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ANTIMAGIC LABELING OF THE UNION OF SUBDIVIDED STARS

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Abstract. Enomoto et al. (1998) defined the concept of a super $(a, 0)$-edge-antimagic total labeling and proposed the conjecture that every tree is a super $(a, 0)$-edge-antimagic total labeling. In support of this conjecture, the present paper deals with different results on antimagicness of subdivided stars and their unions.

Keywords: super (a, d)-EAT labeling, star and subdivision of stars.

AMS Subject Classification: 05C78

1. Introduction

All graphs in this paper are simple, finite and undirected. For a graph G, $V(G)$ and $E(G)$ denote the vertex-set and the edge-set. A (v, e)-graph G is a graph such that $v = |V(G)|$ and $e = |E(G)|$.

In this paper, the domain will be the set of all vertices and edges, and such a labeling is called *a total labeling*. Details on antimagic labeling can be seen in [7]. The subject of edge-magic total labeling of graphs has its origin in the works of Kotzig and Rosa [1, 2] on what they called magic valuations of graphs. The definition of (a, d)-edge-antimagic total labeling was introduced by Simanjuntak, Bertault and Miller in [21] as a natural extension of *edge-magic labeling* defined by Kotzig and Rosa. Enomoto et al. also proposed the following conjecture:

Conjecture 1.1 [6] *Every tree admits a super edge-magic total labeling.*

In favour of this conjecture, many authors have considered super edge-magic total labeling for particular classes of trees for example [3, 4, 5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25]. Lee and Shah [22] verified this conjecture by a computer search for trees with at most 17 vertices. However, this conjecture is still open.

A star is a particular type of tree graph and many authors have proved the magicness for subdivided stars. Lu [24, 25] called the subdivided star $T(m, n, k)$ as a three path trees and proved that it is super edge-magic if n and k are odd, $k = n+1$ or $n+2$. Ngurah et al. [5] proved that $T(m, n, k)$ is also super edge-magic if $k = n + 3$ or $n + 4$. In [3], Salman et al. found the super edge-magic total labeling of a subdivision of a star S_n^m for $m = 1, 2$. Javaid et al. [17] furnished super edge-magic total labeling on subdivided star $K_{1,4}$ and w-trees.

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Definition 1.1 *A graph G is called (a,d)-edge-antimagic total $((a,d)-EAT)$ if there exist integers $a>0$, $d\geq 0$ and a bijection* $\lambda: V(G)\cup E(G)\to\{1,2,...,v+e\}$ *such that $W=\{w(rs): rs\in E(G)\}$ forms an arithmetic progression starting from a with the difference d, where $w(rs)=\lambda(r)+\lambda(s)+\lambda(rs)$ for any $rs\in E(G)$. W is called the set of edge-weights of the graph G.*
Definition 1.2 *A (a,d)-edge-antimagic total labeling λ is called super (a,d)-edge-antimagic total labeling if $\lambda(V(G))=\{1,2,...,v\}$.*
Definition 1.3 For $n_i\geq 1$ and $r\geq 3$, let $G\cong T(n_1,n_2,...,n_r)$ be a graph obtained by inserting n_i-1 vertices to each of the i-th edge of the star $K_{1,r}$, where $1\leq i\leq r$.
Definition 1.4 Two graphs G_1 and G_2 are said to be isomorphic if their exist a bijective function $\lambda: V(G_1)\to V(G_2)$ such that for all $x,y\in V(G_1): xy\in E(G_1)$ if and only if $\lambda(x)\lambda(y)\in E(G_2)$

2. Main Results

We consider the following proposition which we will use frequently in the main results.
Proposition 2.1. [14] If a (v,e)-graph G has a (s,d)-EAV labeling then

(i) G has a super $(s+v+1,d+1)$-EAT labeling,
(ii) G has a super $(s+v+e,d-1)$-EAT labeling. □
Theorem 2.1. For all $n\geq 1$, $G\cong T(n+1,n,n+2,n+3,n_5,...,n_p)$ admits super $(a,0)$-edge-antimagic total labeling with $a=2v+s-1$ and super $(a,2)$-edge-antimagic total labeling with $a=v+s+1$ where $v=|V(G)|$, $s=2(n+3)+\sum_{m=5}^{p}[2^{m-5}(n+2)+1]$ and $n_p=2^{r-4}(n+2)+1$.
Proof. We denote the vertices and edges of G as follows:
$V(G)=\{c\}\cup\{x_i^{l_i}\mid 1\leq i\leq r\ ;\ 1\leq l_i\leq n_i\}$,
$E(G)=\{c\ x_i^1\mid 1\leq i\leq r\}\cup\{x_i^{l_i}x_i^{l_i+1}\mid 1\leq i\leq r\ ;\ 1\leq l_i\leq n_i-1\}$.
Therefore,

