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APPROXIMATION BY STANCU TYPE JAKIMOVSKI-LEVIATAN-PĂLTĂNEA OPERATORS

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Abstract. The present article deals with the general family of summation-integral type operators. Here, we introduce the Stancu type generalization of the Jakimovski-Leviatan-Păltănea operators and study Voronovskaja-type asymptotic theorem, local approximation, weighted approximation, rate of convergence and pointwise estimates using the Lipschitz type maximal function. Also, we propose a king type modification of these operators to obtain better estimates.

Keywords: Voronovskaja-type theorem, K-functional, Appell polynomials, rate of convergence, modulus of continuity, weighted approximation.

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1. Introduction

In approximation theory, the positive approximation processes discovered by Korovkin play a central role and arise in a natural way in many problems connected with functional analysis, harmonic analysis, measure theory, partial differential equations and probability theory. The most useful examples of such operators are Szász [33] operators.
Szász [33] defined the positive linear operators:

$$S_n(f,x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right),$$

where $x \in [0,\infty)$ and $f \in C[0,\infty)$ whenever the above sum converges. Motivated by this work, many researchers have investigated several important properties of the above operators.
Later, Jakimovski and Leviatan [9] gave a generalization of Szász operators by using the Appell polynomials. Let $g(z) = \sum\limits_{k=0}^{\infty} a_k z^k (a_0 \neq 0)$ be an analytic function in the disk

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$|z| < R, (R > 1)$ and $g(1) \neq 0$. It is well known that the Appell polynomials $p_k(x)$ are defined by the following generating functions

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k, \tag{1}$$

with the condition that $p_k(x) \geq 0$ for every $x \in [0, \infty)$. Jakimovski and Leviatan [9] introduced the following positive linear operators:

$$P_n(f, x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right).$$

For the special case $g(z) = 1$ the operators P_n reduced to Szász operators. Karaisa [10] gave Durrmeyer type generalization of these operators and investigate different approximation properties. In order to modify the Phillips operators based on parameter $\rho > 0$ Păltănea [29] proposed the generalization of the well known Phillips operators [30], which provide the link with Szász operators as $\rho \to \infty$.

For $f \in C[0, \infty)$, Verma and Gupta [34] introduced the Jakimovski-Leviatan-Păltănea operators:

$$M_{n,\rho}(f, x) = \sum_{k=1}^{\infty} l_{n,k}(x) \int_0^{\infty} \Theta_{n,k}^{\rho}(x) f(t) dt + l_{n,0}(x) f(0), \tag{2}$$

where $l_{n,k}(x) = \frac{e^{-nx}}{g(1)} p_k(nx)$ and $\Theta_{n,k}^{\rho}(x) = \dfrac{n\rho}{\Gamma(k\rho)} e^{-n\rho t} (n\rho t)^{k\rho - 1}$.

For $g(z) = 1$ and $\rho = 1$ the operators (2) reduced to Phillips operators. In [34] Verma and Gupta studied some approximation properties and asymptotic formula for the operators $M_{n,\rho}$. Very recently, Goyal and Agrawal [8] studied direct approximation theorem and rate of convergence for the functions having a derivative of bounded variation for these operators.

In the recent years, Stancu type generalization of the certain operators introduced by several researchers and obtained different type of approximation properties of many operators, we refer some of the important papers in this direction as [1], [11], [15], [17], [21], [32], [35] etc.

Inspired by the above work, We introduce the Stancu type generalization of the Jakimovski-Leviatan-Păltănea operators as follows:

$$M_{n,\rho}^{\alpha,\beta}(f, x) = \sum_{k=1}^{\infty} l_{n,k}(x) \int_0^{\infty} \Theta_{n,k}^{\rho}(x) f\left(\frac{nt + \alpha}{n + \beta}\right) dt + l_{n,0}(x) f\left(\frac{\alpha}{n + \beta}\right). \tag{3}$$

Taking $\alpha = \beta = 0$ in (3), we get the Jakimovski-Leviatan-Păltănea operators (2).

The aim of this paper is to study the basic convergence theorem, Voronovskaja-type asymptotic formula, rate of convergence, weighted approximation and pointwise estimation of the operators (3). Further, to obtain better approximation, we also propose modification of the operators (3) using King type approach.

2. Auxiliary results

In this section we collect some results about the operators $M_{n,\rho}^{\alpha,\beta}$ useful in the sequel. Let $e_i(t) = t^i,\ i = 0, 1, 2$.

Lemma 2.1. [34] *For $M_{n,\rho}(t^m, x), m = 0, 1, 2,$ we have*

(1) $M_{n,\rho}(e_0, x) = 1$;

(2) $M_{n,\rho}(e_1,x) = x + \frac{g'(1)}{ng(1)}$;
(3) $M_{n,\rho}(e_2,x) = \frac{\rho n^2x^2+nx(1+\rho)}{n^2\rho} + \frac{\rho(2nx+1)+1}{n^2\rho}\frac{g'(1)}{g(1)} + \frac{g''(1)}{n^2g(1)}$.

