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THE BOCHNER VANISHING THEOREMS ON THE CONFORMAL KILLING VECTOR FIELDS

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ABSTRACT. In this paper, the result of the Bochner vanishing theorems, indicating the conditions that every conformal killing vector fields is parallel and there is no nontrivial Conformal Killing vector field, are satisfied under two different modificated Ricci tensors.

Keywords: Ricci curvature, Conformal Killing vector field, Vanishing theorem, Bochner technique.

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1. INRODUCTION

In the 1940s, S. Bochner invented a method to obtain vanishing theorems for some topological and geometric invariants (e.g. Betti number, the dimension of the vector space of Killing vector fields) on a closed compact Riemannian manifold without boundary, under the assumption of the Ricci curvature [2]. In 1953 K. Kodaira used this method to prove the vanishing theorem for harmonic forms with values in a holomorphic vector bundle [6]. Subsequently, the Bochner technique has been extended, on the one hand, to spinor fields and harmonic maps and, on the other, to harmonic functions and harmonic maps on noncompact manifolds. However, Lichnerowicz [8], Mogi [10], Tomogana [11] and Yano [15] have applied this technique to harmonic and Killing tensor fields on the complex manifolds, complete Riemannian manifolds, and Lorentzian manifolds. Another application of this technique is to obtain a lower bound for the first positive eigenvalue of the Laplace operator, under the modified Ricci curvature assumptions [1, 14, 9, 12, 15]. In this paper, the Bochner vanishing theorem for Conformal Killing vector fields are mainly handled.

This article consists of two parts. In section 2, we begin with a section concerning some basic facts about Riemannian geometry. In the following section, two modified Ricci curvature are construct. The first of this is built with the help of the Hessian and Laplacian operator depending on the positive function $f \in C^{\infty}(M, \mathbb{R})$. Accordingly, it is shown that Bochner's vanishing theory for the Conformal Killing vector field was provided using

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the assumption under the modified Ricci curvature constructed in this way. Moreover, under this assumption, assuming that the function f is constant, the assumption $Ric \leq 0$ provides the Bochner vanishing theorem. Also, it has been proven that Bochner vanishing theorem has been provided under the assumption of a modified Ricci tensor constructed with the constant real number κ and the divergence of the conformal killing vector field Z. Fianlly, it has been proven that there are no nontrivial Conformal Killing vector field under two different modified Ricci tensors at any point of the compact, oriented Rieamannian manifolds (M, g).

2. Some Preliminaries

On an n dimensional smooth manifold M, a symmetric and positive definite $(0,2)-$type tensor field

$$g : \chi(M) \times \chi(M) \longrightarrow C^{\infty}(M, \mathbb{R})$$

defines an inner product for any $p \in M$ as follows:

$$g_p : T_pM \times T_pM \longrightarrow \mathbb{R}.$$

Here g is called Riemannian metric and the pair (M, g) is called Riemannian manifold. According to the Riemannian metric g, the definitions of gradient, Hessian and Laplacian are given as follows, respectively.

Definition 2.1. *Gradient is an operator*

$$\begin{array}{rcl} \nabla : C^{\infty}(M, \mathbb{R}) & \longrightarrow & \chi(M) \\ f & \longmapsto & \nabla f \end{array}$$

which makes $g(\nabla f, X) = X(f)$, $\forall X \in \chi(M)$.

Definition 2.2. *Hessian is a* $(0,2)-$*type tensor field which is detoned by* $Hessf$ *and defined as follows*

$$\begin{array}{rcl} Hessf : \chi(M) \times \chi(M) & \longrightarrow & C^{\infty}(M, \mathbb{R}), \\ (X, Y) & \longmapsto & Hessf(X, Y) := g\big(\nabla_X \nabla f, Y\big) \end{array}$$

where $X, Y \in \chi(M)$. *Another definition of hessian is given by*

$$Hessf(X, Y) := g\big(hessf(X), Y\big)$$

where $hessf = \nabla\nabla f : \chi(M) \times \chi(M)^{*} \longrightarrow C^{\infty}(M, \mathbb{R})$.

