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THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE SYSTEM OF THE DIFFERENTIAL EQUATIONS PARTIALLY SOLVED RELATIVELY TO THE DERIVATIVES WITH NON-SQUARE MATRICES

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Abstract. The systems of ordinary differential equations, which are partially resolved relatively to the derivatives, were considered in case of a removable singularity and in case of a pole. The theorems on the existence of at least one analytic solution in the complex domain of the Cauchy problem with an additional condition are established for both cases. Moreover, the asymptotic behavior of these solutions in this domain is studied.

Keywords: Ordinary differential equation, pole, Cauchy's problem; complex domain, singularity, asymptotic behavior

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1. Introduction

One of the methods of researching the systems of differential equations which are not resolved relatively to the derivatives in the real-valued domain was suggested by R.Grabovskaya [1] and J. Diblic [2, 11]. It was developed in the complex domain in the articles by G. Samkova [7], N.Sharay [8,9], E.Michalenko, D.Limanska [3-6] and others. The current article is a continuation of the researching of the systems of differential equations that are not resolved relatively to the derivatives in the complex domain.

Let us consider the system of ordinary differential equations

$$A(z)Y^{'} = B(z)Y + f(z, Y, Y^{'}), \tag{1}$$

where matrices $A, B : D_1 \to \mathbb{C}^{m\times p}, D_1 = \{z : |z| < R_1, R_1 > 0\} \subset \mathbb{C}$, matrices $A(z), B(z)$ are analytic in the domain D_{10}, $D_{10} = D_1\backslash\{0\}$, the pencil of matrices $A(z)\lambda - B(z)$ is singular on the condition that $z \to 0$, function $f : D_1 \times G_1 \times G_2 \to \mathbb{C}^m$,where domains $G_k \subset \mathbb{C}^p, 0 \in G_k, k = 1, 2$, function $f(z, Y, Y^{'})$ is analytic in $D_{10} \times G_{10} \times G_{20}, G_{k0} = G_k\backslash\{0\}, k = 1, 2$.

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Let us research the system of ordinary differential equations (1) on the conditions that $m > p$ and $rangA(z) = p$ on the condition that $z \in D_1$.

Without restricting the generality, let's assume that matrices $A(z), B(z)$ and vector-function $f(z, Y, Y^{'})$ take the forms

$$A(z) = \begin{pmatrix} A_1(z) \\ A_2(z) \end{pmatrix}; B(z) = \begin{pmatrix} B_1(z) \\ B_2(z) \end{pmatrix}; f(z, Y, Y^{'}) = \begin{pmatrix} A_1(z, Y, Y^{'}) \\ A_2(z, Y, Y^{'}) \end{pmatrix} \tag{2}$$

$A_1 : D_1 \to \mathbb{C}^{p\times p}, A_2 : D_1 \to \mathbb{C}^{(m-p)\times p}, B_1 : D_1 \to \mathbb{C}^{p\times p}, B_2 : D_1 \to \mathbb{C}^{(m-p)\times p}, detA_1(z) \neq 0$ on the condition that $z \in D_1, f_1 : D_1 \times G_1 \times G_2 \to \mathbb{C}^p, f_2 : D_1 \times G_1 \times G_2 \to \mathbb{C}^{(m-p)}$.

The system (1) may be written as

$$\begin{cases} Y_1^{'} = A_1^{-1}(z)B_1(z)Y_1 + A_1^{-1}(z)f_1(z, Y, Y_1^{'}), & (3.1) \\ A_2(z)Y^{'} = B_2(z)Y + f_2(z, Y, Y^{'}), & (3.2) \end{cases} \tag{3}$$

where $A_1^{-1}(z)B_1(z)$ is analytic matrix in the domain $D_{10}, A_1^{-1}(z)f_1(z, Y, Y^{'})$ is analytic vector-function in the domain $D_{10} \times G_{10} \times G_{20}$.

