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ON HERMITE-HADAMARD TYPE INEQUALITIES VIA KATUGAMPOLA FRACTIONAL INTEGRALS

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Abstract. In this paper, we give new definitons related to Katugampola fractional integral for two variables functions. We are interested in giving the Hermite–Hadamard inequality for a rectangle in plane via convex functions on co-ordinates involving Katugampola fractional integral.

Keywords: Convex function, co-ordinated convex function, Hermite-Hadamard inequalities, Katugampola fractional integral.

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1. Introduction

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be convex function defined on the interval I of real numbers and $a, b \in I$, with $a < b$. Then the following double inequality is known in the literature as the Hermite-Hadamard's inequality for convex functions [8]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\ dx \leq \frac{f(a)+f(b)}{2}. \qquad (1)$$

The inequalities (1) have become an important cornerstone in mathematical analysis and optimization and many uses of these inequalities have been discovered in a variety of settings. Many generalizations and extensions of the Hermite-Hadamard inequality exist in the literatures (see [5]).

Let us consider a bidimensional interval $\Delta =: [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A function $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on Δ if for all $(x, y), (z, w) \in \Delta$ and $t \in [0, 1]$, it satifies the following inequality:

$$f(tx + (1-t)\ z, ty + (1-t)\ w) \leq t\ f(x, y) + (1-t)\ f(z, w).$$

A modification for convex function on Δ was defined by Dragomir [4], as follows:

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A function $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are convex where defined for all $x\in[a,b]$ and $y\in[c,d]$.

A formal defination for co-ordinated convex function may be stated as follows:

Definition 1.1. *A function $f:\Delta\to\mathbb{R}$ is called co-ordinated convex on Δ, for all $(x,u),(y,v)\in\Delta$ and $t,s\in[0,1]$, if it satifies the following inequality:*

$$\begin{aligned} & f(tx+(1-t)\ y, su+(1-s)\ v) \qquad (2)\\ \leq\ & ts\ f(x,u)+t(1-s)f(x,v)+s(1-t)f(y,u)+(1-t)(1-s)f(y,v). \end{aligned}$$

Note that every convex function $f:\Delta\to\mathbb{R}$ is co-ordinated convex but the converse is not generally true (see, [4]).

For recent developments about Hermite-Hadamard's inequality for some convex functions on the co-ordinates, please refer to ([2],[9],[15],[19]and[20]). Also several inequalities for convex functions on the co-ordinates see the references ([1],[6],[14],[16],[17]).

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. Here, we just point out that the classical Riemann-Liouville fractional integrals I_{a+}^{α} and I_{b-}^{α} of order α defined by (see, [7, 12, 13])

$$\left(I_{a+}^{\alpha}\varphi\right)(x):=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi(t)dt \quad (x>a;\alpha>0) \qquad (3)$$

and

$$\left(I_{b-}^{\alpha}\varphi\right)(x):=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\varphi(t)dt \quad (x<b;\alpha>0) \qquad (4)$$

Later, in [17], Sarikaya presented the following Hermite–Hadamard-type inequalities for Riemann–Liouville fractional integrals by using convex functions of two variables on the co-ordinates:

Theorem 1.1. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a co-ordinated convex on $\Delta:[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $0\leq a<b$, $0\leq c<d$ and $f\in L_1(\Delta)$. Then one has the inequalities:*

$$\begin{aligned} & f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \qquad (5)\\ \leq\ & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\\ & \times\left[J_{a^+,c^+}^{\alpha,\beta}f(b,d)+J_{a^+,d^-}^{\alpha,\beta}f(b,c)+J_{b^-,c^+}^{\alpha,\beta}f(a,d)+J_{b^-,d^-}^{\alpha,\beta}f(a,c)\right]\\ \leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

Theorem 1.2. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a co-ordinated convex on $\Delta : [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $0 \leq a < b$, $0 \leq c < d$ and $f \in L_1(\Delta)$. Then one has the inequalities:*

$$
\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq \; & \frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f\left(b, \frac{c+d}{2}\right) + J_{b^-}^{\alpha} f\left(a, \frac{c+d}{2}\right)\right] \\
& + \frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\left[J_{c^+}^{\beta} f\left(\frac{a+b}{2}, d\right) + J_{d^-}^{\beta} f\left(\frac{a+b}{2}, c\right)\right] \\
\leq \; & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}} \\
& \times \left[J_{a^+,c^+}^{\alpha,\beta} f(b,d) + J_{a^+,d^-}^{\alpha,\beta} f(b,c) + J_{b^-,c^+}^{\alpha,\beta} f(a,d) + J_{b^-,d^-}^{\alpha,\beta} f(a,c)\right] \\
\leq \; & \frac{\Gamma(\alpha+1)}{8(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b,c) + J_{a^+}^{\alpha} f(b,d) + J_{b^-}^{\alpha} f(a,c) + J_{b^-}^{\alpha} f(a,d)\right] \\
& + \frac{\Gamma(\beta+1)}{8(d-c)^{\beta}}\left[J_{c^+}^{\beta} f(a,d) + J_{c^+}^{\beta} f(b,d) + J_{d^-}^{\beta} f(a,c) + J_{d^-}^{\beta} f(b,c)\right] \\
\leq \; & \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}.
\end{aligned} \tag{6}
$$

Recently, Katugampola introduced a new fractional integral that generalizes the Riemann-Liouville and the Hadamard fractional integrals in to a single form(see [10],[11],[18]).

