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THE EIGENVALUE PROBLEM FOR FUZZY DIFFUSION OPERATORS

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Abstract. This paper deals with the eigenvalues of a boundary value problem for a second order fuzzy differential equation. We consider the fuzzy differential operator and give the properties of the eigenvalues of the problem for fuzzy differential pencils. We use the derivative of the functions in the sense of Hukuhara.

Keywords: Fuzzy number, Fuzzy differential equation, Eigenvalue.

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1. Introduction

Fuzzy set theory is a very useful tool to describe the unclear modeling and ambiguous processing in mathematical models. For this reason the fuzzy differential equations are very important from the theoretical point of view [2, 5, 8, 9]. The applications of this class of differential equations are seen in population models, medicine, civil engineering, etc (see for examples [6, 7, 13]). The concept of the fuzzy number has been first given by Zadeh, Dubois and Prade [4, 15]. For more information, we refer the authors to the reference [3, 11, 14].

Consider the fuzzy boundary value problem (FBVP) L of the following form

$$-y'' + (\rho p_1(x) + p_2(x))y = \lambda y, \quad x \in (0, T), \tag{1}$$

$$U(y) := y'(0) - hy(0) = 0, \tag{2}$$

$$V(y) := y'(T) + Hy(T) = 0. \tag{3}$$

The parameters $h, H \in \mathbb{R}_F$ and $\lambda = \rho^2$ is a spectral parameter. Also the real functions $p_1(x)$ and $p_2(x)$ are continuous fuzzy functions.

In this paper, the eigenvalues of the fuzzy differential pencil with two boundary conditions are studied under the approach of Hukuhara differentiability.

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2. Preliminaries

In this section, we bring some definitions and necessary notations which will be needed later.

Definition 2.1. [10, 12] *A fuzzy number is a function $u : \mathbb{R} \longrightarrow [0, 1]$ satisfying the following properties: u is normal, convex fuzzy set, upper semi-continuous on $\mathbb{R}$ and the set $\overline{\{x \in \mathbb{R};\ u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.*

Then $\mathbb{R}_F$ is called the space of fuzzy numbers.

Definition 2.2. [14] *For $0 < \alpha \leq 1$, the α-level set $[u]^\alpha$ of a fuzzy set u on $\mathbb{R}$ is defined as $[u]^\alpha = \{x \in \mathbb{R};\ u(x) \geq \alpha\}$, while its support $[u]^0$ is the closure in topology $\mathbb{R}$ of the union of all level sets, that is*

$$[u]^0 = \overline{\bigcup_{\alpha \in (0,1]} [u]^\alpha} = \overline{\{x \in \mathbb{R};\ u(x) > 0\}}.$$

Then, it is well-known [1], the notation $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$ is a bounded closed interval in $\mathbb{R}$, considering $\underline{u}_\alpha$ as the left-hand endpoint of $[u]^\alpha$ and $\overline{u}_\alpha$ as the right-hand endpoint of $[u]^\alpha$.

Definition 2.3. [11] *For $u, v \in \mathbb{R}_F$ and $\mu \in \mathbb{R}$, the sum u and v and the product μu are defined by $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$ and $[\mu u]^\alpha = \mu [u]^\alpha$.*

The metric on $\mathbb{R}_F$ is written as described below

$$d(u, v) = sup_{\alpha \in [0,1]} d_H([u]^\alpha, [v]^\alpha), \quad u, v \in \mathbb{R}_F,$$

for the Hausdorff distance

$$d_H(A, B) = max\{sup_{a \in A} inf_{b \in B} |a - b|, sup_{b \in B} inf_{a \in A} |a - b|\},$$

in which A and B are two nonempty bounded subsets of $\mathbb{R}$.