Then vector-function $A_1^{-1}(z)f_1(z, Y, Y^{'})$ has an isolated singularity in the point $(0, 0, 0)$. Thus, according to the theorem about an isolated singularity for a function of several complex variables, point $(0, 0, 0)$ is a removable singularity of the function $A_1^{-1}(z)f_1(z, Y, Y^{'})$.

Let us complete definition of vector-function $A_1^{-1}(z)f_1(z, Y, Y^{'})$ in the point $(0, 0, 0)$ thus it became analytic function in the domain $D_1 \times G_1 \times G_2$ and, without restricting the generality, let's assume that $A_1^{-1}(0)f_1(0, 0, 0) = 0$.

Let us consider two cases:

(1) $A_1^{-1}(z)B_1(z)$ is analytic matrix in the domain D_{10} and has a removable singularity in the point $z = 0$
(2) $A_1^{-1}(z)B_1(z)$ is analytic matrix in the domain D_{10} and has a pole of order r in the point $z = 0$

For the first case let us introduce the following notations

$$A_1^{-1}(z)B_1(z) = P^{(1)}(z), A_1^{-1}f_1(z, Y, Y^{'}) = F(z, Y, Y^{'}) \tag{4}$$

then for the first case the system (3.1) may be written as

$$Y^{'} = P^{(1)}(z)Y + F(z, Y, Y^{'}) \tag{5}$$

where $P^{(1)} : D_1 \to \mathbb{C}^{p\times p}, P^{(1)}(z)$ is analytic matrix in the domain $D_1, F^{(1)}(z, Y, Y^{'})$ is analytic vector-function in the domain $D_1 \times G_1 \times G_2$.

For the second case let us introduce the following notations

$$A_1^{-1}(z)B_1(z) = z^{-r}P^{(2)}(z), A_1^{-1}f_1(z, Y, Y^{'}) = F(z, Y, Y^{'}) \tag{6}$$

then for the second case the system (3.1) may be written as

$$Y^{'} = z^{-r}P^{(2)}(z)Y + F(z, Y, Y^{'}) \tag{7}$$

where $P^{(2)} : D_1 \to \mathbb{C}^{p\times p}, P^{(2)}(z)$ is analytic matrix in the domain D_1.

We research the questions of the analytic solutions existence of the system (3) for both cases that satisfy the initial condition

$$Y(z) \to 0 \,\text{on the condition}\; z \to 0, z \in D_{10}, \tag{8}$$

and additional condition

$$Y'(z) \to 0 \,\text{on the condition}\; z \to 0, z \in D_{10}, \tag{9}$$

2. Introduction of some intermediary notation

For arbitrary fixed $t_1 > 0, v_1, v_2 \in \mathbb{R}, v_1 < v_2$, we introduce auxiliary sets: $\breve{I}(t_1) = \{(t,v) \in \mathbb{R}^2 : t \in (0,t_1), v \in (v_1,v_2)\}; L_{v_0}(t_1) = \{(t,v) \in \mathbb{R}^2 : t \in (0,t_1), v = v_0 \in (v_1,v_2)\}, v_0$ is a fixed number. For arbitrary $t_0 \in (0,t_1), O_{t_1}(t_0) = \{(t,v) \in \mathbb{R}^2 : t = t_0, v \in (v_1,v_2)\}$.

On the condition that $z = z(t,v) = te^{iv}$, we associate the set $\breve{I}(t_1) \subset \mathbb{R}^2$ with the set $I(t_1) \subset \mathbb{C} : I(t_1) = \{z = te^{iv} \in \mathbb{C} : t \in (0,t_1), v \in (v_1,v_2)\}$.

