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FUZZY PERFECT EQUITABLE DOMINATION EXCELLENT TREES

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Abstract. A set D of vertices of a fuzzy graph G is a Perfect Dominating set if every vertex not in D is adjacent to exactly one vertex in D. In this paper, we discuss the concept of equitable excellent fuzzy graph, fuzzy equitable excellent dominating set γ^{ef}. We introduce fuzzy perfect equitable excellent dominating set $\gamma_p(G)$ - set and then Construction of perfect equitable excellent fuzzy tree is discussed.

Keywords: Excellent fuzzy graph, fuzzy equitable dominating set γ^{ef}, fuzzy equitable perfect dominating set $\gamma_p(G)$

AMS Subject Classification: 05C72

1. Introduction

G. H. Fricke et. al. [3] call a vertex of a graph G to be good if it is contained in some $\gamma(G)$-set , and bad if it is not. They call a graph G to be γ -excellent if every vertex of G is good. In related work, Mynhardt [5] characterized the vertices that are contained in every $\gamma(T)$ -set and the vertices that are contained in no $\gamma(T)$ for trees T. In [10], new classes of excellent graphs, such as γ -just excellent graphs and γ -very excellent graphs, have been defined.

First, we introduce the basic concepts of fuzzy graph based on [1, 8]. Then, we discuss excellent fuzzy graph, fuzzy equitable dominating set γ^{ef} -set and introduce fuzzy perfect equitable excellent dominating set $\gamma_p(G)$ depending on [2, 6, 9, 10] and construct fuzzy perfect equitable domination excellent tree based on [7].

2. Preliminaries

Definition 2.1. *A fuzzy subset of a nonempty set V is a mapping $\sigma : V \to [0, 1]$. A fuzzy relation on V is a fuzzy subset of $V \times V$. A fuzzy graph $G = (\sigma, \mu)$ is a pair of function σ:$V \to$ [0, 1] and $\mu : V \times V \to [0, 1]$, where μ (u, v) $\leq \sigma(u) \wedge \sigma(v)$ for all*

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$u, v \in V$. The underlying crisp graph of $G = (\sigma, \mu)$ is denoted by $G = (V, E)$, where $V(G) = \{u \in V : \sigma(u) > 0\}$ and $E(G) = \{(u, v) \in V \times V : \mu(u, v) > 0\}$.*

Definition 2.2. *A path P in a fuzzy graph $G = (\sigma, \mu)$ is a sequence of distinct nodes $u_0, u_1, \ldots, u_n$ such that $\mu(u_{i-1}, u_i) > 0, 1 \leq i \leq n$. Degree of membership of weakest arc in the path is defined as its strength. A weakest arc is an arc with least membership value. Here $n \geq 1$ is called length of path P. A single node u may also be considered as a path. In this case the path is of length 0. The consecutive pairs (u_{i-1}, u_i) are called arcs of the path. We call P a cycle if $u_0 = u_n$ and $n \geq 3$.*

Definition 2.3. *The strength of connectedness between two nodes u and v is defined as the maximum of strength of all paths between u and v.*

Definition 2.4. *The degree of vertex u is defined as the sum of the weights of the edges incident at u and is denoted by $deg(u)$.*

Definition 2.5. *Let $G = (\sigma, \mu)$ be a fuzzy graph. A subset D of V is said to be a dominating set of G if for every $v \in V - D$, there exists a $u \in D$ such that u dominates v. The minimum cardinality of such a dominating set is denoted by $\gamma(G)$ and the dominating set is called $\gamma(G)$-set.*

3. Main Results

Definition 3.1. *A subset D of V is called a fuzzy equitable dominating set if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|$ deg(u) - deg(v)$| \leq 1$ and $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$. A fuzzy dominating set D of a fuzzy graph G is called minimal dominating set of G, if for every node $v \in D, D - \{v\}$ is not a dominating set. The minimum cardinalities taken over all minimal dominating sets of vertices of G is denoted by γ^{ef} and is called the fuzzy equitable domination number of G and the dominating set is called γ^{ef} -set.*

Definition 3.2. *A fuzzy graph G is said to be fuzzy equitable excellent if for every vertex of G belongs to γ^{ef} -sets of G. A vertex which belongs to γ^{ef}-set is called fuzzy good. A fuzzy graph G is said to be fuzzy excellent if for every vertex of G is fuzzy Good.*