$$v=(4n+7)+\sum_{m=5}^{p}[2^{m-4}(n+2)+1]$$

and

$$e=v-1.$$

We define the labeling $\lambda: V(G)\to\{1,2,...,v\}$ as follows:

$$\lambda(c)=(3n+5)+\sum_{m=5}^{p}[2^{m-5}(n+2)+1].$$

For odd $1\leq l_i\leq n_i$, where $i=1,2,3,4$ and $5\leq i\leq r$, we define

$$\lambda(u)=\begin{cases}\frac{l_1+1}{2}, & \text{for } u=x_1^{l_1},\\ (n+2)-\frac{l_2+1}{2}, & \text{for } u=x_2^{l_2},\\ n+1+\frac{l_3+1}{2}, & \text{for } u=x_3^{l_3},\\ (2n+5)-\frac{l_4+1}{2}, & \text{for } u=x_4^{l_4}.\end{cases}$$

$$\lambda(x_i^{l_i}) = (2n+5) + \sum_{m=5}^{i}[2^{m-5}(n+2)+1] - \frac{l_i+1}{2} \ \ respectively.$$

For even $1 \le l_i \le n_i$, and $\alpha = 2(n+2) + \sum_{m=5}^{r}[2^{m-5}(n+2)+1]$
For $i = 1, 2, 3, 4$ and $5 \le i \le r$, we define

$$\lambda(u) = \begin{cases} \alpha + \frac{l_1}{2}, & \text{for } \ u = x_1^{l_1}, \\ (\alpha+n+1) - \frac{l_2}{2}, & \text{for } \ u = x_2^{l_2}, \\ (\alpha+n+1) + \frac{l_3}{2}, & \text{for } \ u = x_3^{l_3}, \\ (\alpha+2n+4) - \frac{l_4}{2}, & \text{for } \ u = x_4^{l_4}. \end{cases}$$

and

$$\lambda(x_i^{l_i}) = (\alpha+2n+4) + \sum_{m=5}^{i}[2^{m-5}(n+2)] - \frac{l_i}{2} \ \ respectively.$$

The set of all edge-sums generated by the above formula forms a set of consecutive integer sequence $s = \alpha+2, \alpha+3, \cdots, \alpha+1+e$. Therefore, by Lemma 2.1, λ can be extended to a super $(a,0)$-edge-antimagic total labeling and we obtain the magic constant $a = v+e+s = (10n+29) + \sum_{m=5}^{p}[2^{m-5}5(n+2)+3]$. Similarly by Lemma 2.2, λ can be extended to a super $(a,2)$-edge-antimagic total labeling and we obtain the magic constant $a = v+1+s = (6n+14) + \sum_{m=5}^{p}[2^{m-5}3(n+2)+2]$. □

Theorem 2.2. For all $n \ge 1$ and $r \ge 5$, $G \cong T(n+1, n, n+2, n+3, n_5, ..., n_p)$ admits super $(a,1)$-edge-antimagic total labeling with $a = s + \frac{3v}{2}$ if v is even, where $v = |V(G)|$, $s = 2(n+3) + \sum_{m=5}^{p}[2^{m-5}(n+2)+1]$ and $n_p = 2^{r-4}(n+2)+1$.

Proof. Let us consider the vertices and edges of G, as defined in Theorem 2.6 . Now, we define the labeling $\lambda : V(G) \to \{1, 2, ..., v\}$ as in same theorem. It follows that the edge-weights of all edges of G constitute an arithmetic sequence $s = \alpha+2, \alpha+3, \cdots, \alpha+1+e$ with common difference 1, where $\alpha = 2(n+2) + \sum_{m=5}^{p}[2^{m-5}(n+2)+1]$. We denote it by $A = \{a_i; 1 \le i \le e\}$. Now for G we complete the edge labeling λ for super $(a,1)$-edge-antimagic total labeling with values in the arithmetic sequence $v+1$, $v+2$, $\cdots$, $v+e$ with common difference 1. Let us denote it by $B = \{b_j \ ; \ 1 \le j \le e\}$. Define $C = \{a_{2i-1} + b_{e-i+1} \ ; \ 1 \le i \le \frac{e+1}{2}\} \cup \{a_{2j} + b_{\frac{e-1}{2}-j+1} \ ; \ 1 \le j \le \frac{e+1}{2} - 1\}$. It is easy to see that C constitutes an arithmetic progration with $d = 1$ and $a = s + \frac{3(v)}{2} = \frac{1}{2}(16n+33) + \frac{1}{2}\sum_{m=5}^{p}[2^{m-2}(n+2)+5]$
Consequently, λ is a super $(a,1)$-edge-antimagic total labeling.