Definition 2.4. [14] *Let I be a real interval. The mapping $f : I \longrightarrow \mathbb{R}_F$ is called a fuzzy function and its α-level set is denoted by $[f(t)]^\alpha = [\underline{f}_\alpha(t), \overline{f}_\alpha(t)]$ for $t \in I, \alpha \in [0, 1]$, where $\underline{f}_\alpha(t)$ and $\overline{f}_\alpha(t)$ denote respectively the left-hand endpoint and the right-hand endpoint of $[f(t)]^\alpha$; more precisely,*

$$\underline{f}^\alpha(t) = min\{s \in [w]^\alpha;\ w \in \mathbb{R}_F,\ w = f(t)\},$$
$$\overline{f}^\alpha(t) = max\{s \in [w]^\alpha;\ w \in \mathbb{R}_F,\ w = f(t)\}.$$

Definition 2.5. [12] *Let $x, y \in \mathbb{R}_F$. If there exists $z \in \mathbb{R}_F$ such that $x = y + z$, then z is called the Hukuhara difference of x and y and it is denoted by $x \ominus y$.*

Definition 2.6. [12] *Let I be an open interval in $\mathbb{R}$. A fuzzy function $f : I \longrightarrow \mathbb{R}_F$ is called to be Hukuhara differentiable at $t_0 \in I$, if there exist the limits*

$$\lim_{h \longrightarrow 0^+} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \longrightarrow 0^+} \frac{f(t_0) \ominus f(t_0 - h)}{h},$$

and equal to $f'(t_0)$.

Theorem 2.1. [9] *Let $f : I \longrightarrow \mathbb{R}_F$ be a fuzzy function and denote $[f(t)]^\alpha = [\underline{f}_\alpha(t), \overline{f}_\alpha(t)]$ for each $\alpha \in [0, 1]$. If f is Hukuhara differentiable, then $\underline{f}_\alpha(t)$ and $\overline{f}_\alpha(t)$ are differentiable functions and $[f'(t)]^\alpha = [(\underline{f}_\alpha)'(t), (\overline{f}_\alpha)'(t)]$.*

3. Main result

We consider
$$[y(x,\rho)]^\alpha = [\underline{y}_\alpha(x,\rho), \overline{y}_\alpha(x,\rho)], \tag{4}$$
as the solution of the fuzzy equation (1) in which
$$\underline{y}_\alpha(x,\rho) = h_{11}(\alpha,\rho)y_1(x,\rho) + h_{12}(\alpha,\rho)y_2(x,\rho), \tag{5}$$
$$\overline{y}_\alpha(x,\rho) = h_{21}(\alpha,\rho)y_1(x,\rho) + h_{22}(\alpha,\rho)y_2(x,\rho). \tag{6}$$
Here $y_1(x,\rho)$ and $y_2(x,\rho)$ are two linearly independent functions satisfying the pencil
$$-y'' + (\rho p_1(x) + p_2(x))y = \lambda y, \quad x \in (0,T). \tag{7}$$

Let $[\varphi(x,\rho)]^\alpha = [\underline{\varphi}_\alpha(x,\rho), \overline{\varphi}_\alpha(x,\rho)]$ and $[\psi(x,\rho)]^\alpha = [\underline{\psi}_\alpha(x,\rho), \overline{\psi}_\alpha(x,\rho)]$ be the solutions of the fuzzy equation (1) satisfying the conditions
$$\varphi(0,\rho) = 1, \qquad \varphi'(0,\rho) = h, \tag{8}$$
$$\psi(T,\rho) = 1, \quad \psi'(T,\rho) = -H. \tag{9}$$
For each fixed x, these functions and derivatives are entire in ρ.

Theorem 3.1. *Let $[y(x,\rho)]^\alpha$ and $[z(x,\rho)]^\alpha$ be the solutions of the fuzzy differential equation (1). The Wronskian of these functions is independent of x. In other words,*
$$W(\underline{y}_\alpha, \underline{z}_\alpha)(x,\rho) = \underline{W}_\alpha(\rho), \quad W(\overline{y}_\alpha, \overline{z}_\alpha)(x,\rho) = \overline{W}_\alpha(\rho), \tag{10}$$
where $W(y,z) := yz' - y'z$.