Lemma 2.2. *For the operators $M_{n,\rho}^{\alpha,\beta}(f,x)$ as defined in (3), the following equalities holds*

(1) $M_{n,\rho}^{\alpha,\beta}(e_0,x) = 1$;
(2) $M_{n,\rho}^{\alpha,\beta}(e_1,x) = \frac{nx+\alpha}{n+\beta} + \frac{g'(1)}{(n+\beta)g(1)}$;
(3) $M_{n,\rho}^{\alpha,\beta}(e_2,x) = \frac{\rho n^2x^2+nx(1+\rho+2\alpha\rho)}{\rho(n+\beta)^2} + \frac{\rho(2nx+1)+1}{\rho(n+\beta)^2}\frac{g'(1)}{g(1)} + \frac{g''(1)+2\alpha g'(1)+\alpha^2g(1)}{(n+\beta)^2g(1)}$.

Proof. For $x \in [0,\infty)$, in view of Lemma 2.1, we have $M_{n,\rho}^{\alpha,\beta}(e_0,x) = 1$.
The first order moment is given by

$$M_{n,\rho}^{\alpha,\beta}(e_1,x) = \frac{n}{n+\beta}M_{n,\rho}(e_1,x) + \frac{\alpha}{n+\beta} = \frac{nx+\alpha}{n+\beta} + \frac{g'(1)}{(n+\beta)g(1)}.$$

The second order moment is given by

$$\begin{aligned} M_{n,\rho}^{\alpha,\beta}(e_2,x) &= \left(\frac{n}{n+\beta}\right)^2 M_{n,\rho}(e_2,x) + \frac{2n\alpha}{(n+\beta)^2}M_{n,\rho}(e_1,x) + \left(\frac{\alpha}{n+\beta}\right)^2 \\ &= \frac{\rho n^2x^2+nx(1+\rho+2\alpha\rho)}{\rho(n+\beta)^2} + \frac{\rho(2nx+1)+1}{\rho(n+\beta)^2}\frac{g'(1)}{g(1)} + \frac{g''(1)+2\alpha g'(1)+\alpha^2g(1)}{(n+\beta)^2g(1)}. \end{aligned}$$

□

Lemma 2.3. *For $f \in C_B[0,\infty)$ (space of all real valued bounded and uniformly continuous functions on $[0,\infty)$ endowed with the norm $\| f \| = \sup\{|f(x)| : x \in [0,\infty)\}$), $\| M_{n,\rho}^{\alpha,\beta}(f) \| \leq \| f \|$.*

Proof. In view of (3) and Lemma 2.2, the proof of this lemma easily follows. □

Remark 2.1. *From Lemma 2.2 it follows*

$$M_{n,\rho}^{\alpha,\beta}(t-x,x) = \frac{\alpha-\beta x}{n+\beta} + \frac{g'(1)}{(n+\beta)g(1)}$$

and

$$\begin{aligned} M_{n,\rho}^{\alpha,\beta}\left((t-x)^2,x\right) &= \frac{\beta^2x^2}{(n+\beta)^2} + \left\{\frac{n(1+\rho)-2\rho\alpha\beta}{\rho(n+\beta)^2} - \frac{2\beta}{(n+\beta)^2}\frac{g'(1)}{g(1)}\right\}x \\ &\quad + \frac{\rho(g''(1)+2\alpha g'(1)+\alpha^2g(1))+(\rho+1)g'(1)}{\rho(n+\beta)^2g(1)}. \end{aligned}$$

3. Main Results

In this section we establish some approximation properties in several settings. For the reader's convenience we split up this section in more subsections.

Theorem 3.1. *Let $f \in C[0,\infty)$. Then $\lim\limits_{n\to\infty} M_{n,\rho}^{\alpha,\beta}(f,x) = f(x)$, uniformly in each compact subset of $[0,\infty)$.*

Proof. In view of Lemma 2.2, we get $\lim_{n\to\infty} M_{n,\rho}^{\alpha,\beta}(e_i,x) = x^i$, $i = 0,1,2$, uniformly in each compact subset of $[0,\infty)$. Applying Bohman-Korovkin Theorem, it follows that $\lim\limits_{n\to\infty} M_{n,\rho}^{\alpha,\beta}(f,x) = f(x)$, uniformly in each compact subset of $[0,\infty)$. □

3.1. **Voronovskaja-type theorem.** In this section we prove Voronvoskaja-type asymptotic theorem for the operators $M_{n,\rho}^{\alpha,\beta}$.