Definition 2.1. *Let functions $p, g : \breve{I}(t_1) \to [0,+\infty)$. We say that a function $p(t,v)$ possesses the property Q_1 with respect to the function $g(t,v)$ on the condition that $v = v_0 \in (v_1,v_2)$, if $p(t,v_0)$ is the function of higher order of smallness with respect to the function $g(t,v_0)$ on the condition that $t \to +0$.*

Definition 2.2. *Let functions $p, g : \breve{I}(t_1) \to [0,+\infty)$. We say that a function $p(t,v)$ possesses the property Q_2 with respect to the function $g(t,v)$ on the set $\breve{I}(t_1)$, if there exist numbers $C_1 \geq 0, C_2 \geq 0$, such that the inequality*

$$C_1 g(t,v) \leq p(t,v) \leq C_2 g(t,v)$$

is true on the set $\breve{I}(t_1)$.

We introduce auxiliary vector functions as follows: $\varphi^{(0)}(z) = (\varphi_1^{(0)}(z), ..., \varphi_p^{(0)}(z)), \varphi^{0)} : I(t_1) \to \mathbb{C}^p, \psi^{(0)}(t,v) = (\psi_1^{(0)}(t,v), ..., \psi_p^{(0)}(t,v)), \psi_j^{(0)} : \breve{I}(t_1) \to [0;+\infty), j = \overline{1,p}$, on the condition that $z = z(t,v) = te^{iv}, \psi_j^{(0)}(t,v) =\mid \varphi_j^{(0)}(z(t,v)) \mid, j = \overline{1,p}$.

Let us choose the vector function $\varphi^{(0)}(z)$ that is analytic on the set $I(t_1)$ and, for any $z \in I(t_1)$, the following conditions are true for the vector-function:

$\psi_j^{(0)}(t,v) > 0; (\psi_j^{(0)}(t,v))'_t > 0; (\psi_j^{(0)}(t,v))'_v \geq 0; \psi_j^{(0)}(+0,v) = 0, (\psi_j^{(0)}(+0,v))'_t = 0, j = \overline{1,p}$ uniformly in $v \in (v_1,v_2), (t,v) \in \breve{I}(t_1)$.

3. Transformation of systems (5) and (7) behavior on the segment $L_{v_0}(t_1)$ and along arc $O_{t_1}(t_0)$

Let us consider the systems (5) and (7) on the segment $L_{v_0}(t_1)$ for any fixed $v_0 \in (v_1,v_2)$.

For $z = z(t,v_0) = te^{iv_0}$ we represent each function and matrix in systems (5) and (7) in the algebraic form by separating the real and imaginary parts and introducing the following notation:

$Y(z(t,v_0)) = \widetilde{Y}(t), \widetilde{Y}(t) = \widetilde{Y}_1(t) + i\widetilde{Y}_2(t), \widetilde{Y}_j(t) = col(\widetilde{Y}_{j1}(t), ..., \widetilde{Y}_{jp}(t)), j = 1,2,$
$P^{(\beta)}(z(t,v_0)) = \|\widetilde{p}_{jk}^{(\beta)}(t)\|_{j,k=1}^p = \widetilde{P}_1^{(\beta)}(t) + i\widetilde{P}_2^{(\beta)}(t), \widetilde{P}_s^{(\beta)}(t) = \|\widetilde{p}_{jks}^{(\beta)}(t)\|_{j,k=1}^p, s = 1,2$, where $\widetilde{p}_{jk}^{(\beta)}(t) = \widetilde{p}_{jk1}^{(\beta)}(t) + i\widetilde{p}_{jk2}^{(\beta)}(t), j,k = \overline{1,p}, \beta = 1,2,$
$F(z(t,v_0), Y(z(t,v_0)), Y'(z(t,v_0))) = \widetilde{F}(t,\widetilde{Y},\widetilde{Y}'), \widetilde{F}(t,\widetilde{Y},\widetilde{Y}') = col(\widetilde{F}_1(t,\widetilde{Y},\widetilde{Y}'), ..., \widetilde{F}_p(t,\widetilde{Y},\widetilde{Y}')), \widetilde{F}_j(t,\widetilde{Y},\widetilde{Y}') = \widetilde{F}_{1j}(t,\widetilde{Y},\widetilde{Y}') + i\widetilde{F}_{2j}(t,\widetilde{Y},\widetilde{Y}'), j = \overline{1,p}.$

Let us consider the systems (5) and (7) along arc $O_{t_1}(t_0)$ of the circle for any fixed $t_0 \in (0,t_1)$.

