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SECOND HANKEL DETERMINANT PROBLEM FOR SEVERAL CLASSES OF ANALYTIC FUNCTIONS RELATED TO SHELL-LIKE CURVES CONNECTED WITH FIBONACCI NUMBERS

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Abstract. In this paper, we investigate upper bounds for the second Hankel determinant in several classes of analytic functions in the open unit disc, related to shell-like curves and connected with Fibonacci numbers.

Keywords: Analytic functions, shell-like curve, Fibonacci numbers, starlike functions, convex functions, Hankel determinant.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f which are *analytic* in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and are of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

We say that f is subordinate to F in $\mathbb{U}$, written as $f \prec F$, if and only if $f(z) = F(w(z))$ for some analytic function w such that $|w(z)| \leq |z|$ for all $z \in \mathbb{U}$.

If $f \in \mathcal{A}$ and

$$\frac{zf'(z)}{f(z)} \prec p(z) \quad \text{or} \quad 1 + \frac{zf''(z)}{f'(z)} \prec p(z) \quad \text{or} \quad (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec p(z)$$

where $p(z) = \frac{1+z}{1-z}$, then we say that f is starlike or convex or $\alpha-$convex function, respectively. These functions form known classes denoted by $\mathcal{S}^*$, $\mathcal{C}$ or $\mathcal{M}(\alpha)$, respectively. These classes are very important subclasses of the class $\mathcal{S}$ in geometric function theory.

In [14], Sokół introduced the class $\mathcal{SL}$ of shell-like functions as the set of functions $f \in \mathcal{A}$ which is described in the following definition:

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Definition 1.1. *The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}$ if it satisfies the condition that*

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z)$$

with

$$\tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2},$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

Later, Dziok et al. in [1] and [2] defined and introduced the class $\mathcal{KSL}$ and $\mathcal{SLM}_\alpha$ of convex and $\alpha-$convex functions related to a shell-like curve connected with Fibonacci numbers, respectively. These classes can be given in the following definitions.

Definition 1.2. *The function $f \in \mathcal{A}$ belongs to the class $\mathcal{KSL}$ of convex shell-like functions if it satisfies the condition that*

$$1+\frac{zf''(z)}{f'(z)} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2},$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

Definition 1.3. *The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SLM}_\alpha$, $(0 \le \alpha \le 1)$ if it satisfies the condition that*

$$\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\frac{zf'(z)}{f(z)} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2},$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

The class $\mathcal{SLM}_\alpha$ is related to the class $\mathcal{KSL}$ only through the function $\tilde{p}$ and $\mathcal{SLM}_\alpha \neq \mathcal{KSL}$ for all $\alpha \neq 1$. It is easy to see that $\mathcal{KSL} = \mathcal{SLM}_1$. The function $\tilde{p}$ is not univalent in $\mathbb{U}$, but it is univalent in the disc $|z| < (3-\sqrt{5})/2 \approx 0.38$. For example, $\tilde{p}(0) = \tilde{p}(-1/2\tau) = 1$ and $\tilde{p}(e^{\mp i \arccos(1/4)}) = \sqrt{5}/5$, and it may also be noticed that

$$\frac{1}{|\tau|} = \frac{|\tau|}{1-|\tau|},$$

which shows that the number $|\tau|$ divides $[0,1]$ such that it fulfils the golden section. The image of the unit circle $|z| = 1$ under $\tilde{p}$ is a curve described by the equation given by

$$(10x-\sqrt{5})y^2 = (\sqrt{5}-2x)(\sqrt{5}x-1)^2,$$

which is translated and revolved trisectrix of Maclaurin. The curve $\tilde{p}(re^{it})$ is a closed curve without any loops for $0 < r \le r_0 = (3-\sqrt{5})/2 \approx 0.38$. For $r_0 < r < 1$, it has a loop, and for $r = 1$, it has a vertical asymptote. Since τ satisfies the equation $\tau^2 = 1+\tau$, this expression can be used to obtain higher powers τ^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of τ and 1. The resulting recurrence relationships yield Fibonacci numbers u_n:

$$\tau^n = u_n\tau + u_{n-1}.$$

In 1976, Noonan and Thomas [10] stated the s^{th} Hankel determinant for $s \geq 1$ and $k \geq 1$ as

$$H_s(k) = \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+s-1} \\ a_{k+1} & a_{k+2} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{k+s-1} & \cdots & \cdots & a_{k+2(s-1)} \end{vmatrix}, \tag{2}$$

where $a_1 = 1$.

