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SOLVING LINEAR AND NONLINEAR KLEIN-GORDON EQUATIONS BY NEW PERTURBATION ITERATION TRANSFORM METHOD

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Abstract. We present an effective algorithm to solve the Linear and Nonlinear Klein-Gordon equation, which is based on the Perturbation Iteration Transform Method (PITM). The Klein-Gordon equation is the name given to the equation of motion of a quantum scalar or pseudo scalar field, a field whose quanta are spin-less particles. It describes the quantum amplitude for finding a point particle in various places, the relativistic wave function, but the particle propagates both forwards and backwards in time. The Perturbation Iteration Transform Method (PITM) is a combined form of the Laplace Transform Method and Perturbation Iteration Algorithm. The method provides the solution in the form of a rapidly convergent series. Some numerical examples are used to illustrate the preciseness and effectiveness of the proposed method. The results show that the PITM is very efficient, simple and can be applied to other nonlinear problems.

Keywords: Perturbation Iteration Algorithm, Laplace Transform Method, Linear and Nonlinear Klein-Gordon Equations.

AMS Subject Classification: 15A06, 39B12, 33E30, 35A22

1. Introduction

Nonlinear phenomena, that is found in many areas of scientific research, such as solid state physics, fluid dynamics, plasma physics, mathematical biology and chemical kinematics can be modeled by partial differential equations. The Klein-Gordon equation is an important group of partial differential equations and is present in relativistic quantum mechanics and field theory, which is immensely important for the high energy physicist [1], and is employed for the modeling of various phenomena, including the propagation of dislocations in crystals and the behavior of elementary particles. On the other hand, the one-dimensional Klein-Gordon equation is given through the partial differential equation [2].

In the present paper, we are concerned with the numerical approximation of the following Nonlinear Klein-Gordon equation:

$$u_{tt}(x,t) - u_{xx}(x,t) + au(x,t) = h(x,t) \tag{1}$$

with initial conditions

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$$u(x,0) = f(x); \quad u_t(x,0) = g(x) \tag{2}$$

Klein-Gordon equation and the various other forms of the Nonlinear Klein-Gordon equation are all studied extensively in an assortment of papers. The equation has drawn a great deal of interest in studying Solation's and Soliton Perturbation Theory [3–6]. Biswas et al. studied the adiabatic dynamics of topological and non-topological solitons alike in presence of perturbation terms [7–10]. For one to obtain the exact and numerical solutions for Nonlinear Klein-Gordon Equations, a number of methods have been devised, some of which are the Modified Decomposition Method [11], the Symplectic Finite Difference Approximations Method [12], the Variational Iteration Method [13, 14], the Finite Element Method [15], the Cubic B-Spline Collocation Method [16], the Finite Difference Method [17], the Decomposition Method [18], Exp-Function Method [19, 20], the Homotopy Perturbation Method [21], the Tanh Method [22] and the Jacobi Elliptic Function Method [23], the Soliton Solution [24–28], the Stationary Solutions [29], the Numerical Scheme Based on the Collocation Method [2], and the Traveling Wave Solutions [30].

In this study, we propose a consistent algorithm based on the Perturbation Iteration Method to resolve the Linear and Nonlinear Klein-Gordon Equations. It should be mentioned that this method is a resourceful combination of the Laplace Transformation and New Perturbation Iteration Algorithm [31]. The Perturbation Iteration Transform Method (PITM) yields the solution in a rapid convergent series which results in the solution being in closed form. The prime benefit of the technique is that it grants its users with an analytical approximation, in many cases even an exact solution, in a rapidly convergent sequence with elegantly computed terms.

The paper is presented as follows. In section 2, we start off with some basic information of Perturbation Iteration Algorithm. Section 3 consists of description of Perturbation Iteration Transform Method. In section 4, we apply the Perturbation Iteration Transform Method (PITM) to solve five test examples in order to show its ability and efficiency. Section 5 is a conclusion of all of the above.