With respect to the $hessf$, the definition of the Laplacian is

$$\begin{array}{rcl} \Delta : C^{\infty}(M, \mathbb{R}) & \longrightarrow & C^{\infty}(M, \mathbb{R}) \\ f & \longmapsto & \Delta(f) := tr(hessf). \end{array}$$

Definition 2.3. *Levi–Civita covariant differential of any vector field* X *is a* $(1,1)-$*type tensor field denoted by* ∇X *and defined as*

$$\begin{array}{rcl} \nabla X : \chi(M) & \longrightarrow & \chi(M) \\ Y & \longmapsto & \nabla X(Y) := \nabla_Y X. \end{array}$$

In the following, definitions of the inner product of $(p, q)-$type tensor fields, divergence of any vector field and Ricci curvature are given, respectively. Let A, B be $(p, q)-$type tensor fields. Then the inner product of these tensor fields in any local coordinate system is denoted by $< A, B >$ and defined as folows:

$$< A, B > \; = \; g^{i_1 j_1} ... g^{i_p j_p} g_{k_1 l_1} ... g_{k_q l_q} A^{k_1 ... k_q}_{i_1 ... i_p} B^{l_1 ... l_q}_{j_1 ... j_p}$$

where $g^{ij} = (g_{ij})^{-1}$.

Definition 2.4. *A Ricci tensor is a* $(0,2)-$*type tensor field which is defined as follows:*

$$\begin{array}{rcl} \mathrm{Ric}:\chi(M)\times\chi(M) & \to & \mathcal{C}^{\infty}(M,\mathbb{R}) \\ (V,W) & \mapsto & \mathrm{Ric}(V,W)=g^{\mu\nu}g(R(V,\partial_{\mu})\partial_{\nu},W) \end{array}$$

where $R:\chi(M)\times\chi(M)\chi(M)\longrightarrow\chi(M)$ *is the* $(0,3)-$*type Riemann curvature.*

In the following, the concept of Killing vector fields and Conformal Killing vector fields are introduced, and some formula is given by using Bochner's technique with respect to the Killing vector fields and Conformal Killing vector fieds. Then under the assumptions of modificated Ricci curvature, we study Bochner's vanishing theorems.

3. Conformal Killing Vector Fields

Let (M,g) be an $n-$dimensional Riemannian manifold. Then a Conformal Killing vector field Z is defined as follows:

$$\mathcal{L}_Z g = f.g \tag{1}$$

where $\mathcal{L}_Z g$ is the Lie derivative of the Riemannian metric g with respect to $Z\in\chi(M)$ and the function $f=\frac{2}{n}div(Z)\in C^{\infty}(M,\mathbb{R})$ is the potential function of the Conformal Killing vector field $Z\in\chi(M)$ [1, 4]. Moreover, $\mathcal{L}_Z g$ is the flow of the Conformal Killing vector field Z which consists of the conformal transformation of the Riemannian manifold (M,g) and the flow of the Conformal Killing vector fields preserves the conformal structure of the manifold [3, 5, 15].

Also Conformal Killing vector fields are studied with the Bochner's technique in Riemannian geometry. This technique consist of the vanishing of the Killing vector fields and Conformal Killing vector fields under the assumption of the modified Ricci tensor. On the Riemannian manifold (M,g) a typical example of this technique is given below:

$$\mathrm{div}(\nabla_{\mathrm{X}}\mathrm{X})=\mathrm{Ric}(\mathrm{X},\mathrm{X})+<\nabla\mathrm{X},(\nabla\mathrm{X})^{*}> \tag{2}$$

where ∇X is the covariant derivative of Killing vector field X, and $(\nabla X)^{*}=-\nabla X$ is the anti-selfadjoint of ∇X [13]. Another interesting example is the Conformal Killing vector field Z which is as follows:

$$\mathrm{div}(\nabla_{\mathrm{Z}}\mathrm{Z}) = Ric(Z,Z)+Zdiv(Z)+\frac{2}{n}\big(\mathrm{div}(\mathrm{Z})\big)^{2}-<\nabla\mathrm{Z},\nabla\mathrm{Z}> \tag{3}$$