Definition 1.2. *([10]) Let $[a,b] \subset \mathbb{R}$ be a finite interval. then, the left and right side Katugampola fractional integrals of order $\alpha(>0)$ of $f \in X_c^p(a,b)$ are defined by,*

$$
{}^{\rho}I_{a^+}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^{\rho}-t^{\rho})^{1-a}} f(t)\, dt \quad \text{and} \quad {}^{\rho}I_{b^+}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-a}} f(t)\, dt
$$

with $a < x < b$ and $\rho > 0$, if the integrals exist.

When $\rho = 0$ we arrive at the standard Riemann-Liouville fractional integral.

In [3], Chen and Katugampola proved the following inequality which is Hermite-Hadamard's inequalities for the Katugampola fractional integrals:

Theorem 1.3. *Let $\alpha > 0$ and $\rho > 0$. Let $f : [a^{\rho}, b^{\rho}] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in X_c^p(a^{\rho}, b^{\rho})$. If f is also a convex function on $[a,b]$, then the following inequalities hold:*

$$
f\left(\frac{a^{\rho}+b^{\rho}}{2}\right) \leq \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(b^{\rho}-a^{\rho})^{\alpha}}\left[{}^{\rho}I_{a^+}^{\alpha} f(b^{\rho}) + {}^{\rho} I_{b^-}^{\alpha} f(a^{\rho})\right] \leq \frac{f(a^{\rho}) + f(b^{\rho})}{2} \tag{7}
$$

where the fractional integrals are considered for the function $f(x^{\rho})$ and evaluated at a and b, respectively.

Now, we establish new definitons related to Katugampola fractional integrals for two variables functions:

Definition 1.3. *Let $f \in L_1([a,b]\times[c,d])$. The Katugampola fractional integrals ${}^{\rho,\sigma}I^{\alpha,\beta}_{a^+,c^+}f(x,y)$, ${}^{\rho,\sigma}I^{\alpha,\beta}_{a^+,d^-}f(x,y)$, ${}^{\rho,\sigma}I^{\alpha,\beta}_{b^-,c^+}f(x,y)$, and ${}^{\rho,\sigma}I^{\alpha,\beta}_{b^-,d^-}f(x,y)$ of order $\alpha,\beta>0$ are defined by*

$$\begin{aligned}
{}^{\rho,\sigma}I^{\alpha,\beta}_{a^+,c^+}f(x,y) &:= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_a^x\int_c^y \frac{t^{\rho-1}s^{\sigma-1}}{(x^\rho-t^\rho)^{1-a}(y^\sigma-s^\sigma)^{1-\beta}}f(t,s)dsdt,\\
x &> a,\ y>c,\\
{}^{\rho,\sigma}I^{\alpha,\beta}_{a^+,d^-}f(x,y) &:= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_a^x\int_y^d \frac{t^{\rho-1}s^{\sigma-1}}{(x^\rho-t^\rho)^{1-a}(s^\sigma-y^\sigma)^{1-\beta}}f(t,s)dsdt,\\
x &> a,\ y<d,\\
{}^{\rho,\sigma}I^{\alpha,\beta}_{b^-,c^+}f(x,y) &:= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_x^b\int_c^y \frac{t^{\rho-1}s^{\sigma-1}}{(t^\rho-x^\rho)^{1-a}(y^\sigma-s^\sigma)^{1-\beta}}f(t,s)dsdt,\\
x &< b,\ y>c,
\end{aligned}$$

and

$$\begin{aligned}
{}^{\rho,\sigma}I^{\alpha,\beta}_{b^-,d^-}f(x,y) &:= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_x^b\int_y^d \frac{t^{\rho-1}s^{\sigma-1}}{(t^\rho-x^\rho)^{1-a}(s^\sigma-y^\sigma)^{1-\beta}}f(t,s)dsdt,\\
x &< b,\ y<d.
\end{aligned}$$

with $a<x<b$ and $c<y<d$ with $\rho>0$.

In this paper, we are interested to give the Hermite–Hadamard inequality for a rectangle in plane via convex functions on co-ordinates involving Katugampola fractional integrals. We also study some properties of mappings associated with the Hermite–Hadamard inequality for convex functions on co-ordinates.

2. Hermite Hadamard Type Inequalities for Katugampola fractional integrals

In this section, we will give Hermite-Hadamard type inequalities for the Katugampola fractional integrals by using co-ordinated convex functions.

Theorem 2.1. *Let $\alpha,\beta>0$ and $\rho,\sigma>0$. Let $f:\Delta^{\rho,\sigma}\subset\mathbb{R}^2\to\mathbb{R}$ be a co-ordinated convex on $\Delta^{\rho,\sigma}:=[a^\rho,b^\rho]\times[c^\sigma,d^\sigma]$ in $\mathbb{R}^2$ with $0\le a<b$, $0\le c<d$ and $f\in L_1(\Delta^{\rho,\sigma})$. Then the following inequalities hold:*

$$\begin{aligned}
&f\left(\frac{a^\rho+b^\rho}{2},\frac{c^\sigma+d^\sigma}{2}\right)\\
\le\ & \frac{\rho^\alpha\sigma^\beta\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b^\rho-a^\rho)^\alpha(d^\sigma-c^\sigma)^\beta}\\
&\times\left[{}^{\rho,\sigma}I^{\alpha,\beta}_{a^+,c^+}f(b^\rho,d^\sigma)+{}^{\rho,\sigma}I^{\alpha,\beta}_{a^+,d^-}f(b^\rho,c^\sigma)+{}^{\rho,\sigma}I^{\alpha,\beta}_{b^-,c^+}f(a^\rho,d^\sigma)+{}^{\rho,\sigma}I^{\alpha,\beta}_{b^-,d^-}f(a^\rho,c^\sigma)\right]\\
\le\ & \frac{f(a^\rho,c^\sigma)+f(a^\rho,d^\sigma)+f(b^\rho,c^\sigma)+f(b^\rho,d^\sigma)}{4}.
\end{aligned} \tag{8}$$

with $a < x < b$ and $c < y < d$.