Theorem 3.2. *Let $f \in C_B[0,\infty)$. If f', f'' exists at a fixed point $x \in [0,\infty)$, then we have*

$$\lim_{n\to\infty} n\left(M_{n,\rho}^{\alpha,\beta}(f,x) - f(x)\right) = \left((\alpha-\beta x) + \frac{g'(1)}{g(1)}\right) f'(x) + \frac{x}{2}\left(1+\frac{1}{\rho}\right) f''(x).$$

Proof. Let $x \in [0,\infty)$ be fixed. By Taylor's expansion of f, we can write

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + r(t,x)(t-x)^2, \tag{4}$$

where $r(t,x)$ is the Peano form of remainder, $r(t,x) \in C_B[0,\infty)$ and $\lim_{t\to x} r(t,x) = 0$.
Applying $M_{n,\rho}^{\alpha,\beta}$ on both sides of (4), we have

$$\begin{aligned} n\left(M_{n,\rho}^{\alpha,\beta}(f,x) - f(x)\right) &= nf'(x)M_{n,\rho}^{\alpha,\beta}(t-x,x) + \frac{f''(x)}{2!} nM_{n,\rho}^{\alpha,\beta}\left((t-x)^2,x\right) \\ &\quad + nM_{n,\rho}^{\alpha,\beta}\left(r(t,x)(t-x)^2,x\right). \end{aligned}$$

In view of Remark 2.1, we have

$$\lim_{n\to\infty} nM_{n,\rho}^{\alpha,\beta}(t-x,x) = \left((\alpha-\beta x) + \frac{g'(1)}{g(1)}\right) \tag{5}$$

and

$$\lim_{n\to\infty} nM_{n,\rho}^{\alpha,\beta}\left((t-x)^2,x\right) = x\left(1+\frac{1}{\rho}\right). \tag{6}$$

Now, we shall show that

$$\lim_{n\to\infty} nM_{n,\rho}^{\alpha,\beta}\left(r(t,x)(t-x)^2,x\right) = 0.$$

By using Cauchy-Schwarz inequality, we have

$$M_{n,\rho}^{\alpha,\beta}\left(r(t,x)(t-x)^2,x\right) \le \left(M_{n,\rho}^{\alpha,\beta}(r^2(t,x),x)\right)^{1/2}\left(M_{n,\rho}^{\alpha,\beta}((t-x)^4,x)\right)^{1/2}. \tag{7}$$

We observe that $r^2(x,x) = 0$ and $r^2(.,x) \in C_B[0,\infty)$. Then, it follows that

$$\lim_{n\to\infty} M_{n,\rho}^{\alpha,\beta}(r^2(t,x),x) = r^2(x,x) = 0, \tag{8}$$

in view of fact that $M_{n,\rho}^{\alpha,\beta}((t-x)^4,x) = O\left(\frac{1}{n^2}\right)$.
Now, from (7) and (8) we obtain

$$\lim_{n\to\infty} nM_{n,\rho}^{\alpha,\beta}\left(r(t,x)(t-x)^2,x\right) = 0. \tag{9}$$

From (5), (6) and (9), we get the required result. □

3.2. **Local approximation.** For $C_B[0,\infty)$, let us consider the following K-functional:

$$K_2(f,\delta) = \inf_{g\in W^2}\{\| f-g \| + \delta \| g'' \|\},$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By, p. 177, Theorem 2.4 in [2], there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \le C\omega_2(f,\sqrt{\delta}), \tag{10}$$

where $\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x\in[0,\infty)} \mid f(x+2h)-2f(x+h)+f(x) \mid$ is the second order modulus of smoothness of f. By $\omega_1(f,\delta) = \sup_{0<h\leq\delta} \sup_{x\in[0,\infty)} \mid f(x+h)-f(x) \mid$ we denote the first order modulus of continuity of $f \in C_B[0,\infty)$.

Theorem 3.3. *For $f \in C_B[0,\infty)$, we have*

$$\mid M_{n,\rho}^{\alpha,\beta}(f,x)-f(x) \mid \quad \leq \quad C\omega_2(f,\delta)+\omega_1\left(f,\left|\frac{\alpha-\beta x}{n+\beta}+\frac{g'(1)}{(n+\beta)g(1)}\right|\right),$$

where

$$\delta \quad = \quad \sqrt{M_{n,\rho}^{\alpha,\beta}((t-x)^2,x)+\left(\frac{\alpha-\beta x}{n+\beta}+\frac{g'(1)}{(n+\beta)g(1)}\right)^2}.$$

Proof. For $x \in [0,\infty)$, we consider the auxiliary operators $\overline{M}_{n,\rho}^{\alpha,\beta}$ as follows:

$$\overline{M}_{n,\rho}^{\alpha,\beta}(f;x) \quad = \quad M_{n,\rho}^{\alpha,\beta}(f,x)-f\left(\frac{nx+\alpha}{n+\beta}+\frac{g'(1)}{(n+\beta)g(1)}\right)+f(x). \tag{11}$$