This determinant has also been considered by several authors. For example, Noor [11] determined the rate of growth of $H_s(k)$ as $k \to \infty$ for functions f given by (1) with bounded boundary. Ehrenborg in [3] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [8]. Also, several authors considered the case $s = 2$. Especially, $H_2(1) = a_3 - a_2^2$ is known as Fekete-Szegö functional and this functional is generalized to $a_3 - \mu a_2^2$ where μ is some real number [4]. Estimating for an upper bound of $|a_3 - \mu a_2^2|$ is known as the Fekete-Szegö problem. In [13], Raina and Sokół considered Fekete-Szegö problem for the class $\mathcal{SL}$. In 1969, Keogh and Merkes [7] solved this problem for the classes $\mathcal{S}^*$ and $\mathcal{C}$. The second Hankel determinant is $H_2(2) = a_2a_4 - a_3^2$. Janteng [5] found the sharp upper bound for $|H_2(2)|$ for univalent functions whose derivative has positive real part. Also, in [6] Janteng et al. obtained the bounds for $|H_2(2)|$ for the classes $\mathcal{S}^*$ and $\mathcal{C}$.

Let $\mathcal{P}(\beta)$, $0 \leq \beta < 1$, denote the class of analytic functions p in $\mathbb{U}$ with $p(0) = 1$ and $Re\{p(z)\} > \beta$. Especially, we will use $\mathcal{P}$ instead of $\mathcal{P}(0)$.

Theorem 1.1. *([2]) The function* $\tilde{p}(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2}$ *belongs to the class* $\mathcal{P}(\beta)$ *with* $\beta = \sqrt{5}/10 \approx 0.2236$.

Now we give the following lemmas which will use in proving.

Lemma 1.1. *([12]) Let* $p \in \mathcal{P}$ *with* $p(z) = 1 + c_1z + c_2z^2 + \cdots$, *then*

$$|c_n| \leq 2, \qquad for \qquad n \geq 1. \tag{3}$$

If $|c_1| = 2$, *then* $p(z) \equiv p_1(z) \equiv (1+xz)/(1-xz)$ *with* $x = c_1/2$. *Conversely, if* $p(z) \equiv p_1(z)$ *for some* $|x| = 1$, *then* $c_1 = 2x$. *Furthermore, we have*

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}. \tag{4}$$

If $|c_1| < 2$, *and* $\left| c_2 - \frac{c_1^2}{2} \right| = 2 - \frac{|c_1|^2}{2}$, *then* $p(z) \equiv p_2(z)$, *where*

$$p_2(z) = \frac{1 + \bar{x}wz + z(wz + x)}{1 + \bar{x}wz - z(wz + x)},$$

and $x = \frac{c_1}{2}$, $w = \frac{2c_2 - c_1^2}{4-|c_1|^2}$ *and* $|c_2 - \frac{c_1^2}{2}| = 2 - \frac{|c_1|^2}{2}$.

Lemma 1.2. *([9]) Let* $p \in \mathcal{P}$ *with coefficients* c_n *as above, then*

$$|c_3 - 2c_1c_2 + c_1^3| \leq 2. \tag{5}$$

In this paper, we use ideas and techniques used in geometric function theory. The central problem considered here is the sharp upper bounds for the functional $|a_2a_4 - a_3^2|$ of functions in the classes $\mathcal{SL}$, $\mathcal{KSL}$ and $\mathcal{SLM}_\alpha$, depicted by the Fibonacci numbers, respectively.

2. Main Results

In [13], Raina and Sokół proved the following result:

Theorem 2.1. *If* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, *and* $p \prec \tilde{p}$, *then*

$$|p_1| \leq |\tau| \tag{6}$$

and

$$|p_2| \leq 3\tau^2. \tag{7}$$

The above estimates are sharp.

Now, we prove the following theorem as addition to Theorem 2.1.

Theorem 2.2. *If* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, *and* $p \prec \tilde{p}$, *then*

$$|p_3| \leq 4|\tau|^3. \tag{8}$$

The result is sharp.