Proof. According to (2) with $x^\rho = t^\rho a^\rho + (1-t^\rho)b^\rho$, $y^\rho = (1-t^\rho)a^\rho + t^\rho b^\rho$, $u^\sigma = s^\sigma c^\sigma + (1-s^\sigma)d^\sigma$, $w^\sigma = (1-s^\sigma)c^\sigma + s^\sigma d^\sigma$ and $t = s = \frac{1}{2}$, we find that

$$\begin{aligned} f\left(\frac{a^\rho+b^\rho}{2}, \frac{c^\sigma+d^\sigma}{2}\right) \leq\ & \frac{1}{4}[f(t^\rho a^\rho + (1-t^\rho)b^\rho, s^\sigma c^\sigma + (1-s^\sigma)d^\sigma) \\ & + f(t^\rho a^\rho + (1-t^\rho)b^\rho, (1-s^\sigma)c^\sigma + s^\sigma d^\sigma) \\ & + f((1-t^\rho)a^\rho + t^\rho b^\rho, s^\sigma c^\sigma + (1-s^\sigma)d^\sigma) \\ & + f((1-t^\rho)a^\rho + t^\rho b^\rho, (1-s^\sigma)c^\sigma + s^\sigma d^\sigma)]. \end{aligned} \tag{9}$$

Multiplying both sides of (9) by $t^{\alpha\rho-1}s^{\beta\sigma-1}$, then integrating with respect to (t, s) on $[0,1]\times[0,1]$, we obtain

$$\begin{aligned} & 4f\left(\frac{a^\rho+b^\rho}{2}, \frac{c^\sigma+d^\sigma}{2}\right)\int_0^1\int_0^1 t^{\alpha\rho-1}s^{\beta\sigma-1}dsdt \\ \leq\ & \int_0^1\int_0^1 t^{\alpha\rho-1}s^{\beta\sigma-1}f(t^\rho a^\rho + (1-t^\rho)b^\rho, s^\sigma c^\sigma + (1-s^\sigma)d^\sigma)dsdt \\ & + \int_0^1\int_0^1 t^{\alpha\rho-1}s^{\beta\sigma-1}f(t^\rho a^\rho + (1-t^\rho)b^\rho, (1-s^\sigma)c^\sigma + s^\sigma d^\sigma)dsdt \\ & + \int_0^1\int_0^1 t^{\alpha\rho-1}s^{\beta\sigma-1}f((1-t^\rho)a^\rho + t^\rho b^\rho, s^\sigma c^\sigma + (1-s^\sigma)d^\sigma)dsdt \\ & + \int_0^1\int_0^1 t^{\alpha\rho-1}s^{\beta\sigma-1}f((1-t^\rho)a^\rho + t^\rho b^\rho, (1-s^\sigma)c^\sigma + s^\sigma d^\sigma)dsdt. \end{aligned}$$

Using the change of variable in the last integrals, we have

$$\begin{aligned}
&4f\left(\frac{a^{\rho}+b^{\rho}}{2},\frac{c^{\sigma}+d^{\sigma}}{2}\right)\frac{1}{\alpha\rho}\frac{1}{\beta\sigma}\\
\leq\ & \frac{1}{(b^{\rho}-a^{\rho})^{\alpha}(d^{\sigma}-c^{\sigma})^{\beta}}\left[\int_a^b\int_c^d \frac{x^{\rho-1}}{(b^{\rho}-x^{\rho})^{1-\alpha}}\frac{y^{\sigma-1}}{(d^{\sigma}-y^{\sigma})^{1-\beta}}f(x^{\rho},y^{\sigma})\,dydx\right.\\
&+\int_a^b\int_c^d \frac{x^{\rho-1}}{(b^{\rho}-x^{\rho})^{1-\alpha}}\frac{y^{\sigma-1}}{(y^{\sigma}-c^{\sigma})^{1-\beta}}f(x^{\rho},y^{\sigma})\,dydx+\int_a^b\int_c^d \frac{x^{\rho-1}}{(x^{\rho}-a^{\rho})^{1-\alpha}}\frac{y^{\sigma-1}}{(d^{\sigma}-y^{\sigma})^{1-\beta}}f(x^{\rho},y^{\sigma})\,dydx\\
&\left.+\int_a^b\int_c^d \frac{x^{\rho-1}}{(x^{\rho}-a^{\rho})^{1-\alpha}}\frac{y^{\sigma-1}}{(y^{\sigma}-c^{\sigma})^{1-\beta}}f(x^{\rho},y^{\sigma})\,dydx\right].
\end{aligned}$$

which gives the left hand side inequality in (8). Now we prove the right hand side inequality in (8). For this purpose we first note that if f is a co-ordinated convex on Δ, then we can write by using (2)