For $z = z(t_0,v) = t_0 e^{iv}$ we represent each function and matrix in systems (5) and (7) in the algebraic form by separating the real and imaginary parts and introducing the following notation $Y(z(t_0,v)) = \widehat{Y}(v), \widehat{Y}(v) = \widehat{Y}_1(v) + i\widehat{Y}_2(v), \widehat{Y}_j(v) = col(\widehat{Y}_{j1}(v), ..., \widehat{Y}_{jp}(v)), j = 1,2,$ $P^{(\beta)}(z(t_0,v)) = \|\widehat{p}_{jk}^{(\beta)}(v)\|_{j,k=1}^p = \widehat{P}_1^{(\beta)}(v) + i\widehat{P}_2^{(\beta)}(v), \widehat{P}_s^{(\beta)}(v) = \|\widehat{p}_{jks}^{(\beta)}(v)\|_{j,k=1}^p, s = 1,2,$ where $\widehat{p}_{jk}^{(\beta)}(v) = \widehat{p}_{jk1}^{(\beta)}(v) + i\widehat{p}_{jk2}^{(\beta)}(v), j,k = \overline{1,p}, \beta = 1,2,$

$F(z(t_0,v),Y(z(t_0,v)),Y'(z(t_0,v)))=\widehat{F}(v,\widehat{Y},\widehat{Y}'),\widehat{F}(v,\widehat{Y},\widehat{Y}')=col(\widehat{F}_1(v,\widehat{Y},\widehat{Y}'),...,$
$\widehat{F}_p(v,\widehat{Y},\widehat{Y}')),\widehat{F}_j(v,\widehat{Y},\widehat{Y}')=\widehat{F}_{1j}(v,\widehat{Y},\widehat{Y}')+i\widehat{F}_{2j}(v,\widehat{Y},\widehat{Y}'),j=\overline{1,p}.$

4. Some classes of functions and the properties of systems (5) and (7)

Definition 4.1. *We say that the matrix $P^{(\beta)}(z),\beta\in 1,2$ has the property S_1 with respect to the vector-function $\varphi^{(0)}(z)$, if the following conditions are satisfied:*

1. *For each $v_0\in(v_1,v_2)$ functions $(\psi_j^{(0)}(z(t,v)))_t'$ possess the property Q_1 with respect to functions $\mid \widetilde{p}_{jj}^{(\beta)}(t)\mid \psi_j^{(0)}(z(t,v)),j=\overline{1,p}$, on the conditions that $v=v_0$;*
2. *Functions $(\psi_j^{(0)}(t,v))_v'$ possess the property Q_2 with respect to functions $\mid \widehat{p}_{jj}^{(\beta)}(v)\mid$ $\psi_j^{(0)}(t,v),j=\overline{1,p}$ on the set $\breve{I}(t_2)$ for some $t_2\in(0,t_1]$;*
3. *For each $v_0\in(v_1,v_2)$ functions $\mid \widetilde{p}_{jk}^{(\beta)}(t)|\psi_k^{(0)}(t,v)$ possess the property Q_1 with respect to functions $(\psi_j^{(0)}(t,v))_v',j,k=\overline{1,p},j\neq k$, on the conditions that $v=v_0$;*
4. *Functions $|\widehat{p}_{jk}^{(\beta)}(v)|\psi_k^{(0)}(t,v)$ possess the property Q_2 with respect to functions $(\psi_j^{(0)}(t,v))_v'$, $j,k=\overline{1,p},j\neq k$ on the set $\breve{I}(t_2)$ or some $t_2\in(0,t_1]$.*