From Lemma 2.2, we observe that the operators $\overline{M}_{n,\rho}^{\alpha,\beta}$ are linear and preserve the linear functions.
Hence

$$\overline{M}_{n,\rho}^{\alpha,\beta}(t-x,x) \quad = \quad 0. \tag{12}$$

Let $h \in W^2$ and $x,t \in [0,\infty)$. By Taylor's expansion we have

$$h(t) = h(x)+(t-x)h'(x)+\int_x^t (t-v)h''(v)dv.$$

Applying $\overline{M}_{n,\rho}^{\alpha,\beta}$ on both sides of the above equation and using (12), we get

$$\overline{M}_{n,\rho}^{\alpha,\beta}(h,x)-h(x) \quad = \quad \overline{M}_{n,\rho}^{\alpha,\beta}\left(\int_x^t (t-v)h''(v)dv,x\right).$$

Thus, by (11) we get

$$\begin{aligned}
|\overline{M}_{n,\rho}^{\alpha,\beta}(h,x)-h(x)| \quad &\leq \quad M_{n,\rho}^{\alpha,\beta}\left(\left|\int_x^t (t-v)h''(v)dv\right|,x\right) \\
&\quad +\left|\int_x^{\frac{nx+\alpha}{n+\beta}+\frac{g'(1)}{(n+\beta)g(1)}} \left(\frac{nx+\alpha}{n+\beta}+\frac{g'(1)}{(n+\beta)g(1)}-v\right)h''(v)dv\right| \\
&\leq \quad \left(M_{n,\rho}^{\alpha,\beta}((t-x)^2,x)+\left(\frac{\alpha-\beta x}{n+\beta}+\frac{g'(1)}{(n+\beta)g(1)}\right)^2\right)\parallel h'' \parallel \\
&\leq \quad \delta^2 \parallel h'' \parallel .
\end{aligned}$$

Since $|\overline{M}_{n,\rho}^{\alpha,\beta}(f,x)| \leq\parallel f \parallel$, it follows

$$\begin{aligned}
|M_{n,\rho}^{\alpha,\beta}(f,x)-f(x)| \quad &\leq \quad |\overline{M}_{n,\rho}^{\alpha,\beta}(f-h,x)|+|(f-h)(x)| \\
&\quad +|\overline{M}_{n,\rho}^{\alpha,\beta}(h,x)-h(x)|+\left|f\left(\frac{nx+\alpha}{n+\beta}+\frac{g'(1)}{(n+\beta)g(1)}\right)-f(x)\right| \\
&\leq \quad \parallel f-h \parallel +\delta^2 \parallel h'' \parallel +\left|f\left(\frac{nx+\alpha}{n+\beta}+\frac{g'(1)}{(n+\beta)g(1)}\right)-f(x)\right|.
\end{aligned}$$

Taking infimum over all $h \in W^2$, we get

$$| M_{n,\rho}^{\alpha,\beta}(f,x) - f(x) | \leq K_2\left(f,\delta^2\right) + \omega_1\left(f, \left|\frac{\alpha-\beta x}{n+\beta} + \frac{g'(1)}{(n+\beta)g(1)}\right|\right).$$

In view of (10), we get

$$| M_{n,\rho}^{\alpha,\beta}(f,x) - f(x) | \leq C\omega_2\left(f,\delta\right) + \omega_1\left(f, \left|\frac{\alpha-\beta x}{n+\beta} + \frac{g'(1)}{(n+\beta)g(1)}\right|\right).$$

□

3.3. **Rate of convergence.** Let $\omega_a(f,\delta)$ denote the usual modulus of continuity of f on the closed interval $[0,a], a>0$ and it is given by the relation

$$\omega_a(f,\delta) = \sup_{|t-x|\leq\delta}\ \sup_{x,t\in[0,a]} |f(t)-f(x)|.$$

We observe that for a function $f \in C_B[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero.

Theorem 3.4. *Let $f \in C_B[0,\infty)$ and $\omega_{a+1}(f,\delta)$ be its modulus of continuity on the finite interval $[0,a+1] \subset [0,\infty)$, where $a>0$. Then, we have*

$$|M_{n,\rho}^{\alpha,\beta}(f,x) - f(x)| \leq 6M_f(1+a^2)M_{n,\rho}^{\alpha,\beta}((t-x)^2,x) + 2\omega_{a+1}\left(f, \sqrt{M_{n,\rho}^{\alpha,\beta}((t-x)^2,x)}\right),$$

where M_f is a constant depending only on f.