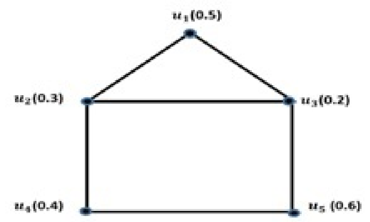

FIGURE 1. γ^{ef}-sets are given by $\{u_1, u_4\}$, $\{u_2, u_4\}$, $\{u_3, u_5\}$

Example 3.1. *In the Figure1,γ^{ef}-sets are given by $\{u_1, u_4\}$, $\{u_2, u_4\}$, $\{u_3, u_5\}$ with cardinality 0.9,0.7,0.8 respectively. All the vertices are good.*

Definition 3.3. *[7] An equitable dominating set D of a fuzzy graph G is said to be a perfect dominating set if for each vertex v not in D, v is adjacent to exactly one vertex of D. Its cardinality is denoted by $\gamma_p(G)$ and the perfect dominating set is called $\gamma_p(G) - set$.*

Definition 3.4. *A fuzzy graph G is said to be γ_p - excellent , if every vertex is in some perfect equitable dominating set $\gamma_p(G) - set$ and $\gamma_p^u(G)$ denotes minimum cardinality of dominating set of G that contains u.*

FIGURE 2. P_4 is γ_p - excellent and $\gamma_p(G) - sets$ are $\{a, d\}, \{b, c\}$ with cardinality 1.2,1 respectively.

The following results are provided based on [6] discussing about perfect equitable domination on complete graph,path and cycle on fuzzy graph.

Theorem 3.1. *Let K_n be a complete fuzzy graph of order $n \geq 2$, then $\gamma_p(K_n) \leq 1$*

Proof. Given a perfect equitable dominating set $\gamma_p(G)$ of K_n and assume $x \in \gamma_p(G)$

Then x dominates all other vertices in K_n and since K_n is complete fuzzy graph for every $y \in K_n$, the vertices x and y are adjacent and $|degx - degy| \leq 1$.

Obviously, x is in $\gamma_p(G) - set$ and $\gamma_p(K_n) \leq 1$

□

Theorem 3.2. *For any integer $n \geq 2, \gamma_p(P_n) \leq \lceil \frac{n}{3} \rceil$*

Proof. Suppose vertex set of $P_n = \{u_1, u_2, ..., u_n\}$. Let u_1 be the first vertex, u_2 be the second vertex,... u_n be the last vertex, where P_n is labelled left to right. Now, u_1 dominates u_2, u_2 dominates u_1 and u_3 and so on. Thus, u_i dominates $u_{i-1}, u_{i+1}, i = 2, 3, ...n - 1$. Let $\gamma_p(P_n) - set$ be perfect equitable dominating set of P_n. Consider the following cases.

Case 1: $P_n = P_{3j-1}, j \in Z^+$

If $j = 1$, then $P_{3(1)-1} = P_2$. Clearly, $\gamma_p(P_n) - set$ is $\{u_1\}$ or $\{u_2\}$. Thus, $\gamma_p(P_2) \leq 1 \leq \lceil \frac{2}{3} \rceil$.

If $j = 2$, then $P_{3(2)-1} = P_5$. Clearly, $\gamma_p(P_n) - set$ is $\{u_1, u_4\}$ or $\{u_2, u_5\}$.
Thus, $\gamma_p(P_5) \leq 2 \leq \lceil \frac{5}{3} \rceil$.

If $j = 3$, then $P_{3(3)-1} = P_2$. Clearly, $\gamma_p(P_n) - set$ is $\{u_1, u_4, u_7\}$ or $\{u_2, u_5, u_8\}$.
Thus, $\gamma_p P_8 \leq 3 = \lceil \frac{8}{3} \rceil$.

In general, $\gamma_P(P_{3j-1}) \leq \frac{3j-1}{3} = \lceil \frac{n}{3} \rceil$

Case 2: $P_n = P_{3j}, j \in Z^+$

If $j = 1$, then $P_{3(1)} = P_3$. Clearly, $\gamma_p(P_3) - set$ is $\{u_2\}$. Thus, $\gamma_p(P_2) \leq 1 = \lceil \frac{2}{3} \rceil$

If $j = 2$, then $P_{3(2)} = P_6$. Clearly, $\gamma_p(P_6) - set$ is $\{u_2, u_5\}$. Thus, $\gamma_p(P_6) \leq 2 = \lceil \frac{5}{3} \rceil$.