2. Perturbation Iteration Algorithm (PIA)

Consider the following non-linear differential equation

$$A(u) - f(r) = 0; \quad r \in \Omega \tag{3}$$

with boundary condition

$$B\Big(u, \frac{\partial u}{\partial n}\Big) = 0; \quad r \in \Gamma \tag{4}$$

Where A, B, $f(r)$ and Γ are general operator, a boundary operator, a known analytic function and the boundary of the domain Ω, respectively. The operator A have both linear and nonlinear terms in it.

$$L[u] + N[u] - f(r) = 0 \tag{5}$$

Introducing ϵ with nonlinear term yield

$$L[u] + \epsilon N[u] - f(r) = 0 \tag{6}$$

By applying Perturbation Iteration Algorithm, discussed in [31], we introduce perturbation expansion with n correction terms in Eq.(6)

$$u = u_\circ + \epsilon u_1 + \epsilon^2 u_2 + ... + \epsilon^n u_n \tag{7}$$

where ϵ is the perturbation parameter. Substitute Eq.(7) in Eq.(6) and expanding Taylor's series with m^{th} order yields

$$\begin{aligned}&A(u_\circ)+\epsilon N(u_\circ)+\Big(A^{'}(u_\circ)+\epsilon N^{'}(u_\circ)\Big)(\epsilon u_1+\epsilon^2 u_2+...+\epsilon^n u_n)+\\&\frac{\Big(A^{''}(u_\circ)+\epsilon N^{''}(u_\circ)\Big)}{2!}(\epsilon u_1+\epsilon^2 u_2+...+\epsilon^n u_n)^2+......+\\&\frac{\Big(A^{(m)}(u_\circ)+\epsilon N^{(m)}(u_\circ)\Big)}{m!}(\epsilon u_1+\epsilon^2 u_2+...+\epsilon^n u_n)^m=f(r)\end{aligned}\tag{8}$$

It is point to remember that since n terms in perturbation expansion and m^{th} order derivatives in the Taylor's series, the Perturbation Iteration Algorithm developed will be $PIA(n,m)$; n should be always less than or equal to m, other unknown correction terms in perturbation expansion can't be determined. Eq.(8) should be grouped with respect to the same order of ϵ, then comparing the coefficient of same power of ϵ gives the unknown correction terms. Substituting back these correction terms in to Eq.(7) yield an algorithm for solution of Eq.(3).

In the present paper the simplest Perturbation Iteration Algorithm $PIA(1,1)$ is used by taking one correction term in the perturbation expansion and correction terms of only first derivatives in the Taylor's series expansion, that is, $n=1$, $m=1$. Consider the second order general partial differential equation

$$F(\ddot{u},\dot{u},u^{''},u^{'},u,\epsilon)=0 \tag{9}$$

where $u=u(x,t)$, $\dot{u}=\frac{\partial u}{\partial t}$, $\ddot{u}=\frac{\partial^2 u}{\partial t^2}$, $u^{'}=\frac{\partial u}{\partial x}$, $u^{''}=\frac{\partial^2 u}{\partial x^2}$ and ϵ the artificially introduced perturbation parameter. In this method only one correction term in the perturbation expansion

$$u_{n+1}=u_n+\epsilon(u_c)_n \tag{10}$$

Substituting Eq.(10) in Eq.(9), expanding in a Taylor's series with first derivative only yields

$$\begin{aligned}&F(\ddot{u},\dot{u},u^{''},u^{'},u,0)+F_{\ddot{u}}(\ddot{u},\dot{u},u^{''},u^{'},u,0)\epsilon(\ddot{u}_c)_n+F_{\dot{u}}(\ddot{u},\dot{u},u^{''},u^{'},u,0)\epsilon(\dot{u}_c)_n+\\&F_{u^{''}}(\ddot{u},\dot{u},u^{''},u^{'},u,0)\epsilon(u_c^{''})_n+F_{u^{'}}(\ddot{u},\dot{u},u^{''},u^{'},u,0)\epsilon(u_c^{'})_n+F_u(\ddot{u},\dot{u},u^{''},u^{'},u,0)\epsilon(u_c)_n+\\&F_\epsilon(\ddot{u},\dot{u},u^{''},u^{'},u,0)\epsilon=0\end{aligned}\tag{11}$$