Definition 4.2. *We say that the vector-function $P^{(\beta)}(z),\beta\in 1,2$ possesses the property S_2 with respect to the vector-function $\varphi^{(0)}(z)$, if the following conditions are satisfied*

1. *For each $v_0\in(v_1,v_2)$ functions $t^r(\psi_j^{(0)}(z(t,v)))_t'$ possess the property Q_1 with respect to functions $\mid \widetilde{p}_{jj}^{(\beta)}(t)\mid \psi_j^{(0)}(z(t,v)),j=\overline{1,p}$, on the conditions that $v=v_0$;*
2. *Functions $t^{r-1}(\psi_j^{(0)}(t,v))_v'$ possess the property Q_2 with respect to functions $\mid \widehat{p}_{jj}^{(\beta)}(v)\mid$ $\psi_j^{(0)}(t,v),j=\overline{1,p}$on the set $\breve{I}(t_2)$ for some $t_2\in(0,t_1]$;*
3. *For each $v_0\in(v_1,v_2)$ functions $\mid \widetilde{p}_{jk}^{(\beta)}(t)\mid \psi_k^{(0)}(t,v)$ possess the property Q_1 with respect to functions $t^r(\psi_j^{(0)}(t,v))_v',j,k=\overline{1,p},j\neq k$, on the conditions that $v=v_0$;*
4. *Functions $\mid \widehat{p}_{jk}^{(\beta)}(v)|\psi_k^{(0)}(t,v)$ possess the property Q_2 with respect to functions $t^{r-1}(\psi_j^{(0)}(t,v))_v',j,k=\overline{1,p},j\neq k$ on the set $\breve{I}(t_2)$ for some $t_2\in(0,t_1]$.*

Denote the sets

$$\widetilde{\Omega}(\delta,\varphi^{(0)}(z(t,v_0)))=\{(t,\widetilde{Y}_1,\widetilde{Y}_2):t\in(0,t_1),\widetilde{Y}_{1j}^2+\widetilde{Y}_{2j}^2<\delta_j^2(\psi_j^{(0)}(t,v_0))^2,j=\overline{1,p}\},$$
$$v_0 \text{ is fixed on } (v_1,v_2) \tag{10}$$
$$\widehat{\Omega}(\tau,\varphi^{(0)}(z(t_0,v)))=\{(v,\widehat{Y}_1,\widehat{Y}_2):v\in(v_1,v_2),\widehat{Y}_{1j}^2+\widehat{Y}_{2j}^2<\tau_j^2(\psi_j^{(0)}(t_0,v))^2,j=\overline{1,p}\},$$
$$t_0 \text{ is fixed on } (0,t_1) \tag{11}$$

where $\delta=(\delta_1,...,\delta_p),\tau=(\tau_1,...,\tau_p),\delta_j,\tau_j\in\mathbb{R}\backslash\{0\},j=\overline{1,p}$.

Definition 4.3. *We say that the vector-function $F(z,Y,Y')$ possesses the property $M_\beta,\beta\in\{1,2\}$ with respect to the vector-function $\varphi^{(0)}(z)$, if the following conditions are satisfied:*

1. *For each $v_0\in(v_1,v_2)$ and for $(t,\widetilde{Y}_1,\widetilde{Y}_2)\in\widetilde{\Omega}(\delta,\varphi^{(0)}(z(t,v_0)))$ functions $\widetilde{F}_{kj}(t,\widetilde{Y}_1,\widetilde{Y}_2,\widetilde{Y}_1',\widetilde{Y}_2')$ possess the property Q_1 with respect to functions $\mid \widetilde{p}_{jj}^{(\beta)}(z(t,v))\mid$ $\times\psi_j^{(0)}(t,v),j=\overline{1,p},k=1,2$ on the conditions that $v=v_0$;*
2. *For $(v,\widehat{Y}_1,\widehat{Y}_2)\in\widehat{\Omega}(\tau,\varphi^{(0)}(z(t_0,v)))$ functions $\widehat{F}_{kj}(v,\widehat{Y}_1,\widehat{Y}_2,\widehat{Y}_1',\widehat{Y}_2')$ possess the property Q_2 with respect to functions $\mid \widehat{p}_{jj}^{(\beta)}(z(t,v))\mid \times\psi_j^{(0)}(t,v)),j=\overline{1,p},k=1,2$ on the set $\breve{I}(t_2)$ for some $t_2\in(0,t_1]$.*