Proof. If $p \prec \tilde{p}$, then there exists an analytic function w such that $|w(z)| \leq |z|$ in $\mathbb{U}$ and $p(z) = \tilde{p}(w(z))$. Therefore, the function

$$h(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \ldots \tag{9}$$

is in the class $\mathcal{P}(0)$. It follows that

$$w(z) = \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{z^3}{2} + \cdots \tag{10}$$

and

$$\begin{aligned}
\tilde{p}(w(z)) \;&=\; 1 + \tilde{p}_1 \left\{\frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{z^3}{2} + \cdots\right\} \\
&\quad + \tilde{p}_2 \left\{\frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{z^3}{2} + \cdots\right\}^2 \\
&\quad + \tilde{p}_3 \left\{\frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{z^3}{2} + \cdots\right\}^3 + \cdots \\
&=\; 1 + \frac{\tilde{p}_1 c_1 z}{2} + \left\{\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_1 + \frac{c_1^2}{4}\tilde{p}_2\right\} z^2 \\
&\quad + \left\{\frac{1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\tilde{p}_1 + \frac{1}{2}c_1\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_2 + \frac{c_1^3}{8}\tilde{p}_3\right\} z^3 + \cdots .
\end{aligned} \tag{11}$$

To find the coefficients $\tilde{p}_n$ of the function $\tilde{p}$, on putting $\tau z = t$, then we have

$$\begin{aligned}
\tilde{p}(z) &= \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2} = \left(t + \frac{1}{t}\right)\frac{t}{1 - t - t^2} \\
&= \frac{1}{\sqrt{5}}\left(t + \frac{1}{t}\right)\left(\frac{1}{1 - (1 - \tau)t} - \frac{1}{1 - \tau t}\right) \\
&= \left(t + \frac{1}{t}\right)\sum_{n=1}^{\infty}\frac{(1 - \tau)^n - \tau^n}{\sqrt{5}} t^n
\end{aligned}$$

$$= \left(t + \frac{1}{t}\right) \sum_{n=1}^{\infty} u_n t^n = 1 + \sum_{n=1}^{\infty} (u_{n-1} + u_{n+1}) \tau^n z^n, \tag{12}$$

where

$$u_n = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}}, \tau = \frac{1-\sqrt{5}}{2} \quad (n = 1, 2, \ldots). \tag{13}$$

This shows that the relevant connection of $\tilde{p}$ with the sequence of Fibonacci numbers u_n, such that $u_0 = 0$, $u_1 = 1$, $u_{n+2} = u_n + u_{n+1}$ for $n = 0, 1, 2, \cdots$. Now using (11), we get

$$\tilde{p}(z) = 1 + \sum_{n=1}^{\infty} \tilde{p}_n z^n$$

$$= 1 + (u_0 + u_2)\tau z + (u_1 + u_3)\tau^2 z^2 + \sum_{n=3}^{\infty} (u_{n-3} + u_{n-2} + u_{n-1} + u_n) \tau^n z^n$$

$$= 1 + \tau z + 3\tau^2 z^2 + 4\tau^3 z^3 + 7\tau^4 z^4 + 11\tau^5 z^5 + \cdots . \tag{14}$$

Thus, $\tilde{p}_1 = \tau,\ \tilde{p}_2 = 3\tau^2$ and

$$\tilde{p}_n = (u_{n-1} + u_{n+1})\tau^n = (u_{n-3} + u_{n-2} + u_{n-1} + u_n)\tau^n = \tau \tilde{p}_{n-1} + \tau^2 \tilde{p}_{n-2} \quad (n = 3, 4, 5, \ldots).$$

If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, then using (10) and (13) , we have

$$p_1 = \frac{c_1}{2}\tau, \tag{15}$$

$$p_2 = \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\tau + \frac{3}{4}c_1^2 \tau^2, \tag{16}$$

and

$$p_3 = \frac{1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\tau + \frac{3}{2}c_1\left(c_2 - \frac{c_1^2}{2}\right)\tau^2 + \frac{1}{2}c_1^3 \tau^3. \tag{17}$$

In [13], Raina and Sokół proved Theorem 2.1 and obtained sharp estimates for $|p_1|$ and $|p_2|$. Now we shall obtain sharp estimate for $|p_3|$. Taking absolute value of (17) we can write

$$\begin{aligned} |p_3| &= \left| \frac{1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\tau + \frac{3}{2}c_1\left(c_2 - \frac{c_1^2}{2}\right)\tau^2 + \frac{1}{2}c_1^3 \tau^3 \right| \\ &= \left| \frac{1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\tau + \frac{3}{2}c_1\left(c_2 - \frac{c_1^2}{2}\right)(\tau + 1) + \frac{1}{2}c_1^3(2\tau + 1) \right| \\ &= \left| \left\{ \frac{1}{2}\left(c_3 - 2c_1 c_2 + c_1^3\right) + \frac{c_1}{4}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{7}{4}c_1 c_2 \right\}\tau + \left\{ \frac{3c_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{c_1^3}{2} \right\} \right| \end{aligned} \tag{18}$$