If $j = 3$, then $P_{3(3)} = P_9$. Clearly, $\gamma_p(P_9) - set$ is $\{u_2, u_5, u_8\}$. Thus, $\gamma_p(P_8) \leq 3 = \lceil \frac{8}{3} \rceil$.

In general, $\gamma_P(P_{3j}) \leq \frac{3j-1}{3} = \lceil \frac{n}{3} \rceil$.

Case 3: $P_n = P_{3j+1}, j \in Z^+$ In a similar manner,it can be seen that $\gamma_P(P_{3j+1}) \leq \frac{3j+1}{3} = \lceil \frac{n}{3} \rceil$

Thus,in all cases, $\gamma_P(P_n) \leq \lceil \frac{n}{3} \rceil$ □

Remark 3.1. *In a path P_n,consecutive vertices of perfect equitable dominating sets are either adjacent or at a distance 3 apart.*

Proof. On the contrary,suppose that the verties of a perfect dominating sets are not adjacent and at a distance two apart,then two vertices dominate exactly one vertex in $V - D$.This is a contradiction to perfect domination property.Hence the proof. □

Theorem 3.3.

$$\gamma_p(C_n) \leq \begin{cases} 2k & if \;\; n = 6k \\ k+1 & if \;\; n = 3k+1 \;\; or \;\; n = 3k-1 \\ k & if \;\; n = 3k, \;\; n \geq 9 \end{cases}$$

Proof. Let vertex of $C_n = \{u_1, u_2, ...u_n\}$.Label the vertex of C_n in clockwise direction such that u_1 is adjacent to u_n, u_2,u_2 is adjacent to $u_1, u_3$and u_{n-1} is adjacent to u_{n-2}, u_1.Consider the following cases:

case1:$n = 6k$ A perfect equitable dominating set $\gamma_p(G) - set$ can be obtained as follows:If $u_i \in \gamma_p(G) - set$ where $\gamma_P(G) - set$ is perfect dominating set,then u_i dominates its two adjacent vertices u_{i-1} and u_{i+1} modulo n.This means that by selecting one vertex of C_n to be a member of $\gamma_P(G)$ three vertices are eliminated from the remaining selection for the next choice of u_i.Thus the process of selecting a member of $\gamma_p(G) - set$ follows the grouping of three consecutive vertices in $6k$ vertices.Thus grouping of four consecutive vertices containing one dominating vertex is not possible or grouping of four consecutive vertices containing two dominating vertices which are both adjacent to another is possible but it is not minimal.Thus grouping of three yields minimum $\gamma_p(G) - set$ .Thus $\gamma_P(C_n) = \frac{6k}{3} = 2k$.

case2:$n = 3k + 1$ Group the vertices of C_n by three.There remains a vertex ,say u_j.Either $u_j \in \gamma_p(G) - set$ or $u_j \notin \gamma_P(G) - set$.If $u_j \in \gamma_p(G) - set$,then perfect dominating set cannot be completed since there remains three consecutive vertices which do not belong to $\gamma_p(G) - set$.Thus,without loss of generality,let $u_2, u_5, u_8, ...u_{3k-1}$ be in $\gamma_p(G) - set$.Now either $u_{3k+1} \in \gamma_p(G) - set$ or $u_{3k+1} \notin \gamma_p(G)$. If $u_{3k+1} \in \gamma_p(G) - set$,then u_1 is adjacent to u_2, u_{3k+1} which is a contradiction to perfect domination property. If $u_{3k+1} \notin \gamma_p(G) - set$,then there does not exist $u_j \in \gamma_p(G)$ adjacent to u_{3k+1}.Thus either $u_1 \in \gamma_p(G) - set$ or $u_3k \in \gamma_p(G) - set$. Hence,$\gamma_p(C_n) \leq \frac{3k}{3} + 1 = k + 1$.