Theorem 3.1. *Let* (M,g) *be a compact, oriented Riemannian manifold and the following assumption is satisfied*

$$Ric+\frac{1}{n+2}\left(\frac{n-2}{f}Hess(f)+\frac{n}{f}(\Delta f)g\right)\leq 0, \tag{4}$$

where $n\geq 2$ *and* $f\in C^{\infty}(M,\mathbb{R}^{+})$. *Then every Conformal Killing vector field is parallel. Furthermore, if*

$$\left(Ric+\frac{1}{n+2}\Big(\frac{n-2}{f}Hess(f)+\frac{n}{f}(\Delta f)g\Big)\right)(p)<0 \tag{5}$$

at the point $p\in M$ *which is giving minimum value of* $f\in C^{\infty}(M,\mathbb{R})$, *then there is no nontrivial Conformal Killing vector field.*

Proof of Theorem 3.1. Let us construct the following vector field denoted by W,

$$W = f\nabla_Z Z + \Big(\frac{n-2}{n+2}\Big)g(\nabla f, Z)Z + \Big(\frac{n}{n+2}\Big)g(Z,Z)\nabla f - f\mathrm{div(Z)Z}, \tag{6}$$

where Z is a Conformal Killing vector field. Taking divergence of the equation (6), one gets

$$\begin{aligned}\mathrm{div}\ (W) \quad = \quad & f\mathrm{div}(\nabla_{\mathrm{Z}}\mathrm{Z}) + \mathrm{g}(\nabla_{\mathrm{Z}}\mathrm{Z}, \nabla\ f) + \Big(\frac{\mathrm{n}-2}{\mathrm{n}+2}\Big)\mathrm{g}(\nabla\ f, \mathrm{Z})\mathrm{div}(\mathrm{Z}) \\ & +\Big(\frac{n-2}{n+2}\Big)g(\nabla_Z\nabla f, Z) + \Big(\frac{n-2}{n+2}\Big)g(\nabla f, \nabla_Z Z) + \Big(\frac{n}{n+2}\Big)(\Delta f)g(Z,Z) \\ & +\Big(\frac{2n}{n+2}\Big)g(\nabla_{\nabla f}Z, Z) - f(\mathrm{div(Z)})^2 - f\mathrm{Z}(\mathrm{div(Z)}) \\ & -div(Z)g(Z,\nabla f).\end{aligned} \tag{7}$$

A Conformal Killing vector field satisfies the equation

$$g(\nabla_V Z, W) + g(\nabla_W Z, V) = \Big(\frac{2}{n}\Big)\mathrm{div(Z)g(V,W)} \tag{8}$$

for all vector fields V, W. Thus, the term $g(\nabla_{\nabla f}Z, Z)$ can be written as

$$g(\nabla_{\nabla f}Z, Z) = -g(\nabla_Z Z, \nabla f) + \Big(\frac{2}{n}\Big)\mathrm{div(Z)g(\nabla\ \mathit{f}, Z)}. \tag{9}$$

By using (3) and (9) in (7), one has

$$\begin{aligned}\mathrm{div}\ (W) \quad = \quad & \Bigg(Ric(Z,Z) + \frac{1}{n+2}\Big(\frac{n-2}{f}Hess(f)(Z,Z) + \frac{n}{f}(\Delta f)g(Z,Z)\Big) \\ & - < \nabla Z, \nabla Z > + \Big(\frac{2}{n} - 1\Big)(\mathrm{div(Z)})^2\Bigg)\ f.\end{aligned} \tag{10}$$