Proof. For $x \in [0,a]$ and $t > a+1$. Since $t-x>1$, we have

$$|f(t)-f(x)| \leq M_f(2+x^2+t^2) \leq M_f(t-x)^2(2+3x^2+2(t-x)^2) \leq 6M_f(1+a^2)(t-x)^2.$$

For $x \in [0,a]$ and $t \leq a+1$, we have

$$|f(t)-f(x)| \leq \omega_{a+1}(f,|t-x|) \leq \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta), \delta>0.$$

From the above, we have

$$|f(t)-f(x)| \leq 6M_f(1+a^2)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta),$$

for $x \in [0,a]$ and $t \geq 0$. Thus

$$\begin{aligned} |M_{n,\rho}^{\alpha,\beta}(f,x) - f(x)| &\leq 6M_f(1+a^2)(M_{n,\rho}^{\alpha,\beta}((t-x)^2,x)) \\ &\quad +\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}(M_{n,\rho}^{\alpha,\beta}((t-x)^2,x))^{\frac{1}{2}}\right) \end{aligned}$$

Applying Cauchy-Schwarz's inequality, we get

$$|M_{n,\rho}^{\alpha,\beta}(f,x) - f(x)| \leq 6M_f(1+a^2)M_{n,\rho}^{\alpha,\beta}((t-x)^2,x) + 2\omega_{a+1}\left(f, \sqrt{M_{n,\rho}^{\alpha,\beta}((t-x)^2,x)}\right),$$

on choosing $\delta = \sqrt{M_{n,\rho}^{\alpha,\beta}((t-x)^2,x)}$. This completes the proof of theorem. □

3.4. **Weighted approximation.** Let $B_\nu[0,\infty)$ denote the weighted space of real-valued functions f defined on $[0,\infty)$ with the property $|f(x)| \leq M_f \nu(x)$ for all $x \in [0,\infty)$, where $\nu(x)$ is a weight function and M_f is a constant depending on the function f. We also consider the weighted subspace $C_\nu[0,\infty)$ of $B_\nu[0,\infty)$ given by $C_\nu[0,\infty) = \{f \in B_\nu[0,\infty) : f$ is continuous on $[0,\infty)\}$ and $C_\nu^*[0,\infty)$ denotes the subspace of all functions $f \in C_\nu[0,\infty)$ for which $\lim\limits_{|x|\to\infty} \dfrac{f(x)}{\nu(x)}$ exists finitely.

It is obvious that $C_\nu^*[0,\infty) \subset C_\nu[0,\infty) \subset B_\nu[0,\infty)$. The space $B_\nu[0,\infty)$ is a normed linear space with the following norm:

$$\| f \|_\nu = \sup_{x\in[0,\infty)} \frac{|f(x)|}{\nu(x)}.$$

The following results on the sequence of positive linear operators in these spaces are given in [3], [4].

Lemma 3.1. *([3], [4]) The sequence of positive linear operators $(L_n)_{n\geq 1}$ which act from $C_\nu[0,\infty)$ to $B_\nu[0,\infty)$ if and only if there exists a positive constant k such that $L_n(\nu, x) \leq k\nu(x)$, i.e. $\| L_n(\nu) \|_\nu \leq k$.*

Theorem 3.5. *([3], [4]) Let $(L_n)_{n\geq 1}$ be the sequence of positive linear operators which act from $C_\nu[0,\infty)$ to $B_\nu[0,\infty)$ satisfying the conditions*

$$\lim_{n\to\infty} \| L_n(t^k) - x^k \|_\nu = 0, \; k = 0, 1, 2,$$

then for any function $f \in C_\nu^*[0,\infty)$

$$\lim_{n\to\infty} \| L_n(f) - f \|_\nu = 0.$$

Lemma 3.2. *Let $\nu(x) = 1 + x^2$ be a weight function. If $f \in C_\nu[0,\infty)$, then*

$$\| M_{n,\rho}^{\alpha,\beta}(\nu) \|_\nu \leq 1 + M.$$

Proof. Using Lemma 2.2, we have

$$\begin{aligned} M_{n,\rho}^{\alpha,\beta}(\nu, x) &= 1 + \frac{n^2x^2}{(n+\beta)^2} + \left(\frac{n(1+\rho+2\alpha\rho)}{\rho(n+\beta)^2} + \frac{2n}{(n+\beta)^2}\frac{g'(1)}{g(1)} \right) x \\ &\quad + \frac{g''(1) + 2\alpha g'(1) + \alpha^2 g(1)}{(n+\beta)^2 g(1)} + \frac{\rho+1}{\rho(n+\beta)^2}\frac{g'(1)}{g(1)}. \end{aligned}$$

Then

$$\begin{aligned} \| M_{n,\rho}^{\alpha,\beta}(\nu) \|_\nu &\leq 1 + \frac{n^2\rho + n(1+\rho+2\alpha\rho)}{\rho(n+\beta)^2} + \frac{2ng'(1)}{(n+\beta)^2 g(1)} \\ &\quad + \frac{\rho(g''(1) + 2\alpha g'(1) + \alpha^2 g(1)) + (\rho+1)g'(1)}{\rho(n+\beta)^2 g(1)} \end{aligned}$$

there exists a positive constant M such that

$$\| M_{n,\rho}^{\alpha,\beta}(\nu) \|_\nu \leq 1 + M$$

so the proof is completed. □

By using Lemma 3.2 we can easily see that the operators $M_{n,\rho}^{\alpha,\beta}$ defined by (3) act from $C_\nu[0,\infty)$ to $B_\nu[0,\infty)$.