$$\begin{aligned}
&f\left(t^{\rho}a^{\rho}+(1-t^{\rho})b^{\rho},s^{\sigma}c^{\sigma}+(1-s^{\sigma})d^{\sigma}\right)\\
\leq\ & t^{\rho}s^{\sigma}f(a^{\rho},c^{\sigma})+(1-t^{\rho})s^{\sigma}f(b^{\rho},c^{\sigma})+t^{\rho}(1-s^{\sigma})f(a^{\rho},d^{\sigma})+(1-t^{\rho})(1-s^{\sigma})f(b^{\rho},d^{\sigma}),
\end{aligned}$$

$$\begin{aligned}
&f(t^{\rho}a^{\rho}+(1-t^{\rho})b^{\rho},(1-s^{\sigma})c^{\sigma}+s^{\sigma}d^{\sigma})\\
\leq\ & t^{\rho}(1-s^{\sigma})f(a^{\rho},c^{\sigma})+(1-t^{\rho})(1-s^{\sigma})f(b^{\rho},c^{\sigma})+t^{\rho}s^{\sigma}f(a^{\rho},d^{\sigma})+(1-t^{\rho})s^{\sigma}f(b^{\rho},d^{\sigma}),
\end{aligned}$$

$$\begin{aligned}
&f((1-t^{\rho})a^{\rho}+t^{\rho}b^{\rho},s^{\sigma}c^{\sigma}+(1-s^{\sigma})d^{\sigma})\\
\leq\ & (1-t^{\rho})s^{\sigma}f(a^{\rho},c^{\sigma})+t^{\rho}s^{\sigma}f(b^{\rho},c^{\sigma})+(1-t^{\rho})(1-s^{\sigma})f(a^{\rho},d^{\sigma})+t^{\rho}(1-s^{\sigma})f(b^{\rho},d^{\sigma}),
\end{aligned}$$

and

$$\begin{aligned}
&f((1-t^{\rho})a^{\rho}+t^{\rho}b^{\rho},(1-s^{\sigma})c^{\sigma}+s^{\sigma}d^{\sigma})\\
\leq\ & (1-t^{\rho})(1-s^{\sigma})f(a^{\rho},c^{\sigma})+t^{\rho}(1-s^{\sigma})f(b^{\rho},c^{\sigma})+(1-t^{\rho})s^{\sigma}f(a^{\rho},d^{\sigma})+t^{\rho}s^{\sigma}f(b^{\rho},d^{\sigma}).
\end{aligned}$$

By adding these inequalities, we get

$$\begin{aligned}
&f\left(t^{\rho}a^{\rho}+(1-t^{\rho})b^{\rho},s^{\sigma}c^{\sigma}+(1-s^{\sigma})d^{\sigma}\right)+f(t^{\rho}a^{\rho}+(1-t^{\rho})b^{\rho},(1-s^{\sigma})c^{\sigma}+s^{\sigma}d^{\sigma})\\
&+f((1-t^{\rho})a^{\rho}+t^{\rho}b^{\rho},s^{\sigma}c^{\sigma}+(1-s^{\sigma})d^{\sigma})+f((1-t^{\rho})a^{\rho}+t^{\rho}b^{\rho},(1-s^{\sigma})c^{\sigma}+s^{\sigma}d^{\sigma}) \qquad (10)\\
\leq\ & f\left(a^{\rho},c^{\sigma}\right)+f\left(a^{\rho},d^{\sigma}\right)+f\left(b^{\rho},c^{\sigma}\right)+f\left(b^{\rho},d^{\sigma}\right).
\end{aligned}$$

Multiplying both sides of (10) by $t^{\alpha\rho-1}s^{\beta\sigma-1}$, then integrating with respect to (t,s) on $[0,1]\times[0,1]$ we obtain

$$
\begin{aligned}
&\int_0^1\int_0^1 t^{\alpha\rho-1}s^{\beta\sigma-1}f(t^\rho a^\rho+(1-t^\rho)b^\rho, s^\sigma c^\sigma+(1-s^\sigma)d^\sigma)dsdt\\
&+\int_0^1\int_0^1 t^{\alpha\rho-1}s^{\beta\sigma-1}f(t^\rho a^\rho+(1-t^\rho)b^\rho,(1-s^\sigma)c^\sigma+s^\sigma d^\sigma)dsdt\\
&+\int_0^1\int_0^1 t^{\alpha\rho-1}s^{\beta\sigma-1}f((1-t^\rho)a^\rho+t^\rho b^\rho, s^\sigma c^\sigma+(1-s^\sigma)d^\sigma)dsdt\\
&+\int_0^1\int_0^1 t^{\alpha\rho-1}s^{\beta\sigma-1}f((1-t^\rho)a^\rho+t^\rho b^\rho,(1-s^\sigma)c^\sigma+s^\sigma d^\sigma)dsdt\\
\le\ &\left[f\left(a^\rho,c^\sigma\right)+f\left(a^\rho,d^\sigma\right)+f\left(b^\rho,c^\sigma\right)+f\left(b^\rho,d^\sigma\right)\right]\int_0^1\int_0^1 t^{\alpha\rho-1}s^{\beta\sigma-1}dsdt.
\end{aligned}
$$