case3:$n = 3k - 1$ Group $3k - 3$ vertices and set aside remaining two vertices.Consider $\{u_1, u_2, ...u_{3k-3}\}$ leaving u_{3k-2} and u_{3k-1}.Without loss of generality,let $u_2, u_5, u_8, ...u_{3k-4} \in \gamma_p(G) - set$,while$u_1, u_3, u_4, ...u_{3k-3}, u_{3k-2}, u_{3k-1} \in V(C_n) - \gamma_p(G) - set$.Consider the remaining vertices u_{3k-2}, u_{3k-1}. If $u_{3k-1} \in \gamma_p(G) - set$,then u_1 is adjacent to u_2 and u_{3k-1} which is a contradiction to the definition of perfect dominating set.If $u_{3k-2} \in \gamma_p(G) - set$,then u_{3k-2} is adjacent to u_{3k-4} and u_{3k-2} which is a contradiction to perfect dominating set.Thus u_{3k-3} and u_{3k-2} be element of $\gamma_p(G) - set$ and $u_{3k-1} \notin \gamma_p(G) - set$ or $u_{3k-3}, u_1 \in \gamma_p(G) - set$ and $u_{3k-2} \notin \gamma_p(G) - set$.Thus,

$\gamma_p(G) \leq \frac{3k-3}{3} + 2 = k + 1$

case4:$n = 3k$ Proof is similar as in case1.

□

4. Construction of fuzzy perfect equitable domination excellent tree

We now provide a constructive characterization of perfect domination equitable excellent fuzzy trees. We accomplish this by defining a family of labelled fuzzy trees. A Tree is acyclic connected fuzzy graph and order of the tree is the number of vertices in it. A leaf has only one vertex in its neighborhood. A support vertex is a vertex adjacent to a leaf. A Tree T is said to be γ_p - equitable excellent, if every vertex is in some perfect equitable dominating set $\gamma_p(T)$ -set and $\gamma_p^u(T)$ denotes minimum cardinality of perfect equitable dominating set of T that contains u. Let $F = \{T_n\}, n \geq 1$ be the family of equitable excellent trees constructed inductively such that T_1 is a γ_p - excellent path P_4 and $T_n = T$,$T' = T_{n-1}$. If $n \geq 2, T_{i+1}$ can be obtained recursively from T_i by one of the two operations F_1, F_2, for $i = 1, 2, \ldots, n - 1$. Then we say that T has length n in F.

Distance between any two vertices is defined to be the number of edges between them. We define the status of a vertex v, denoted $sta(v)$ to be A or B. Initially if $T_1 = P_4$, then $sta(v) = A$ if v is a support vertex and $sta(v) = B$, if v is a leaf. Assign node weight as follows:

$min(a, d) \leq$ node weight of vertex with Status $A \leq max(a, d)$.

$min(b, c) \leq$ node weight of vertex with Status $B \leq max(b, c)$ where a, d and b, c are node weights of P_4 with status A and B respectively. Once a vertex is assigned a status, this status remains unchanged as the tree is constructed. To prove newly constructed tree T from T' is equitable we prove, $|\gamma_p(T) - \gamma_p(T')| \leq 1$.

Operation F_1: Assume $y \in T_n$ and $sta(y) = A$. The tree T_{n+1} is obtained from T_n by adding a path x, w and the edge xy. Let $sta(x) = A$ and $sta(w) = B$, and assign node weight as mentioned.

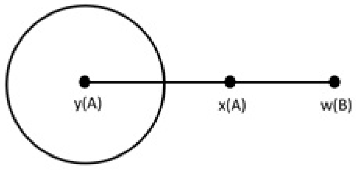

FIGURE 3. Operation F_1

OperationF_2: Assume $y \in T_n$ and $sta(y) = B$. The tree T_{n+1} is obtained from T_n by adding a path x, w, v and the edge xy. Let $sta(x) = sta(w) = A$ and $sta(v) = B$, and assign node weight as mentioned. (shown in Figure4)

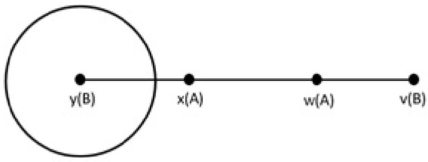

FIGURE 4. Operation F_2

F is closed under the two operations F_1 and F_2. For $T \in F$, let $A(T)$ and $B(T)$ be the sets of vertices of status A and B respectively. We have the following observation, which follow from the construction of F.