Theorem 3.6. *For every $f \in C_\nu^*[0,\infty)$, we have*

$$\lim_{n\to\infty} \| M_{n,\rho}^{\alpha,\beta}(f) - f \|_\nu = 0.$$

Proof. From [3], we know that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \| M_{n,\rho}^{\alpha,\beta}(t^k) - x^k \|_\nu = 0, \ k = 0, 1, 2. \tag{13}$$

Since $M_{n,\rho}^{\alpha,\beta}(1,x) = 1$, the condition in (13) holds for $k = 0$.
For $k = 1$, we have

$$\begin{aligned}\| M_{n,\rho}^{\alpha,\beta}(t) - x \|_\nu &\leq \left|\frac{\beta}{n+\beta}\right| \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \left|\frac{\alpha g(1) + g'(1)}{(n+\beta)g(1)}\right| \sup_{x\in[0,\infty)} \frac{1}{1+x^2} \\ &\leq \left|\frac{(\alpha+\beta)g(1) + g'(1)}{(n+\beta)g(1)}\right|\end{aligned}$$

which implies that $\lim_{n\to\infty} \| M_{n,\rho}^{\alpha,\beta}(t) - x \|_\nu = 0$.
Similarly, we can write for $k = 2$

$$\begin{aligned}\| M_{n,\rho}^{\alpha,\beta}(t^2) - x^2 \|_\nu &\leq \left|\frac{\beta^2 + 2n\beta}{(n+\beta)^2}\right| + \left|\frac{n(1+\rho+2\alpha\rho)}{\rho(n+\beta)^2} + \frac{2n}{(n+\beta)^2}\frac{g'(1)}{g(1)}\right| \\ &\quad + \left|\frac{\rho(g''(1) + 2\alpha g'(1) + \alpha^2 g(1)) + (\rho+1)g'(1)}{\rho(n+\beta)^2 g(1)}\right|,\end{aligned}$$

which implies that $\lim_{n\to\infty} \| M_{n,\rho}^{\alpha,\beta}(t^2) - x^2 \|_\nu = 0$.
This completes the proof of theorem. $\square$

3.5. **Pointwise Estimates.** We know that a function $f \in C[0,\infty)$ is in $\mathrm{Lip}_M(\eta)$ on E, $\eta \in (0,1]$, E$\subset [0,\infty)$ if it satisfies the condition

$$|f(t) - f(x)| \leq M|t-x|^\eta, \ t \in [0,\infty) \ \ \textit{and} \ \ x \in E,$$

where M is a constant depending only on η and f.

Theorem 3.7. *Let $f \in C[0,\infty) \cap Lip_M(\eta)$, $E \subset [0,\infty)$ and $\eta \in (0,1]$. Then, we have*

$$|M_{n,\rho}^{\alpha,\beta}(f,x) - f(x)| \leq M\left(\left(M_{n,\rho}^{\alpha,\beta}((t-x)^2,x)\right)^{\eta/2} + 2d^\eta(x,E)\right), \quad x \in [0,\infty),$$

where M is a constant depending on η and f and $d(x,E)$ is the distance between x and E defined as $d(x,E) = \inf\{|t-x| : t \in E\}$.

Proof. Let $\overline{E}$ be the closure of E in $[0,\infty)$. Then, there exists at least one point $x_0 \in \overline{E}$ such that $d(x,E) = |x - x_0|$. By our hypothesis and the monotonicity of $M_{n,\rho}^{\alpha,\beta}$, we get

$$\begin{aligned}|M_{n,\rho}^{\alpha,\beta}(f,x) - f(x)| &\leq M_{n,\rho}^{\alpha,\beta}(|f(t) - f(x_0)|, x) + M_{n,\rho}^{\alpha,\beta}(|f(x) - f(x_0)|, x) \\ &\leq M\left(M_{n,\rho}^{\alpha,\beta}(|t-x_0|^\eta, x) + |x-x_0|^\eta\right) \\ &\leq M\left(M_{n,\rho}^{\alpha,\beta}(|t-x|^\eta, x) + 2|x-x_0|^\eta\right).\end{aligned}$$

Now, applying Hölder's inequality with $p = \dfrac{2}{\eta}$ and $q = \dfrac{2}{2-\eta}$, we obtain

$$|M_{n,\rho}^{\alpha,\beta}(f,x) - f(x)| \leq M\left(\left(M_{n,\rho}^{\alpha,\beta}(|t-x|^2,x)\right)^{\eta/2} + 2d^\eta(x,E)\right),$$

from which the desired result immediate. $\square$

Next, we obtain the local direct estimate of the operators defined in (3), using the Lipschitz-type maximal function of order η introduced by B. Lenze [19] as

$$\widetilde{\omega}_\eta(f,x)=\sup_{t\neq x,\ t\in[0,\infty)}\frac{|f(t)-f(x)|}{|t-x|^\eta},\quad x\in[0,\infty)\ \text{ and }\ \eta\in(0,1]. \tag{14}$$