where $u=u(x,t)$, $F_{\ddot{u}}=\frac{\partial F}{\partial \ddot{u}}$, $F_{\dot{u}}=\frac{\partial F}{\partial \dot{u}}$, $F_{u^{''}}=\frac{\partial F}{\partial u^{''}}$, $F_{u^{'}}=\frac{\partial F}{\partial u^{'}}$, $F_u=\frac{\partial F}{\partial u}$, $F_\epsilon=\frac{\partial F}{\partial \epsilon}$ and ϵ the all derivative are evaluated at $\epsilon=0$. Starting with the initial condition $u_\circ$ first $(u_c)_\circ$ has been calculated by the help of Eq.(7). Then we substitute $(u_c)_\circ$ into Eq.(6) to find u_1. Iteration process is repeated using Eq.(6) and Eq.(7) until we obtain a satisfactory result.

3. Perturbation Iteration Transform Method (PITM)

To illustrate the basic idea of this method, we consider a general nonlinear partial differential equation with boundary conditions of the form:

$$Du(x,t)+Ru(x,t)+Nu(x,t)=g(x,t) \tag{12}$$

with

$$u(x,0)=h(x); \quad u_t(x,0)=f(x) \tag{13}$$

Where D is the second order linear differential operator $D = \frac{\partial^2}{\partial t^2}$, R is the linear differential operator of less order than D, N represents the general nonlinear differential operator and $g(x,t)$ is the source term. Taking the Laplace Transform (denote in this paper by $\mathcal{L}$) on both sides of Eq.(12), we get

$$\mathcal{L}[Du(x,t)] + \mathcal{L}[Ru(x,t)] + \mathcal{L}[Nu(x,t)] = \mathcal{L}[g(x,t)] \tag{14}$$

Using the differential property of the Laplace Transform, we have

$$\mathcal{L}[Du(x,t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} + \frac{1}{s^2}\mathcal{L}[g(x,t)] - \frac{1}{s^2}\mathcal{L}[Ru(x,t)] - \frac{1}{s^2}\mathcal{L}[Nu(x,t)] \tag{15}$$

Operating with the Laplace Inverse Transform on both sides of Eq. (15) gives

$$u(x,t) = G(x,t) - \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[Ru(x,t) + Nu(x,t)]\Big] \tag{16}$$

where $G(x,t)$ represent the term arising from the source term and the prescribed initial conditions. Now we apply the Perturbation Iteration Method. Take Eq.(16) and arranging the equation and make the following form

$$u(x,t) - G(x,t) + \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[Ru(x,t) + Nu(x,t)]\Big]\epsilon = 0 \tag{17}$$

where ϵ is the perturbation parameter

$$u(x,t) - G(x,t) + u_c(x,t)\epsilon + \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[Ru(x,t) + Nu(x,t)]\Big]\epsilon = 0 \tag{18}$$

and

$$u_c(x,t) = \frac{G(x,t) - u(x,t)}{\epsilon} - \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[Ru(x,t) + Nu(x,t)]\Big] \tag{19}$$

which is the coupling of the Laplace Transform and the Perturbation Iteration Method. Starting with an initial guess $u_\circ$. First $(u_c)_\circ$ is calculated from Eq.(19) and then substituted in Eq.(10) for calculating u_1. The iteration algorithm procedure is repeated using Eq.(19) and Eq.(2) until a satisfactory result is obtained. This iteration algorithm may produce similar results with the variation algorithm explained in [14].

$$\begin{aligned}
u_\circ(x,t) =& G(x,t)\\
u_1(x,t) =& -u_\circ(x,t) + G(x,t) - \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[Ru_\circ(x,t) + Nu_\circ(x,t)]\Big]\\
u_2(x,t) =& -u_1(x,t) + G(x,t) - \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[Ru_1(x,t) + Nu_1(x,t)]\Big]\\
&.\\
&.\\
&.
\end{aligned} \tag{20}$$

and so on. The approximate solution thus obtained by

$$u(x,t) = u_\circ(x,t) + u_1(x,t) + u_2(x,t) + ... \tag{21}$$

4. Numerical Applications

In this section, we use Perturbation Iteration Transform Method (PITM) in solving the Linear and Nonlinear Klein-Gordon Equations.