Then by using the change of variable we have

$$
\begin{aligned}
&\frac{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}{\rho^{1-\alpha}\sigma^{1-\beta}(b^\rho-a^\rho)^\alpha\left(d^\sigma-c^\sigma\right)^\beta}\\
&\times\left[{}^{\rho,\sigma}I^{\alpha,\beta}_{a^+,c^+}f\left(b^\rho,d^\sigma\right)+{}^{\rho,\sigma}I^{\alpha,\beta}_{a^+,d^-}f\left(b^\rho,c^\sigma\right)+{}^{\rho,\sigma}I^{\alpha,\beta}_{b^-,c^+}f\left(a^\rho,d^\sigma\right)+{}^{\rho,\sigma}I^{\alpha,\beta}_{b^-,d^-}f\left(a^\rho,c^\sigma\right)\right]\\
\le\ &\frac{1}{\alpha\rho}\frac{1}{\beta\sigma}\left[f\left(a^\rho,c^\sigma\right)+f\left(a^\rho,d^\sigma\right)+f\left(b^\rho,c^\sigma\right)+f\left(b^\rho,d^\sigma\right)\right].
\end{aligned}
$$

In this way the proof is completed. □

Remark 2.1. *If we set $\rho,\sigma=1$ in Theorem 2.1, then the inequalities (8) become the inequalities (5).*

Theorem 2.2. *Let $\alpha,\beta>0$ and $\rho,\sigma>0$. Let $f:\Delta^{\rho,\sigma}\subset\mathbb{R}^2\to\mathbb{R}$ be a co-ordinated convex on $\Delta^{\rho,\sigma}:=[a^\rho,b^\rho]\times[c^\sigma,d^\sigma]$ in $\mathbb{R}^2$ with $0\le a<b$, $0\le c<d$ and $f\in L_1(\Delta^{\rho,\sigma})$.*

Then the following inequalities hold:

$$
\begin{aligned}
& f\left(\frac{a^{\rho}+b^{\rho}}{2},\frac{c^{\sigma}+d^{\sigma}}{2}\right) \\
\leq\; & \frac{\rho^{\alpha}\Gamma(\alpha+1)}{4(b^{\rho}-a^{\rho})^{\alpha}}\left[{}^{\rho,\sigma}I_{a+}^{\alpha}f\left(b,\frac{c^{\sigma}+d^{\sigma}}{2}\right)+{}^{\rho,\sigma}I_{b-}^{\alpha}f\left(a,\frac{c^{\sigma}+d^{\sigma}}{2}\right)\right] \\
& +\frac{\sigma^{\beta}\Gamma(\beta+1)}{4(d^{\sigma}-c^{\sigma})^{\beta}}\left[{}^{\rho,\sigma}I_{c+}^{\beta}f\left(\frac{a^{\rho}+b^{\rho}}{2},d\right)+{}^{\rho,\sigma}I_{d-}^{\beta}f\left(\frac{a^{\rho}+b^{\rho}}{2},c\right)\right] \\
\leq\; & \frac{\rho^{\alpha}\sigma^{\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b^{\rho}-a^{\rho})^{\alpha}(d^{\sigma}-c^{\sigma})^{\beta}} \\
& \times\left[{}^{\rho,\sigma}I_{a^{+},c^{+}}^{\alpha,\beta}f(b^{\rho},d^{\sigma})+{}^{\rho,\sigma}I_{a^{+},d^{-}}^{\alpha,\beta}f(b^{\rho},c^{\sigma})+{}^{\rho,\sigma}I_{b^{-},c^{+}}^{\alpha,\beta}f(a^{\rho},d^{\sigma})+{}^{\rho,\sigma}I_{b^{-},d^{-}}^{\alpha,\beta}f(a^{\rho},c^{\sigma})\right] \\
\leq\; & \frac{\rho^{\alpha}\Gamma(\alpha+1)}{8(b^{\rho}-a^{\rho})^{\alpha}}\left[{}^{\rho,\sigma}I_{a+}^{\alpha}f(b,c^{\sigma})+{}^{\rho,\sigma}I_{a+}^{\alpha}f(b,d^{\sigma})+{}^{\rho,\sigma}I_{b-}^{\alpha}f(a,c^{\sigma})+{}^{\rho,\sigma}I_{b-}^{\alpha}f(a,d^{\sigma})\right] \\
& +\frac{\sigma^{\beta}\Gamma(\beta+1)}{8(d^{\sigma}-c^{\sigma})^{\beta}}\left[{}^{\rho,\sigma}I_{c+}^{\beta}f(a^{\rho},d)+{}^{\rho,\sigma}I_{d-}^{\beta}f(a^{\rho},c)+{}^{\rho,\sigma}I_{c+}^{\beta}f(b^{\rho},d)+{}^{\rho,\sigma}I_{d-}^{\beta}f(b^{\rho},c)\right] \\
\leq\; & \frac{f(a^{\rho},c^{\sigma})+f(a^{\rho},d^{\sigma})+f(b^{\rho},c^{\sigma})+f(b^{\rho},d^{\sigma})}{4}.
\end{aligned} \tag{11}
$$

with $a<x<b$ and $c<y<d$.