Integrating (10), one has

$$\begin{aligned}0 \quad = \quad & \int_M \Bigg(Ric(Z,Z) + \frac{1}{n+2}\Big(\frac{n-2}{f}Hess(f)(Z,Z) + \frac{n}{f}(\Delta f)g(Z,Z)\Big) \\ & - < \nabla Z, \nabla Z > + \Big(\frac{2}{n} - 1\Big)(\mathrm{div(Z)})^2\Bigg)\ f\,\mathrm{dvol}.\end{aligned} \tag{11}$$

Using the assumption (4) given in Theorem 1 and $(\frac{2}{n} - 1)(divZ)^2)f \leq 0$ in the integral expression (11) we get

$$\int_M \big(\langle\nabla Z, \nabla Z\rangle f\big)dvol \leq 0. \tag{12}$$

On the other hand since $\langle\nabla Z, \nabla Z\rangle \geq 0$ and $f > 0$, we get

$$\int_M \big(\langle\nabla Z, \nabla Z\rangle f\big)dvol \geq 0. \tag{13}$$

According to (12)and (13), $\langle\nabla Z, \nabla Z\rangle$ is vanished. This means $\nabla Z = 0$.

Proof of Theorem 3.1. Under the assumption (4) we obtained that every Conformal Killing vector field is parallel. Since this result is satisfied under the assumption (5) we get $\nabla Z = 0$. With the aid of $\nabla Z = 0$ we obtain $Ric(Z, Z) = 0$. Then we have

$$\left(\left(\frac{1}{n+2}\right)\left(\frac{n-2}{f}Hess(f) + \frac{n}{f}(\Delta f)g\right)\right)(p) < 0 \tag{14}$$

at the point $p \in M$ which gives the minimum value of f. On the other hand, at this point $(\Delta f)(p) \geq 0$ and $(Hess(f))_p \geq 0$ which means,

$$\left(\left(\frac{1}{n+2}\right)\left(\frac{n-2}{f}Hess(f) + \frac{n}{f}(\Delta f)g\right)\right)(p) > 0. \tag{15}$$

Therefore, (15) contradicts with (5). Hence, there is no nontrivial Conformal Killing vector field.

In the following by using different assumption the same results are obtained.

Theorem 3.2. *Let (M, g) be $n-$dimensional compact, oriented Riemannian manifold and satisfies*

$$Ric + k\left(\frac{2}{n} + 1\right)\mathrm{div(Z)g} \leq 0, \tag{16}$$

where $n \geq 2$, Z is the Conformal Killing vector field and k is a constant. Then every Conformal Killing vector field is parallel. Furthermore, if there is a negative constant $C < 0$ satisfying the following inequality

$$Ric + k\left(\frac{2}{n} + 1\right)\mathrm{div(Z)g} \leq \mathrm{C}, \tag{17}$$

then there is no nontrivial Conformal Killing vector field.

Proof of Theorem 3.2. Let the vector field W be given as follows

$$W = \nabla_Z Z + kg(Z, Z)Z - \mathrm{div(Z)Z}, \tag{18}$$

where Z is a Conformal Killing vector field. The divergence of W is first determined as

$$\begin{aligned}\mathrm{div}\ (W) \;&=\; \mathrm{div(\nabla_Z Z) + kg(Z,Z)div(Z) + 2kg(\nabla_Z Z, Z)} - \big(\mathrm{div(Z)}\big)^2 \\ &\quad -Z\mathrm{div(Z)}).\end{aligned} \tag{19}$$