4.1. **Example 1.** Consider the following Linear Klein-Gordon Equation

$$u_{tt}(x,t) - u_{xx}(x,t) + u(x,t) = 0 \tag{22}$$

with the boundary conditions

$$u(x,0) = 0; \quad u_t(x,0) = x \tag{23}$$

Applying Laplace Transform on both sides of Eq.(22) subject to the boundary conditions Eq.(23), we have

$$\mathcal{L}[u(x,t)] = \frac{x}{s} + \frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u(x,t)] \tag{24}$$

The Inverse of Laplace Transform implies that

$$u(x,t) = xt + \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u(x,t)]\Big] \tag{25}$$

Now, applying the Perturbation Iteration Method, we get

$$u(x,t) - xt - \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u(x,t)]\Big]\epsilon = 0 \tag{26}$$

$$u(x,t) - xt + u_c(x,t)\epsilon - \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u(x,t)]\Big]\epsilon = 0 \tag{27}$$

$$u_c(x,t) = \frac{-u(x,t) + xt}{\epsilon} + \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u(x,t)]\Big] \tag{28}$$

we have

$$\begin{aligned}
u_\circ(x,t) =& xt\\
u_1(x,t) =& -\frac{1}{3!}xt^3\\
u_2(x,t) =& \frac{1}{5!}xt^5\\
u_3(x,t) =& -\frac{1}{7!}xt^7\\
u_4(x,t) =& \frac{1}{9!}xt^9\\
&.\\
&.\\
&.
\end{aligned} \tag{29}$$

and so on. Therefore the solution $u(x,t)$ is given by

$$u(x,t) = x\Big(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \frac{1}{9!}t^9 - ....\Big) \tag{30}$$

In series form, we have $u(x,t) = x\sin t$

4.2. **Example 2.** Consider the following Nonlinear Klein-Gordon Equation

$$u_{tt}(x,t) - u_{xx}(x,t) + u(x,t) = 2\sin x \tag{31}$$

with the boundary conditions

$$u(x,0) = \sin x; \quad u_t(x,0) = 1 \tag{32}$$

Applying Laplace Transform on both sides of Eq.(31) subject to the boundary conditions Eq.(32), we have

$$\mathcal{L}[u(x,t)] = \frac{2\sin x}{s^3} + \frac{\sin x}{s} + \frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u(x,t)] \tag{33}$$

The Inverse of Laplace Transform implies that

$$u(x,t) = t^2 \sin x + \sin x + t + \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u(x,t)]\Big] \tag{34}$$

Now, applying the Perturbation Iteration Method, we get

$$u(x,t) - t^2 \sin x - \sin x - t - \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u(x,t)]\Big]\epsilon = 0 \tag{35}$$

$$u_c(x,t) = \frac{-u(x,t) + \sin x + t + t^2 \sin x}{\epsilon} + \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u(x,t)]\Big] \tag{36}$$

we have

$$\begin{aligned}
u_\circ(x,t) =& t + \sin x + t^2 \sin x \\
u_1(x,t) =& -\frac{1}{3!}t^3 - t^2 \sin x - \frac{1}{3!}t^4 \sin x \\
u_2(x,t) =& \frac{1}{5!}t^5 + \frac{1}{3!}t^4 \sin x + \frac{1}{90}t^6 \sin x \\
u_3(x,t) =& -\frac{1}{7!}t^7 - \frac{1}{90}t^6 \sin x - \frac{1}{2520}t^8 \sin x \\
u_4(x,t) =& \frac{1}{9!}t^9 + \frac{1}{2520}t^8 \sin x + \frac{1}{113400}t^{10} \sin x \\
&. \\
&. \\
&.
\end{aligned} \tag{37}$$

and so on. Therefore the solution $u(x,t)$ is given by

$$u(x,t) = \sin x + \Big(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \frac{1}{9!}t^9 - ....\Big) \tag{38}$$