It is known that

$$\forall n \in \mathbb{N},\ \tau = \frac{\tau^n}{u_n} - x_n, \quad x_n = \frac{u_{n-1}}{u_n}, \quad \lim_{n\to\infty} \frac{u_{n-1}}{u_n} = |\tau| \approx 0.618. \tag{19}$$

Therefore, we have

$$\begin{aligned}|p_3| &= \left|\left\{\frac{1}{2}\left(c_3 - 2c_1c_2 + c_1^3\right) + \frac{1}{4}c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{7}{4}c_1c_2\right\}\frac{\tau^n}{u_n}\right.\\ &\quad\left. + \left\{-\frac{1}{2}\left(c_3 - 2c_1c_2 + c_1^3\right)x_n + \frac{2-x_n}{4}c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{4-7x_n}{4}c_1c_2\right\}\right|\\ &\le \left|\frac{1}{2}\left(c_3 - 2c_1c_2 + c_1^3\right) + \frac{1}{4}c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{7}{4}c_1c_2\right|\frac{|\tau|^n}{u_n}\\ &\quad + \left|-\frac{1}{2}\left(c_3 - 2c_1c_2 + c_1^3\right)x_n + \frac{2-x_n}{4}c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{4-7x_n}{4}c_1c_2\right|\\ &\le \left\{\frac{1}{2}|c_3 - 2c_1c_2 + c_1^3| + \frac{1}{4}|c_1|\left|c_2 - \frac{c_1^2}{2}\right| + \frac{7}{4}|c_1c_2|\right\}\frac{|\tau|^n}{u_n}\\ &\quad + \left\{\frac{|c_3 - 2c_1c_2 + c_1^3|}{2}x_n + \frac{2-x_n}{4}|c_1|\left|c_2 - \frac{c_1^2}{2}\right| + \frac{|4-7x_n|}{4}|c_1||c_2|\right\}.\end{aligned}$$

By (19), for sufficiently large n we have $|4-7x_n| = 7x_n - 4$. Therefore, from (3), (4) and (5) we can write

$$\begin{aligned}|p_3| &\le \left\{1 + \frac{1}{4}|c_1|\left(2 - \frac{|c_1|^2}{2}\right) + \frac{7}{2}|c_1|\right\}\frac{|\tau|^n}{u_n} + \left\{x_n + \frac{2-x_n}{4}|c_1|\left(2 - \frac{|c_1|^2}{2}\right) + \frac{7x_n-4}{2}|c_1|\right\}\\ &= \left\{1 + 4|c_1| - \frac{|c_1|^3}{8}\right\}\frac{|\tau|^n}{u_n} + \left\{x_n + (3x_n - 1)|c_1| - \frac{2-x_n}{8}|c_1|^3\right\}.\end{aligned}$$

We have

$$\max_{y\in[0,2]}\left\{1 + 4y - \frac{y^3}{8}\right\} = 8 \quad \text{at} \quad y = 2,$$

since

$$\lim_{n\to\infty}\left\{1 + 4|c_1| - \frac{|c_1|^3}{8}\right\}\frac{|\tau|^n}{u_n} = 0.$$

Furthermore, for sufficiently large n we have

$$\max_{y\in[0,2]}\left\{x_n + (3x_n - 1)y - \frac{2-x_n}{8}y^3\right\} = 8x_n - 4 \quad \text{at} \quad y = 2,$$

so

$$\lim_{n\to\infty}\max_{y\in[0,2]}\left\{x_n + (3x_n - 1)y - \frac{2-x_n}{8}y^3\right\} = 8|\tau| - 4 = 4|\tau|^3.$$

Therefore, we get

$$\lim_{n\to\infty}\left[\left\{1 + 4|c_1| - \frac{|c_1|^3}{8}\right\}\frac{|\tau|^n}{u_n} + \left\{x_n + (3x_n - 1)|c_1| - \frac{2-x_n}{8}|c_1|^3\right\}\right] \le 4|\tau|^3$$

which shows that

$$|p_3| \le 4|\tau|^3.$$

If we take in (9)

$$h(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \dots,$$

then putting $c_1 = c_2 = c_3 = 2$ in (17) gives $p_3 = 4\tau^3$ and it shows that (8) is sharp. It completes the proof.