Observation 4.1. *Let $T \in F$ and $v \in V(T)$.*

1. *If $sta(v) = A$, then v is adjacent to exactly one vertex of $B(T)$ and at least one vertex of $A(T)$.*
2. *If $sta(v) = B$, then $N(v)$ is a subset of $A(T)$.*
3. *If v is a support vertex, then $sta(v) = A$.*
4. *If v is a leaf, then $sta(v) = B$.*
5. *$|A(T)| \geq |B(T)|$*
6. *Distance between any two vertices in $B(T)$ is at least three.*

Theorem 4.1. *If $T \in F$, then $B(T)$ is a $\gamma_p(T)$ -set. Moreover, if T is obtained from $T' \in F$ using operation $F_1 or F_2$, then $|\gamma_p(T) - \gamma_p(T')| \leq 1$.*

Proof. By observation4.1 and remark3.1, it is clear that $B(T)$ is a perfect dominating set. Now we prove that, $B(T)$ is equitable. We proceed by induction on the length n of the sequence of trees needed to construct the tree T. Suppose $n = 1$, then $T = P_4$, belongs to F. Let the vertices of P_4 be labeled as a, b, c, d and node weight are assigned in such a way that maximum difference between perfect dominating sets is 1. Then P_4 will be $\gamma_p(G)$ -set and also $B(P_4) = \{a, d\}$ and is a$\gamma_p(P_4)$-set. This establishes the base case.

Assume that the result holds for all trees in F that can be constructed from a sequence of fewer than n trees where $n \geq 2$. Let $T \in F$ be obtained from a sequence $T_1, T_2, \dots T_n$ of n trees, where $T' = T_{n-1}$ and $T = T_n$. By our inductive hypothesis, $B(T')$ is equitable and excellent $\gamma_p(T')$. We now consider two possibilities depending on whether T is obtained from $T'$by operation $F_1 or F_2$.

Case 1: T is obtained from T' by operation F_1. Suppose T is obtained from T' by adding a path y, x, w of length 2 to the vertex $y \in V(T')$ and assigned node weight as mentioned. Any $\gamma_p(T')$ -set can be extended to a $\gamma_p(T)$-set by adding to it the vertex w which is of status B. Hence $B(T) = B(T') \bigcup \{w\}$ is a $\gamma_p(T)$ -set.
$\gamma_p(T) = \gamma_p(T')$ + node weight of w.
$\leq \gamma_p(T') + 1$ (1)
Also, $\gamma_p(T') \leq \gamma_p(T) \leq \gamma_p(T) + 1$
i. e, $\gamma_p(T') - 1 \leq \gamma_p(T)$ (2)
By combining inequalities (1) and (2) , we get $|\gamma_p(T) - \gamma_p(T')| \leq 1$.

Case 2: T is obtained from T' by operation F_2. Assume $y \in T'$ and $sta(y) = B$. The tree T is obtained from T' by adding a path x, w, v and the edge xy. Let $sta(x) = sta(w) = A$ and $sta(v) = B$, and assign node weight as mentioned.

Subcase2.1: If $y \in T'$ is in any $\gamma_p^y(T')$ -set, then it can be extended to a $\gamma_p^y(T')$ - set of T by adding the vertex v, which is of status B. Since T' is γ_p -set and thus we have, $\gamma_p(T) = \gamma_p(T')$+node weight of v.
$\leq \gamma_p(T') + 1$ (3)
Also, $\gamma_p(T') \leq \gamma_p(T) \leq \gamma_p(T) + 1$.
i.e. $\gamma_p(T') - 1 \leq \gamma_p(T)$ (4)
By combining inequalities (3) and (4) we get, $|\gamma_p(T) - \gamma_p(T')| \leq 1$.

Subcase2.2: If $y \in T'$ is not in $\gamma_p^z(T')$ -set for any $z \neq y$ in T', then any $\gamma_p^z(T')$ -set can be extended to a $\gamma_p(T)$ -set by adding to it the vertex x and w, which is of status A. Thus,
$\gamma_p(T)$= $\gamma_p(T')$ + node weight of x and w in T - node weight of a vertex in T'.
$\leq \gamma_p(T') + 1$. (5)
Also, $\gamma_p(T') \leq \gamma_p(T) \leq \gamma_p(T) + 1$
i.e., $\gamma_p(T') - 1 \leq \gamma_p(T)$ (6)
By combining inequalities (5) and (6), we get, $|\gamma_p(T) - \gamma_p(T')| \leq 1$ Thus , if T is obtained from $T' \in F$ using operation F_1 or F_2, it follows that $|\gamma_p(T) - \gamma_p(T')| \leq 1$.