□

Theorem 2.3. For all positive integers n, $G \cong 2T(n+1, n, n, n+1, n_5, n_6, ..., n_p)$ admits super $(a,0)$-edge-antimagic total labeling with $a = 2v+s-2$ and super $(a,2)$-edge-antimagic total labeling with $a = v+s+1$ where $n_i = 2^{i-4}(n+1) fori = 5, 6, ..., p-1$, $n_p = 2^{i-4}(n+1)-1$ and $v = |V(G)|$.

Proof. We suppose the vertex-set and the edge-set of G as follows: $V(G) = \{c_j \mid 1 \leq j \leq 2\} \cup \{x_{ij}^{l_i} \mid 1 \leq i \leq p\ ;\ 1 \leq l_i \leq n_i;\ 1 \leq j \leq 2\}$,
$E(G) = \{c_j x_i^1 | 1 \leq i \leq p\ ;\ 1 \leq j \leq 2\} \cup$
$\{x_{ij}^{l_i} x_{ij}^{l_i+1} \mid 1 \leq i \leq p\ ; 1 \leq l_i \leq n_i - 1\ ;\ 1 \leq j \leq 2\}$.

If $v = |V(G)|$ and $e = |E(G)|$ then

$$v = 4n + 2^{p-2}(n+1)$$

and

$$e = 4n - 2 + 2^{p-2}(n+1).$$

Now, we define the vertex labeling $\lambda : V(G) \to \{1, 2, ..., v\}$ as follows:

$$\lambda(c_j) = 3(n+1) + 2^{p-3}(n+1) + [(n-1) + 2^{p-4}(n+1)](j-1),\ j = 1, 2.$$

For odd l_i $1 \leq l_i \leq n_i$, we define

$$\lambda(u) = \begin{cases} \frac{l_1+1}{2} + [(n+1) + 2^{p-4}(n+1)](j-1), & \text{for } u = x_{1j}^{l_1}, \\ \frac{2n+3-l_2}{2} \\ +[(n+1) + 2^{p-4}(n+1)](j-1), & \text{for } u = x_{2j}^{l_2}, \\ \frac{(2n+3)+l_3}{2} \\ +[(n+1) + 2^{p-4}(n+1)](j-1), & \text{for } u = x_{3j}^{l_3}, \\ \frac{4n+5-l_4}{2} \\ +[(n+1) + 2^{p-4}(n+1)](j-1), & \text{for } u = x_{4j}^{l_4}, \\ (n+1+2^{p-4}(n+1))j, & \text{for } u = x_{pj}^{l_p}, \\ & \text{for } l_p = 1, \\ \frac{2n+3+2^{k-3}(n+1)-l_k}{2} \\ +[(n+1) + 2^{p-4}(n+1)](j-1), & \text{for } u = x_{kj}^{l_k}, \\ & \text{for } k = 5, 6, ..., p-1, \\ \frac{2n+3+2^{p-3}(n+1)-l_p}{2} \\ +[3n + 2^{p-4}(3n+3)](j-1), & \text{for } u = x_{pj}^{l_p}, \\ & \text{for } 4 \leq l_p \leq n_p. \end{cases}$$

For even l_i , $1 \le l_i \le n_i$,we define

$$\lambda(u) = \begin{cases} \frac{4(n+1)+2^{p-2}(n+1)+l_1}{2} & \\ +[(n-1)+2^{p-4}(n+1)](j-1), & \text{for } u = x_{1j}^{l_1}, \\ \frac{6(n+1)+2^{p-2}(n+1)-l_2}{2} & \\ +[(n-1)+2^{p-4}(n+1)](j-1), & \text{for } u = x_{2j}^{l_2}, \\ \frac{6(n+1)+2^{p-2}(n+1)+l_3}{2} & \\ +[(n-1)+2^{p-4}(n+1)](j-1), & \text{for } u = x_{3j}^{l_3}, \\ \frac{6(n+1)+(2^{p-2}+2^{k-3})(n+1)-l_k}{2} & \\ +[(n-1)+2^{p-4}(n+1)](j-1), & \text{for } u = x_{kj}^{l_k}, \\ & \text{for } k = 4,5,...,p-1, \\ \frac{6(n+1)-2+2^{p-3}3(n+1)-l_p}{2} & \\ -[(n+2^{p-4}(n+1)](j-1), & \text{for } u = x_{pj}^{l_p}, \\ & \text{for } 2 \le l_p \le n_p. \end{cases}$$