Without restricting the generality, let us suppose that $0 < t_2 \leq t_1 \leq R_1$. Further, we introduce the following domains $\Lambda^{(\beta)}_{+.k}(t_2), k \in \{+,-\}, \beta = 1,2$

$$\Lambda^{(\beta)}_{+.+}(t_2) = \{(t,v) : cos((r-1)v + \widetilde{a}^{(\beta)}_{jk}(t)) > 0,$$
$$sin((r-1)v + \widehat{a}^{(\beta)}_{jk}(v)) > 0, j = \overline{1,p}, t \in (0,t_2), v \in (v_1,v_2)\}; \beta = 1,2$$
$$\Lambda^{(\beta)}_{+.-}(t_2) = \{(t,v) : cos((r-1)v + \widetilde{a}^{(\beta)}_{jk}(t)) > 0,$$
$$sin((r-1)v + \widehat{a}^{(\beta)}_{jk}(v)) < 0, j = \overline{1,p}, t \in (0,t_2), v \in (v_1,v_2)\}; \beta = 1,2$$

where functions $\widetilde{a}^{(\beta)}_{jk}(t), \widehat{a}^{(\beta)}_{jk}(v), j,k = \overline{1,p}, \beta = 1,2$ can be defined as

$$cos(\widetilde{a}^{(\beta)}_{jk}(t)) = \frac{\widetilde{p}^{(\beta)}_{jk1}(t)}{\sqrt{\widetilde{p}^{(\beta)}_{jk1}(t)^2 + \widetilde{p}^{(\beta)}_{jk2}(t)^2}}, j,k = \overline{1,p}, \beta = 1,2$$

$$sin(\widetilde{a}^{(\beta)}_{jk}(t)) = \frac{\widetilde{p}^{(\beta)}_{jk2}(t)}{\sqrt{\widetilde{p}^{(\beta)}_{jk1}(t)^2 + \widetilde{p}^{(\beta)}_{jk2}(t)^2}}, j,k = \overline{1,p}, \beta = 1,2$$

$$cos(\widehat{a}^{(\beta)}_{jk}(t)) = \frac{\widehat{p}^{(\beta)}_{jk1}(t)}{\sqrt{\widehat{p}^{(\beta)}_{jk1}(t)^2 + \widehat{p}^{(\beta)}_{jk2}(t)^2}}, j,k = \overline{1,p}, \beta = 1,2$$

$$sin(\widehat{a}^{(\beta)}_{jk}(t)) = \frac{\widehat{p}^{(\beta)}_{jk2}(t)}{\sqrt{\widehat{p}^{(\beta)}_{jk1}(t)^2 + \widehat{p}^{(\beta)}_{jk2}(t)^2}}, j,k = \overline{1,p}, \beta = 1,2$$

Definition 4.4. *We say that system (5) belongs to the class $C^{(1)}_{+.k}, k \in \{+,-\}$, if the matrix $P^{(1)}(z) = P^{(1)}(te^{iv})$ is such that $(t,v) \in \Lambda^{(1)}_{+.k}(t_2), k \in \{+,-\}$.*

Definition 4.5. *We say that system (7) belongs to the class $C^{(2)}_{+.k}, k \in \{+,-\}$, if the matrix $P^{(2)}(z) = P^{(2)}(te^{iv})$ is such that $(t,v) \in \Lambda^{(2)}_{+.k}(t_2), k \in \{+,-\}$.*

5. Main results

Let us introduce the following domains $G^{(\beta)}_{+.k}(t_2) = \{z = z(t,v) : 0 < |z| < t_2, (t,v) \in \Lambda^{(\beta)}_{+.k}(t_2)\}, k \in \{+,-\}, \beta = 1,2$.