In series form, we have $u(x,t) = \sin x + \sin t$

4.3. **Example 3.** Consider the following Nonlinear Klein-Gordon equation

$$u_{tt}(x,t) - u_{xx}(x,t) + u^2(x,t) = x^2 t^2 \tag{39}$$

with the boundary conditions

$$u(x,0) = 0; \quad u_t(x,0) = x \tag{40}$$

Applying Laplace Transform on both sides of Eq.(39) subject to the boundary conditions Eq.(40), we have

$$\mathcal{L}[u(x,t)] = \frac{x}{s^2} + \frac{2x^2}{s^5} + \frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u^2(x,t)] \tag{41}$$

The Inverse of Laplace Transform implies that

$$u(x,t) = xt + \frac{x^2 t^4}{12} + \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u^2(x,t)]\Big] = 0 \tag{42}$$

Now, applying the Perturbation Iteration Method, we get

$$u(x,t) - xt - \frac{x^2t^4}{12}\epsilon - \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u^2(x,t)]\Big]\epsilon = 0 \quad (43)$$

$$u_c(x,t) = \frac{-u(x,t) + xt}{\epsilon} + \frac{x^2t^4}{12} + \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u^2(x,t)]\Big] \quad (44)$$

we have

$$\begin{aligned}
u_\circ(x,t) =& xt + \frac{1}{12}x^2t^4\\
u_1(x,t) =& \frac{1}{180}t^6 - \frac{1}{12}x^2t^4 - \frac{1}{252}x^3t^7 - \frac{1}{12960}x^4t^{10}\\
u_2(x,t) =& -\frac{1}{180}t^6 - \frac{1}{5896800}t^{14} - \frac{11}{22680}xt^9 - \frac{1}{142560}x^2t^{12} + \frac{1}{252}x^3t^7 + \frac{1}{4762800}x^3t^{15} +\\
& \frac{1}{6048}x^4t^{10} + \frac{1}{356918400}x^4t^{18} + \frac{1}{1010880}x^5t^{13} - \frac{1}{15240960}x^6t^{16} - \frac{1}{558472320}x^7t^{19} -\\
& \frac{1}{77598259200}x^8t^{22}\\
& .\\
& .\\
& .
\end{aligned} \quad (45)$$

and so on. Therefore the solution $u(x,t)$ in series form is given by $u(x,t) = xt$

4.4. **Example 4.** Consider the following Nonlinear Klein-Gordon equation

$$u_{tt}(x,t) - u_{xx}(x,t) + u^2(x,t) = 2x^2 - 2t^2 + x^4t^4 \quad (46)$$

with the boundary conditions

$$u(x,0) = 0; \quad u_t(x,0) = 0 \quad (47)$$

Applying Laplace Transform on both sides of Eq.(46) subject to the boundary conditions Eq.(47), we have

$$\mathcal{L}[u(x,t)] = -\frac{4}{s^5} + \frac{24x^4}{s^7} + \frac{2x^2}{s^3} - \frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u^2(x,t)] \quad (48)$$

The Inverse of Laplace Transform implies that

$$u(x,t) = x^2t^2 - \frac{1}{6}t^4 + \frac{1}{30}x^4t^6 - \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u^2(x,t)]\Big] \quad (49)$$