Proof. Since $f:\Delta^{\rho,\sigma}\to\mathbb{R}$ is co-ordinated convex on $\Delta^{\rho}:=[a^{\rho},b^{\rho}]\times[c^{\sigma},d^{\sigma}]$ in $\mathbb{R}^2$ with $0\leq a<b$, $0\leq c<d$, it follows that the mapping $g_x:[c,d]\to\mathbb{R}$, $g_x(y)=f(x,y)$, is convex on $[c,d]$ for all $x\in[a,b]$. Then by using inequalities ([3]), we can write

$$
\begin{aligned}
g_x\left(\frac{c^{\sigma}+d^{\sigma}}{2}\right) &\leq \frac{\sigma^{\beta}\Gamma(\beta+1)}{2(d^{\sigma}-c^{\sigma})^{\beta}}\left[{}^{\rho,\sigma}I_{c+}^{\beta}g_x(d^{\sigma})+{}^{\rho,\sigma}I_{d-}^{\beta}g_x(c^{\sigma})\right] \\
&\leq \frac{g_x(c^{\sigma})+g_x(d^{\sigma})}{2},\ x\in[a,b].
\end{aligned}
$$

That is,

$$
\begin{aligned}
& f\left(x,\frac{c^{\sigma}+d^{\sigma}}{2}\right) \\
\leq\; & \frac{\rho\beta}{2(d^{\sigma}-c^{\sigma})^{\beta}}\left[\int_c^d\frac{y^{\sigma-1}}{(d^{\sigma}-y^{\sigma})^{1-\beta}}f(x^{\rho},y^{\sigma})\,dy+\int_c^d\frac{y^{\sigma-1}}{(y^{\sigma}-c^{\sigma})^{1-\beta}}f(x^{\rho},y^{\sigma})\,dy\right] \\
\leq\; & \frac{f(x,c^{\sigma})+f(x,d^{\sigma})}{2},\ x\in[a,b].
\end{aligned} \tag{12}
$$

Then multiplying both sides of (12) by $\frac{\rho\alpha}{2(b^\rho-a^\rho)^\alpha}\frac{x^{\rho-1}}{(b^\rho-x^\rho)^{1-\alpha}}$ and $\frac{\rho\alpha}{2(b^\rho-a^\rho)^\alpha}\frac{x^{\rho-1}}{(x^\rho-a^\rho)^{1-\alpha}}$, integrating with respect to x over $[a,b]$, respectively, we get

$$\begin{aligned}
&\frac{\rho\alpha}{2(b^\rho-a^\rho)^\alpha}\int_a^b \frac{x^{\rho-1}}{(b^\rho-x^\rho)^{1-\alpha}} f\left(x,\frac{c^\sigma+d^\sigma}{2}\right)dx \\
\leq\;& \frac{\rho\alpha}{2(b^\rho-a^\rho)^\alpha}\frac{\sigma\beta}{2(d^\sigma-c^\sigma)^\beta}\left[\int_a^b\int_c^d \frac{x^{\rho-1}}{(b^\rho-x^\rho)^{1-\alpha}}\frac{y^{\sigma-1}}{(d^\sigma-y^\sigma)^{1-\beta}} f(x^\rho,y^\sigma)\,dydx\right. \\
&\left.+\int_a^b\int_c^d \frac{x^{\rho-1}}{(b^\rho-x^\rho)^{1-\alpha}}\frac{y^{\sigma-1}}{(y^\sigma-c^\sigma)^{1-\beta}} f(x^\rho,y^\sigma)\,dydx\right] \\
\leq\;& \frac{\rho\alpha}{2(b^\rho-a^\rho)^\alpha}\left[\int_a^b \frac{x^{\rho-1}}{(b^\rho-x^\rho)^{1-\alpha}} f(x,c^\sigma)dx+\int_a^b \frac{x^{\rho-1}}{(b^\rho-x^\rho)^{1-\alpha}} f(x,d^\sigma)dx\right],
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
&\frac{\rho\alpha}{2(b^\rho-a^\rho)^\alpha}\int_a^b \frac{x^{\rho-1}}{(x^\rho-a^\rho)^{1-\alpha}} f\left(x,\frac{c^\sigma+d^\sigma}{2}\right)dx \\
\leq\;& \frac{\rho\alpha}{2(b^\rho-a^\rho)^\alpha}\frac{\sigma\beta}{2(d^\sigma-c^\sigma)^\beta}\left[\int_a^b\int_c^d \frac{x^{\rho-1}}{(x^\rho-a^\rho)^{1-\alpha}}\frac{y^{\sigma-1}}{(d^\sigma-y^\sigma)^{1-\beta}} f(x^\rho,y^\sigma)\,dydx\right. \\
&\left.+\int_a^b\int_c^d \frac{x^{\rho-1}}{(x^\rho-a^\rho)^{1-\alpha}}\frac{y^{\sigma-1}}{(y^\sigma-c^\sigma)^{1-\beta}} f(x^\rho,y^\sigma)\,dydx\right] \\
\leq\;& \frac{\rho\alpha}{2(b^\rho-a^\rho)^\alpha}\left[\int_a^b \frac{x^{\rho-1}}{(x^\rho-a^\rho)^{1-\alpha}} f(x,c^\sigma)dx+\int_a^b \frac{x^{\rho-1}}{(x^\rho-a^\rho)^{1-\alpha}} f(x,d^\sigma)dx\right].
\end{aligned} \tag{14}$$

