left(g_1 - \frac{a}{2}\right)\text{sech}^2\zeta. \tag{25}$$

The term on the right-hand side causes the secularity in the solution. In order to remove such a secularity we must have

$$g_1 - \frac{a}{2} = 0, \quad \text{or} \quad g_1 = \frac{a}{2}. \tag{26}$$

As a matter of fact, this result is equivalent to $S_0(u_0) = 0$ for the case of a travelling wave solution. Thus, without loosing the generality of the problem we may take $S_0(u_0) = 0$. Thus (20) reduces to

$$\frac{\partial u_1}{\partial \tau} + 3\frac{\partial}{\partial \xi}(u_0 u_1) - \frac{1}{2}\frac{\partial^3 u_1}{\partial \xi^3} = 0. \tag{27}$$

In this work we shall assume that the solution of the homogeneous equation (27) is zero, i. e. $u_1 = 0$.

2.3. $O(\epsilon^3)$ Equation

To obtain the solution for this order equation, in (9) we set $u_1 = 0$ and get the following equation:

$$\frac{\partial}{\partial \xi}\left[\frac{\partial u_2}{\partial \tau} + g_2\frac{\partial u_0}{\partial \tau} - \frac{1}{2}\frac{\partial^3 u_2}{\partial \xi^3} + 3\frac{\partial}{\partial \xi}(u_0 u_2)\right] - g_1\frac{\partial^2 u_0}{\partial \tau^2} = 0. \tag{28}$$

Inserting the expression $\partial^2 u_0/\partial\tau^2$ from (17) into (28) we have

$$\frac{\partial}{\partial \xi}\left[\frac{\partial u_2}{\partial \tau} + g_2\frac{\partial u_0}{\partial \tau} - \frac{1}{2}\frac{\partial^3 u_2}{\partial \xi^3} + 3\frac{\partial}{\partial \xi}(u_0 u_2)\right] = g_1\frac{\partial^2}{\partial \xi^2}\left[3u_0^3 - \frac{3}{4}\left(\frac{\partial u_0}{\partial \xi}\right)^2 - 3u_0\frac{\partial^2 u_0}{\partial \xi^2} + \frac{1}{4}\frac{\partial^4 u_0}{\partial^4 \xi^4}\right],$$

or, integrating the result with respect to ξ, we have

$$\frac{\partial u_2}{\partial \tau} + 3\frac{\partial}{\partial \xi}\frac{\partial u_2}{\partial \tau} + 3\frac{\partial}{\partial \xi}(u_0 u_2) - \frac{1}{2}\frac{\partial^3 u_2}{\partial \xi^3} = g_1\frac{\partial}{\partial \xi}\left[3u_0^3 - \frac{3}{4}\left(\frac{\partial u_0}{\partial \xi}\right)^2 - 3u_0\frac{\partial^2 u_0}{\partial \xi^2} + \frac{1}{4}\frac{\partial^4 u_0}{\partial \xi^4}\right] + g_2\frac{\partial}{\partial \xi}\left[\frac{3}{2}u_0^2 - \frac{1}{2}\frac{\partial^2 u_0}{\partial \xi^2}\right]. \tag{29}$$

Noticing that $S_0(u_0) = 0$ for such a travelling wave solution, (29) becomes

$$\frac{\partial u_2}{\partial \tau} + 3\frac{\partial}{\partial \xi}(u_0 u_2) - \frac{1}{2}\frac{\partial^3 u_2}{\partial \xi^3} = \frac{\partial S_1(u_0)}{\partial \xi}, \tag{30}$$

where $S_1(u_0)$ is defined by

$$S_1(u_0) = -3g_1^2 u_0^2 + g_1^2\frac{\partial^2 u_0}{\partial \xi^2} + \frac{3}{2}g_2 u_0^2 - \frac{g_2}{2}\frac{\partial^2 u_0}{\partial \xi^2}. \tag{31}$$

In order to remove the secularity of the solution we must have

$$g_2 = 2g_1^2 = \frac{a^2}{2}. \tag{32}$$

Thus, the solution takes the form

$$u = -\epsilon\, a\, \text{sech}^2\zeta \tag{33}$$

with

$$\zeta = \epsilon^{1/2}\left(\frac{a}{2}\right)^{1/2}\left[x - t + \epsilon a t + \epsilon^2\frac{a^2 t}{2} + \epsilon^3\frac{a^3}{2}t + \dots\right]. \tag{34}$$

The exact solution of the Boussinesq equation exists in the literature and is given by

$$u = -2b^2\,\text{sech}^2\left[b\left(x - \sqrt{1-4b^2}\right)t\right], \tag{35}$$

where b is a constant. Comparing the solutions given in (33) and (35) we have

$$b = \epsilon^{\frac{1}{2}}\left(\frac{a}{2}\right)^{\frac{1}{2}}, \tag{36}$$

and the solution (35) becomes

$$u = -\epsilon\, a\,\text{sech}^2\left[\epsilon^{\frac{1}{2}}\left(\frac{a}{2}\right)^{\frac{1}{2}}\left(x - \sqrt{1-2\epsilon a}\right)t\right]. \tag{37}$$

Expanding $\sqrt{1-2\epsilon a}$ into a Maclaurin series in ϵ we obtain

$$\sqrt{1-2\epsilon a} = 1 - \epsilon a - \frac{\epsilon^2}{2}a^2 - \epsilon^3\frac{a^3}{2} - \dots. \tag{38}$$

Hence, the exact solution, up to ϵ^3 order terms, takes the form

$$u = -\epsilon\, a \operatorname{sech}^2 \left[\epsilon^{1/2}\left(\frac{a}{2}\right)^{1/2} \cdot \left(1 - t + \epsilon a t + \epsilon^2 \frac{a^2}{2} t + \epsilon^3 \frac{a^3}{2}\right)\right] t. \tag{39}$$

This solution is exactly the same to that obtained by us via the use of the modified reductive perturbation method.