□

Theorem 4.2. *If $T \in F$have length n, then T is a γ_p - excellent tree.*

Proof. Since T has length n in F, T can be obtained from a sequence $T_1, T_2, \dots, T_n$ of trees such that T_1 is a path P_4 and $T_n = T$, a tree. If $n \geq 2, T_{i+1}$ can be obtained from T_i by one of the two operations F_1, F_2 for $i = 1, 2, \cdots, n-1$. To prove the desired result, we proceed by induction on the length n of the sequence of trees needed to construct the tree T.

If $n = 1$, then $T = P_4 \in F$ and therefore, T is γ_p -set. Hence the lemma is true for the base case. Assume that the result holds for all trees in F of length less than n, where $n \geq 2$. Let $T \in F$ be obtained from a sequence $T_1, T_2, \dots, T_n$ of n trees. For convenience, we denote T_{n-1} by T'. We now consider two possibilities depending on whether T is obtained from T' by operation $F_1 or F_2$. By above the lemma,$|\gamma_p(T) - \gamma_p(T')| \leq 1$.

Case 1: Let $u \in V(T)$ be arbitrary is obtained from T' by operation F_1. Suppose T is obtained from T' by adding a path y, x, w of length 2 to vertex $y \in V(T')$ and assign node weight as mentioned. Since T' is γ_p -excellent, any $\gamma_p^y(T')$ -set can be extended to a perfect dominating set of T by adding the vertex x or w. Thus any $\gamma_p^y(T')$ -set of T' can be extended to $\gamma_p^u(T)$ -set of T.

$\gamma_p^u(T) = \gamma_p^y(T')$ + node weight of x or $w \leq \gamma_p(T') + 1 \leq \gamma_p(T) + 1$. Consequently, we have $|\gamma_p^u(T) - \gamma_p(T)| \leq 1$, for any vertex T. Hence T is γ_p-excellent.

Case 2: Let $u \in V(T)$ be arbitrary is obtained from T' by operation F_2. Assume $y \in T'$ and sta(y) = B. The tree T is obtained from T' by adding a path x, w, v and the edge xy to the vertex y. Let $sta(x) = sta(w) = A$ and $sta(v) = B$, and assign node weight as mentioned.

Subcase 1: If $y \in T'$ is in any $\gamma_p^y(T')$-set, then it can be extended to a $\gamma_p^u(T)$ - set of $T$by adding the vertex v, Since T' is γ_p-excellent and so,

$\gamma_p^u(T) = \gamma_p(T')$ + node weight of v

$\leq \gamma_p(T') + 1$

$\leq \gamma_p(T) + 1.$

Hence T is γ_p-excellent.

Subcase 2: If $y \in T'$ is not in $\gamma_p^z(T')$ -set for any $z \neq y$ in T', then any $\gamma_p^z(T')$-set can be extended to a $\gamma_p(T)$-set by adding to it, the vertex x and w, which is of status A. Thus, any $\gamma_p^z(T')$ -set can be extended to a $\gamma_p^u(T)$ - set of T,

$\gamma_p^u(T) = \gamma_p^z(T')$ + node weight of x and w in T-node weight of a vertex in T'.

$\leq \gamma_p(T') + 1$

$\leq \gamma_p(T) + 1$

Thus, we have $|\gamma_p^u(T) - \gamma_p(T)| \leq 1$, for any vertex in T. Hence T is γ_p-equitable excellent.

□

Theorem 4.3. *If T is a tree obtained from an equitable perfect dominating tree T' by adding a path x, w or a path x, w, v and an edge joining x to the vertex y of T', then* $|\gamma_p(T) - \gamma_p(T')| \leq 1$.

Proof. **Case1:** Suppose T is a tree obtained from a perfect dominating tree T' by adding a path x, w and an edge joining x to the vertex y of T', then any $gamma_p(T')$ -set can be extended to a perfect dominating set of T by adding x or w and so $\gamma_p(T) = \gamma_p(T')$ + node weight of x or w

$\leq \gamma_p(T') + 1$ (7)

Also, $\gamma_p(T') \leq \gamma_p(T) \leq (\gamma_p(T) + 1.$ (8)

By combining inequalities (7) and (8), we get, $|\gamma_p(T) - \gamma_p(T')| \leq 1$.

Case2: If T is a tree obtained from a perfect dominating tree T' by adding a path x, w, v and an edge joining x to the vertex y of T' .