Now, applying the Perturbation Iteration Method, we get

$$u(x,t) - x^2t^2\epsilon + \frac{1}{6}t^4\epsilon - \frac{1}{30}x^4t^6\epsilon + \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u^2(x,t)]\Big]\epsilon = 0 \quad (50)$$

$$u_c(x,t) = \frac{-u(x,t)}{\epsilon} + x^2t^2 - \frac{1}{6}t^4 + \frac{1}{30}x^4t^6 - \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u^2(x,t)]\Big] \quad (51)$$

we have

$$
\begin{aligned}
u_\circ(x,t) =& x^2t^2 - \frac{1}{6}t^4 + \frac{1}{30}x^4t^6 \\
u_1(x,t) =& -\frac{1}{6}t^4 + \frac{1}{3240}t^{10} - \frac{11}{840}x^2t^8 + \frac{1}{30}x^4t^6 - \frac{1}{11880}x^4t^{12} + \frac{1}{1350}x^6t^{10} + \frac{1}{163800}x^8t^{14} \\
& . \\
& . \\
& .
\end{aligned} \tag{52}
$$

and so on. Therefore the solution $u(x,t)$ in series form is given by $u(x,t) = x^2t^2$

4.5. **Example 5.** Consider the following Nonlinear Klein-Gordon equation

$$u_{tt}(x,t) - u_{xx}(x,t) + u^2(x,t) = 6xt(x^2 - t^2) + x^6t^6 \tag{53}$$

with the boundary conditions

$$u(x,0) = 0; \quad u_t(x,0) = 0 \tag{54}$$

Applying Laplace Transform on both sides of Eq.(53) subject to the boundary conditions Eq.(54), we have

$$\mathcal{L}[u(x,t)] = -\frac{36x}{s^6} + \frac{6x^3}{s^4} + \frac{720x^6}{s^9} + \frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u^2(x,t)] \tag{55}$$

The Inverse of Laplace Transform implies that

$$u(x,t) = -\frac{3}{10}xt^5 + \frac{3}{2}x^3t^3 + \frac{1}{56}x^6t^8 + \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u^2(x,t)]\Big] \tag{56}$$

Now, applying the Perturbation Iteration Method, we get

$$u(x,t) + \frac{3}{10}xt^5\epsilon - \frac{3}{2}x^3t^3\epsilon - \frac{1}{56}x^6t^8\epsilon - \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u^2(x,t)]\Big]\epsilon = 0 \tag{57}$$

$$u_c(x,t) = \frac{-u(x,t)}{\epsilon} - \frac{3}{10}xt^5 + \frac{3}{2}x^3t^3 + \frac{1}{56}x^6t^8 + \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[u_{xx}(x,t) - u^2(x,t)]\Big] \tag{58}$$

we have

$$
\begin{aligned}
u_\circ(x,t) =& -\frac{3}{10}xt^5 + \frac{3}{2}x^3t^3 + \frac{1}{56}x^6t^8 \\
u_1(x,t) =& \frac{9}{20}xt^5 - \frac{3}{4400}x^2t^{12} + \frac{67}{4200}x^4t^{10} - \frac{9}{224}x^6t^8 + \frac{1}{19600}x^7t^{15} - \frac{1}{2912}x^9t^{13} - \frac{1}{959616}x^{12}t^{18} \\
& . \\
& . \\
& .
\end{aligned} \tag{59}
$$

and so on. Therefore the solution $u(x,t)$ in series form is given by $u(x,t) = x^3t^3$

5. Concluding Remarks

The recently derived Perturbation Iteration Transform Method is applied to Linear and Nonlinear Klein-Gordon Equations for the first time. The theory is first developed and then applied to five different problems. On the basis of this study, one may reach the conclusion that $PITM$ is a powerful and capable process that helps profoundly in finding exact and approximate solutions for linear and nonlinear differential equations. It is important to note that the method is capable of reducing the volume of the computational work as compared to the classical methods while maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach. That the Perturbation Iteration Transform Method solves Linear and Nonlinear problems without needing Adomian Polynomials is a clear advantage of this technique over the Decomposition Method. Conclusively, the $PITM$ could be considered as a welcome refinement in existing numerical techniques and may open doors to wider applications.

6. Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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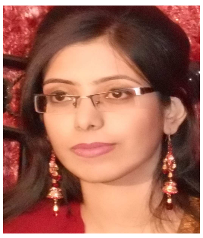

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