3. Dressed Solitary Wave Formalism for the Boussinesq Equation

In order to see the novelty of the present formulation we shall study the same problem by use of the so called "dressed solitary wave formalism". For that purpose, we shall examine a solution of (1) of the following type:

$$u = F(\xi), \quad \xi = k(x - Vt), \tag{40}$$

where k is the wave number and V is the speed of propagation. Introducing (40) into (1) we have

$$V^2 \frac{\mathrm{d}^2 F}{\mathrm{d}\xi^2} - \frac{\mathrm{d}^2 F}{\mathrm{d}\xi^2} + k^2 \frac{\mathrm{d}^4 F}{\mathrm{d}\xi^2} - 3\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}(F^2) = 0. \tag{41}$$

In this work we are concerned with a localized travelling wave solution, i. e., the function $F(\xi)$ and its various order derivatives vanish as $\xi \to \mp\infty$. Integrating (41) with respect to ξ twice and using the localization condition we obtain

$$(V^2 - 1)F + k^2 \frac{\mathrm{d}^2 F}{\mathrm{d}\xi^2} - 3F^2 = 0. \tag{42}$$

In this work, we shall assume that the wavelength is large, i. e., the wave number k is small. We shall further assume that the wave speed and the wave function F can be expanded into asymptotic series as follows:

$$\begin{aligned} F &= k^2 F_0 + k^4 F_1 + k^6 F_2 + \ldots, \\ V &= 1 + V_0(k^2 + k^4 V_1 + k^6 V_2 + \ldots). \end{aligned} \tag{43}$$

Introducing (43) into (42) and setting the coefficients with similar powers of k equal to zero, we obtain the following the set of differential equations:

$O(k^2)$ equation:

$$2V_0 F_0 + \frac{\mathrm{d}^2 F_0}{\mathrm{d}\xi^2} - 3F_0^2 = 0; \tag{44}$$

$O(k^4)$ equation:

$$(V_0^2 + 2V_0 V_1)F_0 + 2V_0 F_1 + \frac{\mathrm{d}^2 F_1}{\mathrm{d}\xi^2} - 6F_0 F_1 = 0; \tag{45}$$

$O(k^6)$ equation:

$$\begin{aligned} &2(V_0 V_2 + V_0^2 V_1)F_0 + (V_0^2 + 2V_0 V_1)F_1 \\ &+ 2V_0 F_2 + \frac{\mathrm{d}^2 F_2}{\mathrm{d}\xi^2} - 3(F_1^2 + 2F_0 F_2) = 0. \end{aligned} \tag{46}$$

For the differential equation (44) we shall propose a solution of the form

$$F_0 = -a \operatorname{sech}^2 \zeta, \quad \zeta = \alpha\xi, \tag{47}$$

where a is the amplitude of the solitary wave and α is a constant to be determined from the solution. Introducing (47) into (44) and setting the coefficients with similar powers of $\operatorname{sech}\zeta$ equal to zero we obtain

$$\alpha = \left(\frac{a}{2}\right)^{1/2}, \quad V_0 = -a. \tag{48}$$

Introducing the solutions (47) and (48) into (45) we have

$$\frac{\mathrm{d}^2 F_1}{\mathrm{d}\zeta^2} + 4(\operatorname{sech}^2\zeta - 1)F_1 = 2a(a - 2V_1)\operatorname{sech}^2\zeta. \tag{49}$$

The term on the right-hand side is the one that causes the secularities. If we choose $(a - 2V_1) = 0$, the secularity will disappear. Thus, we obtain the correction term to the speed as

$$V_1 = \frac{a}{2}. \tag{50}$$

Hence, without loosing the generality of the problem we may take $F_1 = 0$.

Introducing (47) into (46) and keeping in mind that F_1 is chosen to be zero, we have

$$\frac{\mathrm{d}^2 F_2}{\mathrm{d}\zeta^2} + 4(3\operatorname{sech}^2\zeta - 1)F_2 = 4\left(V_2 - \frac{a^2}{2}\right)\operatorname{sech}^2\zeta. \tag{51}$$

Again, the term on the right-hand side causes the secularity. In order to remove the secularity, we must have

$$V_2 = \frac{a^2}{2}. \tag{52}$$

Without loosing the generality of the problem we may set $F_2 = 0$. Thus, the solution takes the form

$$u = -k^2\, a \operatorname{sech}^2\zeta, \tag{53}$$

with

$$\zeta = k\left(\frac{a}{2}\right)^{\frac{1}{2}}\left[x-t+k^2at+k^4\frac{a^2}{2}t+k^6\frac{a^3}{2}t\right]. \quad (54)$$

Here, we see that this solution is exactly the same presented by us in equations (33) and (34).

4. Conclusion

The study of effects of higher-order terms in the perturbation expansion of the field variables through the use of the classical reductive perturbation method leads to some secularities. To eliminate such secularities various methods, like the renormalization method of Kodama and Taniuti [4] and the multiple scale expansion method by Kraenkel and Manna [6], have been presented in the current literature. The results of the present work and of those given in [12] and [13] proved that the "modified reductive perturbation method", presented by us, is the most simple and effective one. The present problem has been studied by Kraenkel et al. [14] but the method they used is quite complicated compared to ours. The presented method is rather simple and it is based on the idea of balancing higher-order nonlinearities with higher-order dispersive effects.

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