By similar argument applied for the mapping $g_y : [a,b] \to \mathbb{R}$, $g_y(x) = f(x,y)$, we have

$$\begin{aligned}
&\frac{\sigma\beta}{2(d^\sigma - c^\sigma)^\beta} \int\limits_c^d \frac{y^{\sigma-1}}{(y^\sigma - c^\sigma)^{1-\beta}} f\left(\frac{a^\rho + b^\rho}{2}, y\right) dy \\
\leq\ & \frac{\rho\alpha}{2(b^\rho - a^\rho)^\alpha} \frac{\sigma\beta}{2(d^\rho - c^\rho)^\beta} \left[\int\limits_a^b \int\limits_c^d \frac{x^{\rho-1}}{(x^\rho - a^\rho)^{1-\alpha}} \frac{y^{\sigma-1}}{(y^\sigma - c^\sigma)^{1-\beta}} f(x^\rho, y^\sigma)\, dydx \right. \\
& \left. + \int\limits_a^b \int\limits_c^d \frac{x^{\rho-1}}{(b^\rho - x^\rho)^{1-\alpha}} \frac{y^{\sigma-1}}{(y^\sigma - c^\sigma)^{1-\beta}} f(x^\rho, y^\sigma)\, dydx \right] \\
\leq\ & \frac{\sigma\beta}{2(d^\sigma - c^\sigma)^\beta} \left[\int\limits_c^d \frac{y^{\sigma-1}}{(y^\sigma - c^\sigma)^{1-\beta}} f(a^\rho, y) dy + \int\limits_c^d \frac{y^{\sigma-1}}{(y^\sigma - c^\sigma)^{1-\beta}} f(b^\rho, y) dy \right],
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
&\frac{\sigma\beta}{2(d^\sigma - c^\sigma)^\beta} \int\limits_c^d \frac{y^{\sigma-1}}{(d^\sigma - y^\sigma)^{1-\beta}} f\left(\frac{a^\rho + b^\rho}{2}, y\right) dy \\
\leq\ & \frac{\rho\alpha}{2(b^\rho - a^\rho)^\alpha} \frac{\sigma\beta}{2(d^\sigma - c^\sigma)^\beta} \left[\int\limits_a^b \int\limits_c^d \frac{x^{\rho-1}}{(x^\rho - a^\rho)^{1-\alpha}} \frac{y^{\sigma-1}}{(d^\sigma - y^\sigma)^{1-\beta}} f(x^\rho, y^\sigma)\, dydx \right. \\
& \left. + \int\limits_a^b \int\limits_c^d \frac{x^{\rho-1}}{(b^\rho - x^\rho)^{1-\alpha}} \frac{y^{\sigma-1}}{(d^\sigma - y^\sigma)^{1-\beta}} f(x^\rho, y^\sigma)\, dydx \right] \\
\leq\ & \frac{\sigma\beta}{2(d^\sigma - c^\sigma)^\beta} \left[\int\limits_c^d \frac{y^{\sigma-1}}{(d^\sigma - y^\sigma)^{1-\beta}} f(a^\rho, y) dy + \int\limits_c^d \frac{y^{\sigma-1}}{(d^\sigma - y^\sigma)^{1-\beta}} f(b^\rho, y) dy \right].
\end{aligned} \tag{16}$$

Adding the inequalities (13)-(16), we obtain

$$\begin{aligned}
&\frac{\rho^\alpha \Gamma(\alpha+1)}{4(b^\rho - a^\rho)^\alpha} \left[{}^{\rho,\sigma}I^\alpha_{a^+} f\left(b, \frac{c^\sigma + d^\sigma}{2}\right) + {}^{\rho,\sigma} I^\alpha_{b^-} f\left(a, \frac{c^\sigma + d^\sigma}{2}\right) \right] \\
&+ \frac{\sigma^\beta \Gamma(\beta+1)}{4(d^\sigma - c^\sigma)^\beta} \left[{}^{\rho,\sigma}I^\beta_{c^+} f\left(\frac{a^\rho + b^\rho}{2}, d\right) + {}^{\rho,\sigma} I^\beta_{d^-} f\left(\frac{a^\rho + b^\rho}{2}, c\right) \right] \\
\leq\ & \frac{\rho^\alpha \sigma^\beta \Gamma(\alpha+1)\Gamma(\beta+1)}{4(b^\rho - a^\rho)^\alpha (d^\sigma - c^\sigma)^\beta} \\
&\times \left[{}^{\rho,\sigma}I^{\alpha,\beta}_{a^+,c^+} f(b^\rho, d^\sigma) + {}^{\rho,\sigma} I^{\alpha,\beta}_{a^+,d^-} f(b^\rho, c^\sigma) + {}^{\rho,\sigma} I^{\alpha,\beta}_{b^-,c^+} f(a^\rho, d^\sigma) + {}^{\rho,\sigma} I^{\alpha,\beta}_{b^-,d^-} f(a^\rho, c^\sigma) \right]
\end{aligned}$$

$$\leq \frac{\rho^{\alpha}\Gamma(\alpha+1)}{8(b^{\rho}-a^{\rho})^{\alpha}}\left[{}^{\rho,\sigma}I_{a+}^{\alpha}f(b,c^{\sigma})+{}^{\rho,\sigma}I_{a+}^{\alpha}f(b,d^{\sigma})+{}^{\rho,\sigma}I_{b-}^{\alpha}f(a,c^{\sigma})+{}^{\rho,\sigma}I_{b-}^{\alpha}f(a,d^{\sigma})\right]$$

$$+\frac{\sigma^{\beta}\Gamma(\beta+1)}{8(d^{\sigma}-c^{\sigma})^{\beta}}\left[{}^{\rho,\sigma}I_{c+}^{\beta}f(a^{\rho},d)+{}^{\rho,\sigma}I_{c+}^{\beta}f(b^{\rho},d)+{}^{\rho,\sigma}I_{d-}^{\beta}f(a^{\rho},c)+{}^{\rho,\sigma}I_{d-}^{\beta}f(b^{\rho},c)\right].$$

Thus, we proved the second and the third inequalities in (11).