Theorem 3.8. *Let $f\in C_B[0,\infty)$ and $0<\eta\leq 1$. Then, for all $x\in[0,\infty)$ we have*

$$|M_{n,\rho}^{\alpha,\beta}(f,x)-f(x)|\leq \widetilde{\omega}_\eta(f,x)\left(M_{n,\rho}^{\alpha,\beta}((t-x)^2,x)\right)^{\eta/2}.$$

Proof. From the equation (14), we have

$$|M_{n,\rho}^{\alpha,\beta}(f,x)-f(x)|\leq \widetilde{\omega}_\eta(f,x)M_{n,\rho}^{\alpha,\beta}(|t-x|^\eta,x).$$

Applying the Hölder's inequality with $p=\dfrac{2}{\eta}$ and $q=\dfrac{2}{2-\eta}$, we get

$$|M_{n,\rho}^{\alpha,\beta}(f,x)-f(x)|\leq \widetilde{\omega}_\eta(f,x)M_{n,\rho}^{\alpha,\beta}((t-x)^2,x)^{\frac{\eta}{2}}\leq \widetilde{\omega}_\eta(f,x)\left(M_{n,\rho}^{\alpha,\beta}((t-x)^2,x)\right)^{\eta/2}.$$

Thus, the proof is completed. □

For $a,b>0$, Özarslan and Aktuğlu [28] consider the Lipschitz-type space with two parameters:

$$Lip_M^{(a,b)}(\eta)=\left(f\in C[0,\infty):|f(t)-f(x)|\leq M\frac{|t-x|^\eta}{(t+ax^2+bx)^{\eta/2}};\ x,t\in[0,\infty)\right),$$

where M is any positive constant and $0<\eta\leq 1$.

Theorem 3.9. *For $f\in Lip_M^{(a,b)}(\eta)$. Then, for all $x>0$, we have*

$$|M_{n,\rho}^{\alpha,\beta}(f,x)-f(x)|\leq M\left(\frac{M_{n,\rho}^{\alpha,\beta}((t-x)^2,x)}{ax^2+bx}\right)^{\eta/2}.$$

Proof. First we prove the theorem for $\eta=1$. Then, for $f\in Lip_M^{(a,b)}(1)$, and $x\in[0,\infty)$, we have

$$\begin{aligned}|M_{n,\rho}^{\alpha,\beta}(f,x)-f(x)| &\leq MM_{n,\rho}^{\alpha,\beta}\left(\frac{|t-x|}{(t+ax^2+bx)^{1/2}},x\right)\\ &\leq \frac{M}{(ax^2+bx)^{1/2}}M_{n,\rho}^{\alpha,\beta}(|t-x|,x).\end{aligned}$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}|M_{n,\rho}^{\alpha,\beta}(f,x)-f(x)| &\leq \frac{M}{(ax^2+bx)^{1/2}}\left(M_{n,\rho}^{\alpha,\beta}((t-x)^2,x)\right)^{1/2}\\ &\leq M\left(\frac{M_{n,\rho}^{\alpha,\beta}((t-x)^2,x)}{ax^2+bx}\right)^{1/2}.\end{aligned}$$

Now, we prove that the result is true for $0<\eta<1$. Then, for $f\in Lip_M^{(a,b)}(\eta)$, and $x\in[0,\infty)$, we get

$$|M_{n,\rho}^{\alpha,\beta}(f,x)-f(x)| \leq \frac{M}{(ax^2+bx)^{\eta/2}}M_{n,\rho}^{\alpha,\beta}(|t-x|^\eta,x).$$

Taking $p=\frac{1}{\eta}$ and $q=\frac{2}{2-\eta}$, applying the Hölder's inequality, we have

$$|M_{n,\rho}^{\alpha,\beta}(f,x)-f(x)|\leq\frac{M}{(ax^2+bx)^{\eta/2}}\left(M_{n,\rho}^{\alpha,\beta}(|t-x|,x)\right)^{\eta}.$$

Finally by Cauchy-Schwarz inequality, we get

$$|M_{n,\rho}^{\alpha,\beta}(f,x)-f(x)| \quad\leq\quad M\left(\frac{M_{n,\rho}^{\alpha,\beta}((t-x)^2,x)}{ax^2+bx}\right)^{\eta/2}.$$