Theorem 5.1. *Let us suppose that $m > p, A(z)$ is analytic matrix in the domain D_1 and $rangA(z) = p$ on the condition that $z \in D_1$. Suppose that the system (1) can be reduced to the form (5)-(3.2). Moreover, suppose that the system (5) satisfies the following conditions:*

1. *The matrix $P^{(1)}(z)$ is analytic in the domain D_1 and has the property S_1 with respect to the analytic vector-function $\varphi^{(0)}(z)$;*
2. *The vector-function $F(z,Y,Y')$ is analytic in the domain $D_1 \times G_1 \times G_2, F(0,0,0) = 0$ and has the property M_1 with respect to the analytic vector function $\varphi^{(0)}(z)$;*
3. *The system (5) belongs to one of the classes $C^{(1)}_{+.k}, k \in \{+,-\}$;*
4. *The matrices $A_2(z), B_2(z)$ and the vector-function $f_2(z,Y,Y')$ have such form that the compatibility conditions with (3.2) are fulfilled along the solution of the system (5) on the condition that $z \in D_1, D_1 \cap G^{(1)}_{+.k}(t^*) \neq \emptyset, k \in \{+,-\}$;*

Then, for each $k \in \{+,-\}$ and for some $t^* \in (0,t_2)$ there exist such solutions of the system (1) $Y(z)$, satisfying the initial conditions $Y(z_0) = Y_0$ for $z_0 \in G^{(1)}_{+.k}(t^*), Y_0 \in \{Y : |Y_j(z_0)| < \delta_j \mid \varphi^{(0)}_j(z_0) \mid, \delta_j > 0, j = \overline{1,p}\}$, are analytic in the domain $D_1 \cap G^{(1)}_{+.k}(t^*)$ and these solutions admit the estimates

$$\mid Y_j(z) \mid^2 < \delta_j^2 \mid \varphi_j^{(0)}(z) \mid^2, j = \overline{1,p} \tag{12}$$

in the domain $D_1 \cap G^{(1)}_{+.k}(t^*)$.

Proof. Taking into account the representations (2), let us reduce the system (1) to the system (3). Assuming that $A_1^{-1}(z)B_1(z)$ is analytic matrix in the domain D_{10} and has removable singularity in the point $z = 0$, the system (3) may be written as (5)-(3.2).

Let us adapt the results of the theorem 1 [3] for the system (1). We get that the system (5) that satisfies the initial condition $Y_0 = Y(z_0)$ for $z_0 \in G^{(1)}_{+.k}(t^*), Y_0 \in \{Y :\mid Y_j(z_0) \mid < \delta_j \mid \varphi^{(0)}_j(z_0) \mid, \delta_j > 0, j = \overline{1,p}\}, k \in \{+,-\}$, has at least one analytic solution in the domain $D_1 \cap G^{(1)}_{+.k}(t^*)$. Moreover, inequalities (12) are true for every solution in this domain.

If the matrices $A_2(z), B_2(z)$ and the vector-function $f_2(z, Y, Y^{'})$ have such form that the compatibility conditions with (3.2) are fulfilled along the solution of the system (5) on the condition that $z \in D_1, D_1 \cap G^{(1)}_{+.k}(t^*) \neq \emptyset, k \in \{+,-\}$, then the system (1), that satisfies the initial condition $Y_0 = Y(z_0)$, has at least one analytic solution in the domain $D_1 \cap G^{(1)}_{+.k}(t^*)$. Moreover, inequalities (12) are true for every solution in this domain. The theorem is proved.