Comjecture 2.1. *If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, and $p \prec \tilde{p}$, then*

$$|p_n| \leq (u_{n-1} + u_{n+1})|\tau|^n, \quad n = 1, 2, 3, \ldots,$$

where $u_0 = 0$, $u_1 = 1$, $u_{n+2} = u_n + u_{n+1}$ for $n = 0, 1, 2, \ldots$, is the Fibonacci sequence. This bound would be sharp for the function (14).

This conjecture has been just verified for $n = 3$ in last Theorem 2.2, while for $n = 1, 2$ it was proved in [13].

Theorem 2.3. *If $f(z) = z + a_2 z^2 + \ldots$ belongs to $\mathcal{SL}$, then*

$$|a_2 a_4 - a_3^2| \leq \frac{11}{3}\tau^4. \tag{20}$$

Proof. For given $f \in \mathcal{SL}$, define $p(z) = 1 + p_1 z + p_2^2 z^2 + \cdots$, by

$$\frac{zf'(z)}{f(z)} = p(z)$$

where $p \prec \tilde{p}$. Hence

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 - 3a_2 a_3 + a_2^3)z^3 + \cdots = 1 + p_1 z + p_2^2 z^2 + \cdots$$

and

$$a_2 = p_1, \quad a_3 = \frac{p_1^2 + p_2}{2}, \quad a_4 = \frac{p_1^3 + 3p_1 p_2 + 2p_3}{6}.$$

Therefore,

$$a_2 a_4 - a_3^2 \;=\; \frac{1}{12}(-p_1^4 + 4p_1 p_3 - 3p_2^2). \tag{21}$$

Using Theorem 2.1 and Theorem 2.2, we obtain

$$\begin{aligned} |a_2 a_4 - a_3^2| &= \left|\frac{1}{12}(-p_1^4 + 4p_1 p_3 - 3p_2^2)\right| \\ &\leq \frac{1}{12}\left(|p_1|^4 + 4|p_1||p_3| + 3|p_2|^2\right) \\ &\leq \frac{1}{12}\left(|\tau|^4 + 4|\tau|4|\tau|^3 + 3(3\tau^2)^2\right) \\ &= \frac{1}{12}\left(|\tau|^4 + 16|\tau|^4 + 27|\tau|^4\right) \\ &= \frac{11}{3}\tau^4. \end{aligned}$$

□

The bound in 20 is not sharp. So we give the following conjecture for sharpness.

Comjecture 2.2. *If $f(z) = z + a_2 z^2 + \ldots$ belongs to $\mathcal{SL}$, then*

$$|a_2 a_4 - a_3^2| \leq \tau^4. \tag{22}$$

The bound is sharp.

Theorem 2.4. *If $f(z) = z + a_2 z^2 + \ldots$ belongs to $\mathcal{KSL}$, then*

$$|a_2 a_4 - a_3^2| \leq \frac{4}{9}\tau^4.$$

Proof. For given $f \in \mathcal{KSL}$, define $p(z) = 1 + p_1 z + p_2^2 z^2 + \cdots$, by

$$1 + \frac{zf''(z)}{f'(z)} = p(z)$$

where $p \prec \tilde{p}$ in $\mathbb{U}$. Hence

$$1 + \frac{zf''(z)}{f'(z)} = 1 + 2a_2 z + (6a_3 - 4a_2^2)z^2 + (12a_4 - 18a_2a_3 + 8a_2^3)z^3 + \cdots = 1 + p_1 z + p_2^2 z^2 + \cdots$$

and

$$a_2 = \frac{p_1}{2}, \quad a_3 = \frac{p_1^2 + p_2}{6}, \quad a_4 = \frac{p_1^3 + 3p_1p_2 + 2p_3}{24}.$$

Therefore, using Theorem 2.1 and Theorem 2.2, we obtain

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}\tau^4.$$

□

Theorem 2.5. *If $f(z) = z + a_2z^2 + ...$ belongs to $\mathcal{SLM}_\alpha$, then*

$$|a_2a_4 - a_3^2| \leq \frac{145\alpha^5 + 625\alpha^4 + 1061\alpha^3 + 867\alpha^2 + 330\alpha + 44}{12(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)}\tau^4.$$

Proof. For given $f \in \mathcal{SLM}_\alpha$, define $p(z) = 1 + p_1 z + p_2^2 z^2 + \cdots$, by