The set of all edge-sums generated by the above formula forms a set of consecutive integer sequence $S = \{(2n+3)+2^{p-3}(n+1)+1, (2n+3)+2^{p-3}(n+1)+2, ..., (2n+3)+2^{p-3}(n+1)+e\}$, where $s = min(S)$. Therefore, by Proposition 2.1, λ can be extended to a super $(a,0)$-edge-antimagic total labeling and we obtain the magic constant $a = 2v+s-2 = 2(5n+1)+5(n+1)2^{p-3}$. Similarly by Proposition 2.1, λ can be extended to a super $(a,2)$-edge-antimagic total labeling and we obtain the magic constant $a = v+1+s = 6n+5+3(n+1)2^{p-3}$. $\square$

Theorem 2.4. For all positive integers n, $G \cong 2T(n+1,n,n,(n+1),n_5,...,n_p)$ admits super $(a,1)$-edge-antimagic total labeling with $a = v+s+e$ and super $(a,3)$-edge-antimagic total labeling with $a = v+s+1$ where $v = |V(G)|$, $s = 4$, $n_i = 2^{i-4}(n+1)$ for $i = 5,6,...,n_p$ and $n_p = 2^{i-3}(n+1)-1$.

Proof. We suppose the vertex-set and the edge-set of G as follows: $V(G) = \{c_j \mid 1 \le j \le 2\} \cup \{x_{ij}^{l_i} \mid 1 \le i \le 5 \ ; \ 1 \le l_i \le n_i; \ 1 \le j \le 2\}$,

$E(G) = \{c_j x_i^1 | 1 \le i \le 5 \ ; \ 1 \le j \le 2\} \cup$
$\{x_{ij}^{l_i} x_{ij}^{l_i+1} \mid 1 \le i \le 5 \ ; 1 \le l_i \le n_i - 1 \ ; \ 1 \le j \le 2\}$.

If $v = |V(G)|$ and $e = |E(G)|$ then

$$v = 2(2n+1) + 2^{p-2}(n+1)$$

and

$$e = 4n + 2^{p-2}(n+1)$$

Now, we define the vertex labeling $\lambda : V(G) \to \{1,2,...,v\}$ as follows:

$$\lambda(c_j) = 2(2n+1) + j, \ j = 1,2.$$

For all l_i $1 \leq l_i \leq n_i$, we define

$$\lambda(u) = \begin{cases} 2(l_1 - 1) + j, & \text{for } u = x_{1j}^{l_1}, \\ 2(2n+1) - 2l_2 + j, & \text{for } u = x_{2j}^{l_2}, \\ 2(2n+3) - 2l_3 + j, & \text{for } u = x_{3j}^{l_3}, \\ (10n+12) + j - 2l_4, & \text{for } u = x_{4j}^{l_4}, \\ 2(4n+3) + \sum_{m=5}^{i} [2^{m-3}(n+1)] - 2l_i + j, & \text{for } u = x_{ij}^{l_i},\ i \geq 5. \end{cases}$$

The set of all edge-sums generated by the above formula forms a set of consecutive integer sequence $s = \{4, 4+2, ..., 4+2(e-1)\}$, where $s = min(S)$. Therefore, by Proposition 2.1, λ can be extended to a super $(a, 1)$-edge-antimagic total labeling and we obtain the magic constant $a = v + e + s = 2(4n + 3) + 2^{p-1}(n + 1)$. Similarly by Proposition 2.1, λ can be extended to a super $(a, 3)$-edge-antimagic total labeling and we obtain the magic constant $a = v + 1 + s = 4n + 7 + 2^{p-2(n+1)}$. □

3. Conclusion

In this paper, we have shown that a subclass of trees, namely subdivided stars $G \cong 2T(n + 1, n, n, n + 1, n_5, n_6, ..., n_p)$ admits super (a,d)-edge-antimagic total labeling for $d = 0, 1, 2, 3$, for all positive integers n. However the problem of the magicness is still open for different values of magic constant (minimum edge-weight a).

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