Subcase 1: If $y \in T'$ is in $\gamma_p(T'$-set, then any $\gamma_p(T')$-set can be extended to a$\gamma_p(T)$-set by adding to it the vertex w, and thus we have,

$\gamma_p(T) = \gamma_p(T')$ + node weight of w

$\leq \gamma_p(T') + 1$ (9)

Also, $\gamma_p(T') \leq \ \gamma_p(T) \leq \gamma_p(T) + 1$. Thus,

$\gamma_p(T') - 1 \leq \gamma_p(T)$ (10)

By combining inequalities (9) and (10) we get, $|\gamma_p(T) - \gamma_p(T')| \leq 1$.

Subcase 2: If $y \in T'$ is not in $\gamma_p^z(T')$set for any $z \neq y$ in T', then any $\gamma_p^z(T')$ set can be extended to a $\gamma_p(T)$ -set by adding to it the vertex x and w. Thus any $\gamma_p^z(T')$-set can be extended to a $\gamma_p^u(T)$ - set of T. Thus,

$\gamma_p(T) = \gamma_p(T')$ +node weight of x and w in T-node weight of a vertex in T'
$\leq \gamma_p(T')$ +1. (11)
Also, $\gamma_p(T') \leq \gamma_p(T) \leq \gamma_p(T) + 1$
i.e., $\gamma_p(T') - 1 \leq \gamma_p(T)$ (12)
By combining inequalities (11) and (12) we get, $|\gamma_p(T) - \gamma_p(T')| \leq 1$

□

Theorem 4.4. *A tree T of order $n \geq 4$ is γ_p - excellent if and only if $T \in F$.*

Proof. By theorem4.2, it is sufficient to prove that the condition is necessary. We proceed by induction on the order n of a γ_p-excellent tree T. For $n = 4, T = P_4$ is γ_p -excellent and also it belongs to the family F. Assume that $n \geq 5$ and all γ_p -excellent trees with order less than n belong to F. Let T be a γ_p-excellent tree of order n. Let $P : v_1, v_2, \ldots, v_k$ be a longest path in T. obviously v_1, v_k are leaf vertices. $sta(v1) = sta(vk) = B$. We consider two possibilities.

Case 1: v_3 is a support vertex.

Let $T' = T - \{v_1, v_2\}$. We prove that T' is γ_p-set. Let $u \in T' \subset$ T and T is γ_p-excellent, there exists a $\gamma_p^u(T)$ set such that $|\gamma_p(T) - \gamma_p(T')| \leq 1$. Let S be a $\gamma_p^u(T)$-set and $S' = S \cap V(T')$. Then S' is a perfect dominating set of T'. Also, $\gamma_p(T') = |S'| = |S|$ -node weight of v_1 or $v_2 \geq \gamma_p(T) - 1$, by proposition 3.4,

$\gamma_p(T') + 1 \geq \gamma_p(T) \geq \gamma_p^u(T')$. (13)

Also, $\gamma_p(T') \leq \gamma_p^u(T') \leq \gamma_p^u(T') + 1$. (14)

By combining inequalities (13) and (14) we get, $|\gamma_p^u(T') - \gamma_p(T')| \leq 1$. Thus T' is γ_p -excellent. Hence by the inductive hypothesis $T' \in F$, since $|V(T')| < |V(T)|$. The $sta(v_3) = A$ in T', because v_3 is a support vertex. Thus, T is obtained from $T' \in F$ by the operation F_1. Hence $T \in F$ as desired.

Case 2: v_3 is not a support vertex.

Let $T' = T - \{v_1, v_2, v_3\}$. As in Case 1, we can prove that T' is γ_p-excellent. Since $|V(T')| < |V(T)|, T' \in F$ by the inductive hypothesis.

If v_4 is a support vertex or has a neighbor which is a support vertex in T', then v_3 is present in none of the γ_p-sets of T. So, T cannot be γ_p-excellent. Hence v_4 is a leaf of T' so that $v_4 \in B(T')$. Thus, T can be obtained from T' by the operation F_2. Hence $T \in F$.

□

5. Conclusion

In this paper, we have given a maximum bound of perfect domination number in a path and cycle. Then, we have constructed a perfect domination equitable excellent tree from path of length four by inductive method and characterized a fuzzy perfect domination equitable excellent tree. we would further extend this work in construction of total domination tree, mixed domination tree in various types of fuzzy graphs.

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