To simplify this equation, it is used to Conformal Killing vector field Z satisfiying the equation

$$L_Z g = fg \Leftrightarrow g(\nabla_V Z, W) + g(\nabla_W Z, V) = 2\frac{\mathrm{div(Z)}}{n}g(V, W) \tag{20}$$

for all vector fields V, W. Thus, the term $g(\nabla_Z Z, Z)$ can be written as

$$\begin{aligned}g(\nabla_Z Z, Z) + g(\nabla_Z Z, Z) &= 2\frac{\mathrm{div(Z)}}{n}g(Z, Z) \\ 2g(\nabla_Z Z, Z) &= 2\frac{\mathrm{div(Z)}}{n}g(Z, Z).\end{aligned} \tag{21}$$

Combining (21) with (19), one gets

$$\begin{aligned}\mathrm{div}\ (W) \;&=\; div(\nabla_Z Z) + kg(Z, Z)\mathrm{div(Z)} + \left(\frac{\mathrm{2k}}{\mathrm{n}}\right)\mathrm{div(Z)g(Z,Z)} \\ &\quad -(\mathrm{div(Z)})^2 - \mathrm{Z(div(Z))}.\end{aligned} \tag{22}$$

Inserting (3) into (22), we get

$$\begin{aligned} \text{div } (W) &= Ric(Z,Z) + \Big(\frac{2}{n} - 1\Big)(\text{div(Z)})^2 - \langle \nabla \text{Z}, \nabla \text{Z} \rangle \\ &+ \Big(k + \frac{2k}{n}\Big) g(Z,Z)\text{div(Z)}. \end{aligned} \tag{23}$$

Integrating both sides of (22), we obtain

$$\begin{aligned} 0 &= \int_M \Bigg(Ric(Z,Z) + \Big(\frac{2}{n} - 1\Big)(\text{div(Z)})^2 - \langle \nabla \text{Z}, \nabla \text{Z} \rangle \\ &+ \Big(k + \frac{2k}{n}\Big) g(Z,Z)\text{div(Z)} \Bigg) \text{dvol}. \end{aligned} \tag{24}$$

Using the assumption (16) given in Theorem 2 and $\left(\frac{2}{n} - 1\right)(\text{div(Z)})^2 \leq 0$ in the integral expression (24 we get

$$\int_M \big(\langle \nabla Z, \nabla Z \rangle u \big) dvol. \leq 0 \tag{25}$$

On the other hand, since $\langle \nabla Z, \nabla Z \rangle \geq 0$, $\int_M < \nabla Z, \nabla Z > dvol \geq 0$. Then $\langle \nabla Z, \nabla Z \rangle = 0$ iff $\nabla Z = 0$.

Proof of Theorem 3.2. Let us consider integral expression of (19) which is given in (24). By using the assumption (17) given in Theorem 2.2 in the integral expression (24) we get

$$0 \leq \int_M \Bigg(Cg(Z,Z) + \Big(\frac{2-n}{n}\Big)(\text{div(Z)})^2 - \langle \nabla \text{Z}, \nabla \text{Z} \rangle \Bigg) \text{dvol} \tag{26}$$

On the other hand, Since

$$Cg(Z,Z) + \Big(\frac{2-n}{n}\Big)(\text{div(Z)})^2 - \langle \nabla \text{Z}, \nabla \text{Z} \rangle \leq 0, \tag{27}$$

$$0 \geq \int_M \Bigg(Cg(Z,Z) + \Big(\frac{2-n}{n}\Big)(\text{div(Z)})^2 - \langle \nabla \text{Z}, \nabla \text{Z} \rangle \Bigg) \text{dvol}. \tag{28}$$

Thus we have

$$Cg(Z,Z) + \Big(\frac{2-n}{n}\Big)(\text{div(Z)})^2 - \langle \nabla \text{Z}, \nabla \text{Z} \rangle = 0 \tag{29}$$

which means $Cg(Z,Z) = 0$, $\left(\frac{2-n}{n}\right)(\text{div(Z)})^2 \leq 0$ and $\langle \nabla Z, \nabla Z \rangle = 0$. Since C negative, we obtain get $g(Z,Z) =$ iff $Z = 0$ Hence, there is no nontrivial Conformal Killing vector field.

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