Now, using the left side inequality in (7), we also have

$$f\left(\frac{a^{\rho}+b^{\rho}}{2},\frac{c^{\sigma}+d^{\sigma}}{2}\right)\leq\frac{\rho\alpha}{2(b^{\rho}-a^{\rho})^{\alpha}}$$

$$\times\left[\int_a^b\frac{x^{\rho-1}}{(x^{\rho}-a^{\rho})^{1-\alpha}}f\left(x,\frac{c^{\sigma}+d^{\sigma}}{2}\right)dx+\int_a^b\frac{x^{\rho-1}}{(b^{\rho}-x^{\rho})^{1-\alpha}}f\left(x,\frac{c^{\sigma}+d^{\sigma}}{2}\right)dx\right]$$

and

$$f\left(\frac{a^{\rho}+b^{\rho}}{2},\frac{c^{\sigma}+d^{\sigma}}{2}\right)\leq\frac{\sigma\beta}{2(d^{\sigma}-c^{\sigma})^{\beta}}$$

$$\times\left[\int_c^d\frac{y^{\sigma-1}}{(y^{\sigma}-c^{\sigma})^{1-\beta}}f\left(\frac{a^{\rho}+b^{\rho}}{2},y\right)dy+\int_c^d\frac{y^{\sigma-1}}{(d^{\sigma}-y^{\sigma})^{1-\beta}}f\left(\frac{a^{\rho}+b^{\rho}}{2},y\right)dy\right].$$

By adding these inequalities, we get

$$f\left(\frac{a^{\rho}+b^{\rho}}{2},\frac{c^{\sigma}+d^{\sigma}}{2}\right)\leq\frac{\rho^{\alpha}\Gamma(\alpha+1)}{4(b^{\rho}-a^{\rho})^{\alpha}}\left[{}^{\rho,\sigma}I_{a+}^{\alpha}f\left(b,\frac{c^{\sigma}+d^{\sigma}}{2}\right)+{}^{\rho,\sigma}I_{b-}^{\alpha}f\left(a,\frac{c^{\sigma}+d^{\sigma}}{2}\right)\right]$$

$$+\frac{\sigma^{\beta}\Gamma(\beta+1)}{4(d^{\sigma}-c^{\sigma})^{\beta}}\left[{}^{\rho,\sigma}I_{c+}^{\beta}f\left(\frac{a^{\rho}+b^{\rho}}{2},d\right)+{}^{\rho,\sigma}I_{d-}^{\beta}f\left(\frac{a^{\rho}+b^{\rho}}{2},c\right)\right]$$

which gives the first inequality in (11).

Finally, using the right-hand side inequality in (7), we can state

$$\frac{\rho\alpha}{2(b^{\rho}-a^{\rho})^{\alpha}}\left[\int_a^b\frac{x^{\rho-1}}{(b^{\rho}-x^{\rho})^{1-\alpha}}f(x,c^{\sigma})dx+\int_a^b\frac{x^{\rho-1}}{(x^{\rho}-a^{\rho})^{1-\alpha}}f(x,c^{\sigma})dx\right]\leq\frac{f(a^{\rho},c^{\sigma})+f(b^{\rho},c^{\sigma})}{2}, \tag{17}$$

$$\frac{\rho\alpha}{2(b^{\rho}-a^{\rho})^{\alpha}}\left[\int_a^b\frac{x^{\rho-1}}{(b^{\rho}-x^{\rho})^{1-\alpha}}f(x,d^{\sigma})dx+\int_a^b\frac{x^{\rho-1}}{(x^{\rho}-a^{\rho})^{1-\alpha}}f(x,d^{\sigma})dx\right]\leq\frac{f(a^{\rho},d^{\sigma})+f(b^{\rho},d^{\sigma})}{2}, \tag{18}$$

$$\frac{\sigma\beta}{2(d^{\sigma}-c^{\sigma})^{\beta}}\left[\int_c^d\frac{y^{\sigma-1}}{(d^{\sigma}-y^{\sigma})^{1-\beta}}f(a^{\rho},y)dy+\int_c^d\frac{y^{\sigma-1}}{(y^{\sigma}-c^{\sigma})^{1-\beta}}f(a^{\rho},y)dy\right]\leq\frac{f(a^{\rho},c^{\sigma})+f(a^{\rho},d^{\sigma})}{2} \tag{19}$$

and

$$\frac{\sigma\beta}{2(d^\sigma - c^\sigma)^\beta}\left[\int_c^d \frac{y^{\sigma-1}}{(d^\sigma - y^\sigma)^{1-\beta}} f(b^\rho, y)dy + \int_c^d \frac{y^{\sigma-1}}{(y^\sigma - c^\sigma)^{1-\beta}} f(b^\rho, y)dy\right] \leq \frac{f(b^\rho, c^\sigma) + f(b^\rho, d^\sigma)}{2}. \tag{20}$$

which give, by addition (17)-(20), the last inequality in (11). □

Remark 2.2. *If we set $\rho, \sigma = 1$ in Theorem 2.2, then the inequalities (11) become the inequalities (6).*

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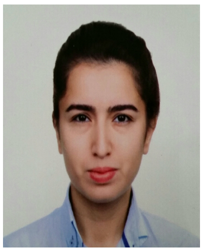

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