Theorem 5.2. *Let us suppose that $m > p, A(z)$ is analytic matrix in the domain D_1 and $rangA(z) = p$ on the condition that $z \in D_1$. Suppose that the system (1) can be reduced to the form (7)-(3.2). Moreover, suppose that the system (7) satisfies the following conditions:*

1. *The matrix $P^{(2)}(z)$ is analytic in the domain D_1 and has the property S_2 with respect to the analytic vector-function $\varphi^{(0)}(z)$;*
2. *The vector-function $F(z, Y, Y^{'})$ is analytic in the domain $D_1 \times G_1 \times G_2, F(0,0,0) = 0$ and has the property M_2 with respect to the analytic vector function $\varphi^{(0)}(z)$;*
3. *The system (7) belongs to one of the classes $C^{(2)}_{+.k}, k \in \{+,-\}$;*
4. *The matrices $A_2(z), B_2(z)$ and the vector-function $f_2(z, Y, Y^{'})$ have such form that the compatibility conditions with (3.2) are fulfilled along the solution of the system (7) on the condition that $z \in D_1, D_1 \cap G^{(2)}_{+.k}(t^*) \neq \emptyset, k \in \{+,-\}$;*

Then, for each $k \in \{+,-\}$ and for some $t^* \in (0,t_2)$ there exist such solutions of the system (1) $Y(z)$, satisfying the initial conditions $Y(z_0) = Y_0$ for $z_0 \in G^{(2)}_{+.k}(t^*), Y_0 \in \{Y : |Y_j(z_0)| < \delta_j \mid \varphi^{(0)}_j(z_0) \mid, \delta_j > 0, j = \overline{1,p}\}$, are analytic in the domain $D_1 \cap G^{(2)}_{+.k}(t^*)$ and these solutions admit the estimates (12) in the domain $D_1 \cap G^{(2)}_{+.k}(t^*)$.

Proof

Taking into account the representations (2), let us reduce the system (1) to the system (3). Assuming that $A_1^{-1}(z)B_1(z)$ is analytic matrix in the domain D_{10} and has removable singularity in the point $z = 0$, the system (3) may be written as (7)-(3.2).

Let us adapt the results of the theorem 1 [4] for the system (1). We get that the system (7) that satisfies the initial condition $Y_0 = Y(z_0)$ for $z_0 \in G^{(2)}_{+.k}(t^*), Y_0 \in \{Y :\mid Y_j(z_0) \mid <$

$\delta_j \mid \varphi_j^{(0)}(z_0) \mid, \delta_j > 0, j = \overline{1,p}\}, k \in \{+,-\}$, has at least one analytic solution in the domain $D_1 \cap G_{+.k}^{(2)}(t^*)$. Moreover, inequalities (12) are true for every solution in this domain.

If the matrices $A_2(z), B_2(z)$ and the vector-function $f_2(z, Y, Y')$ have such form that the compatibility conditions with (3.2) are fulfilled along the solution of the system (7) on the condition that $z \in D_1, D_1 \cap G_{+.k}^{(2)}(t^*) \neq \emptyset, k \in \{+,-\}$, then the system (1), that satisfies the initial condition $Y_0 = Y(z_0)$, has at least one analytic solution in the domain $D_1 \cap G_{+.k}^{(2)}(t^*)$. Moreover, inequalities (12) are true for every solution in this domain. The theorem is proved. □

6. Conclusions

The sufficient conditions of the existence of the analytical solutions for systems of differential equations (5) and (7), partially solved relatively to the derivatives, with non-square matrices, in the presence of a removable singularity or a pole $z = 0$, were found. It was found an estimate for these solutions in the neighborhood of the point $z = 0$.

Theorems of the existence of the solutions for the system (1) were proved. These solutions are analytic in the domain with the zero-point on a border and have estimates in this domain.

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