□

4. King's Approach

To make the convergence faster, King [18] proposed an approach to modify the classical Bernstein polynomial, so that the sequence preserve test functions e_0 and e_2, where $e_i(t)=t^i, i=0,1,2$. After this approach many researcher contributed in this direction.
As the operator $M_{n,\rho}^{\alpha,\beta}(f,x)$ defined in (3) preserve only the constant functions so further modification of these operators is proposed to be made so that the modified operators preserve the constant as well as linear functions.
For this purpose the modification of (3) is defined as

$$\hat{M}_{n,\rho}^{\alpha,\beta}(f,x) \quad=\quad \sum_{k=1}^{\infty}l_{n,k}(r_n(x))\int_0^{\infty}\Theta_{n,k}^{\rho}(x)f\left(\frac{nt+\alpha}{n+\beta}\right)dt+l_{n,0}(r_n(x))f\left(\frac{\alpha}{n+\beta}\right) \tag{15}$$

where $r_n(x)=\frac{(n+\beta)x-\alpha}{n}-\frac{g'(1)}{ng(1)}$ for $x\in I_n=[\frac{\alpha}{n+\beta},\infty)$.

Lemma 4.1. *For every $x\in I_n$, we have*

(1) $\hat{M}_{n,\rho}^{\alpha,\beta}(e_0,x)=1$;
(2) $\hat{M}_{n,\rho}^{\alpha,\beta}(e_1,x)=x$;
(3) $\hat{M}_{n,\rho}^{\alpha,\beta}(e_2,x)=x^2+\dfrac{1+\rho}{\rho(n+\beta)}x-\dfrac{\alpha^2\rho+\alpha\rho+\alpha}{\rho(n+\beta)^2}-\dfrac{1+\rho+3\alpha\rho}{\rho(n+\beta)^2}\dfrac{g'(1)}{g(1)}+\left(\dfrac{g'(1)}{(n+\beta)g(1)}\right)^2.$

Consequently, for each $x\in I_n$, we have the following equalities

$$\hat{M}_{n,\rho}^{\alpha,\beta}(t-x,x)=0$$

$$\hat{M}_{n,\rho}^{\alpha,\beta}((t-x)^2,x)=\frac{1+\rho}{\rho(n+\beta)}x-\frac{\alpha^2\rho+\alpha\rho+\alpha}{\rho(n+\beta)^2}-\frac{1+\rho+3\alpha\rho}{\rho(n+\beta)^2}\frac{g'(1)}{g(1)}+\left(\frac{g'(1)}{(n+\beta)g(1)}\right)^2.$$

Theorem 4.1. *For $f\in C_B(I_n), C'>0$, we have*

$$|\hat{M}_{n,\rho}^{\alpha,\beta}(f,x)-f(x)|\leq C'\omega_2\left(f,\sqrt{\hat{M}_{n,\rho}^{\alpha,\beta}((t-x)^2,x)}\right),$$

Proof. Let $h\in W^2$ and $x,t\in I_n$. Using the Taylor's expansion we have

$$h(t)=h(x)+(t-x)h'(x)+\int_x^t(t-v)h''(v)dv.$$

Applying $\hat{M}_{n,\rho}^{\alpha,\beta}$ on both sides and using Lemma 3.1, we get

$$\hat{M}_{n,\rho}^{\alpha,\beta}(h,x)-h(x) \quad=\quad \hat{M}_{n,\rho}^{\alpha,\beta}\left(\int_x^t(t-v)h''(v)dv,x\right).$$

Obviously, we have $\left|\int_x^t (t-v)h''(v)dv\right| \leq (t-x)^2\|h''\|$.
Therefore

$$\mid \hat{M}_{n,\rho}^{\alpha,\beta}(h,x) - h(x) \mid \leq \hat{M}_{n,\rho}^{\alpha,\beta}((t-x)^2, x) \parallel h'' \parallel .$$

Since $\mid \hat{M}_{n,\rho}^{\alpha,\beta}(f,x) \mid \leq \|f\|$, we get

$$\begin{aligned} \mid \hat{M}_{n,\rho}^{\alpha,\beta}(f,x) - f(x) \mid &\leq \mid \hat{M}_{n,\rho}^{\alpha,\beta}(f-h,x) \mid + \mid (f-h)(x) \mid + \mid \hat{M}_{n,\rho}^{\alpha,\beta}(h,x) - h(x) \mid \\ &\leq \|f-h\| + \hat{M}_{n,\rho}^{\alpha,\beta}((t-x)^2, x)\|h''\|. \end{aligned}$$

Finally, taking the infimum over all $h \in W^2$ and using (10) we obtain

$$\mid \hat{M}_{n,\rho}^{\alpha,\beta}(f,x) - f(x) \mid \leq C'\omega_2\left(f, \sqrt{\hat{M}_{n,\rho}^{\alpha,\beta}((t-x)^2, x)}\right).$$

□

Theorem 4.2. *Let $f \in C_B(I_n)$. If f'' exists at a fixed point $x \in I_n$, then we have*

$$\lim_{n\to\infty} n\left(\hat{M}_{n,\rho}^{\alpha,\beta}(f,x) - f(x)\right) = \frac{x}{2}\left(1 + \frac{1}{\rho}\right) f''(x).$$

The proof follows along the lines of Theorem 3.2.

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