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = p(z)$$

where $p \prec \tilde{p}$ in $\mathbb{U}$. Hence

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + (1+\alpha)a_2 z + [2(1+2\alpha)a_3 - (1+3\alpha)a_2^2]z^2$$

$$+[3(1+3\alpha)a_4 - 3(1+5\alpha)a_2a_3 + (1+7\alpha)a_2^3]z^3 + \cdots = 1 + p_1 z + p_2^2 z^2 + \cdots$$

and

$$a_2 = \frac{p_1}{1+\alpha}, \quad a_3 = \frac{(1+3\alpha)p_1^2 + (1+\alpha)^2 p_2}{2(1+\alpha)^2(1+2\alpha)},$$

$$a_4 = \frac{3(1+3\alpha)(1+5\alpha)p_1^3 + 3(1+\alpha)^2(1+5\alpha)p_1p_2 - 2(1+2\alpha)(1+7\alpha)p_1^3 + 2(1+\alpha)^3(1+2\alpha)p_3}{6(1+\alpha)^3(1+2\alpha)(1+3\alpha)}.$$

Therefore, using Theorem 2.1 and Theorem 2.2, we obtain

$$|a_2a_4 - a_3^2| \leq \frac{145\alpha^5 + 625\alpha^4 + 1061\alpha^3 + 867\alpha^2 + 330\alpha + 44}{12(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)}\tau^4.$$

□

It is clear that if we take $\alpha = 0$ and $\alpha = 1$ in Theorem 2.5, we obtain the results of Theorem 2.3 and Theorem 2.4, respectively.

3. Concluding Remarks and Observations

In our present article, we have obtained sharp estimates for second Hankel determinants of several classes of analytic functions related to shell-like curves connected with Fibonacci numbers. Firstly, we have found a sharp bound estimate for third coefficient of a function with positive real part which is subordinate to a shell-like curve and have given a conjecture for general case. Secondly, we have studied the problem of finding the upper bounds associated with the second Hankel determinant $H_2(2)$ for these classes. We have also considered several results which are closely related to our investigation in this paper. However, we give some conjectures for sharpness of bounds.

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References

[1] Dziok, J., Raina, R. K., Sokół, J., 61(2011) 2605-2613, Certain results for a class of convex functions related to a shell-like curve connected with Fibonacci numbers, Comp. Math. Appl.
[2] Dziok, J., Raina, R. K., Sokół, 218 (2011) 996–1002, On $\alpha-$convex functions related to a shell-like curve connected with Fibonacci numbers, Appl. Math. Comp.
[3] Ehrenborg, R., 107 (2000): 557–560, The Hankel determinant of exponential polynomials, Amer. Math. Monthly.
[4] Fekete, M., Szegö, G., Eine Bemerkung, 8 (1933) 85–89, Über ungerade schlichte Functionen, J. London Math. Soc.
[5] Janteng, A., Halim, S., Darus, M., 7 (2) (2006), Article 50, Coefficient inequality for a function whose derivative has a positive real part, J. Inequal. Pure Appl. Math.
[6] Janteng, A., Halim, S., Darus, M., 1 (13) (2007) 619–625, Hankel determinant for starlike and convex functions, Int. J. Math. Anal.
[7] Keogh, F. R., Merkes,E. P., 20 (1969) 8-12, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc.
[8] Layman, J. W., 4 (2001): 1-11. The Hankel transform and some of its properties, J. Integer Sequences
[9] Libera, R. J., Złotkiewicz, E. J., 87 (2) (1983) 251-257, Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$, Proc. Amer. Math. Soc.
[10] Noonan, J. W., Thomas, D. K., 223 (2) (1976) 337-346, On the second Hankel determinant of a really mean p-valent functions, Trans. Amer. Math. Soc.
[11] Noor, K. I., 28 (8) (1983) 731-739, Hankel determinant problem for the class of functions with bounded boundary rotation, Rev. Roum. Math. Pures Appl.
[12] Pommerenke, Ch., 1975, Univalent Functions, Math. Math, Lehrbucher, Vandenhoeck and Ruprecht, Göttingen.
[13] Raina, R. K., Sokół, J., 66 (2016) 135-140, Fekete-Szegö problem for some starlike functions related to shell-like curves, Math. Slovaca.
[14] Sokół, J., 175 (1999) 111-116, On starlike functions connected with Fibonacci numbers, Folia Scient. Univ. Tech. Resoviensis.

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