Proof. It is sufficient to show that $W'(\underline{y}_\alpha, \underline{z}_\alpha)(x,\rho) = 0$ and $W'(\overline{y}_\alpha, \overline{z}_\alpha)(x,\rho) = 0$. Since
$$W(\underline{y}_\alpha, \underline{z}_\alpha)(x,\rho) = \underline{y}_\alpha(x,\rho)\underline{z}'_\alpha(x,\rho) - \underline{y}'_\alpha(x,\rho)\underline{z}_\alpha(x,\rho),$$
by differentiation of the above equation with respect to x, i.e.,
$$\frac{d}{dx}W(\underline{y}_\alpha, \underline{z}_\alpha)(x,\rho) = \frac{d}{dx}(\underline{y}_\alpha(x,\rho)\underline{z}'_\alpha(x,\rho) - \underline{y}'_\alpha(x,\rho)\underline{z}_\alpha(x,\rho)),$$
we have
$$\frac{d}{dx}W(\underline{y}_\alpha, \underline{z}_\alpha)(x,\rho) = \underline{y}_\alpha(x,\rho)\underline{z}''_\alpha(x,\rho) - \underline{y}''_\alpha(x,\rho)\underline{z}_\alpha(x,\rho).$$
Since the functions $\underline{y}_\alpha(x,\rho)$ and $\underline{z}_\alpha(x,\rho)$ are the solutions of the fuzzy differential equation (1), we get
$$W'(\underline{y}_\alpha, \underline{z}_\alpha)(x,\rho) = 0.$$
Similarly we can prove $W'(\overline{y}_\alpha, \overline{z}_\alpha)(x,\rho) = 0$. This completes the proof. □

Now we study the properties of the eigenvalues of (1)-(3). Denote the characteristic function of (1)-(3) by
$$\Delta_\alpha(\rho) = [\underline{W}_\alpha(\rho), \overline{W}_\alpha(\rho)], \tag{11}$$
where
$$\underline{W}_\alpha(\rho) = W(\underline{\varphi}_\alpha, \underline{\psi}_\alpha)(x,\rho), \quad \overline{W}_\alpha(\rho) = W(\overline{\varphi}_\alpha, \overline{\psi}_\alpha)(x,\rho). \tag{12}$$

Definition 3.1. *The values of the parameter λ are called the eigenvalues of (1)-(3) if the fuzzy equation (1) has the nontrivial solutions $[y]^\alpha$ satisfying (2)-(3). These corresponding solutions are called eigenfunctions of L.*

Theorem 3.2. *The roots of $\Delta_\alpha(\rho)$ coincide with the eigenvalues of the FBVP (L).*

Proof. Let ρ_0 be the zero's of the $\Delta_\alpha(\rho)$. From (11) and (12), we have

$$\underline{\varphi}_\alpha(x,\rho_0) = k_1\underline{\psi}_\alpha(x,\rho_0), \quad \overline{\varphi}_\alpha(x,\rho_0) = k_2\overline{\psi}_\alpha(x,\rho_0), \tag{13}$$

for $k_1,\ k_2 \neq 0$. Since the functions $\underline{\psi}_\alpha(x,\rho_0)$ and $\overline{\psi}_\alpha(x,\rho_0)$ satisfy (3), from (13), we can result that the solutions $\underline{\varphi}_\alpha(x,\rho_0)$ and $\overline{\varphi}_\alpha(x,\rho_0)$ also satisfy (3). So $[\varphi(x,\rho_0)]^\alpha$ is the nontrivial solution of FBVP(L). Thus $\rho=\rho_0$ is an eigenvalue. Conversely, let ρ_0 be an eigenvalue and $[y(x,\rho_0)]^\alpha$ be a corresponding eigenfunction of the fuzzy boundary value problem L. It is trivial that $U([y(x,\rho_0)]^\alpha)=0$ and $V([y(x,\rho_0)]^\alpha)=0$. Thus we infer that

$$[y(x,\rho_0)]^\alpha = \gamma_1[\varphi(x,\rho_0)]^\alpha, \qquad [y(x,\rho_0)]^\alpha = \gamma_2[\psi(x,\rho_0)]^\alpha, \tag{14}$$

and so $U([\psi(x,\rho_0)]^\alpha)=0$. By using (2) and (8), we can give

$$\begin{aligned} U([\psi(x,\rho_0)]^\alpha) &= ([\psi]^\alpha)'(0,\rho_0) - h[\psi]^\alpha(0,\rho_0) \\ &= [\varphi]^\alpha(0,\rho_0)([\psi]^\alpha)'(0,\rho_0) - ([\varphi]^\alpha)'(0,\rho_0)[\psi]^\alpha(0,\rho_0) \\ &= W([\varphi]^\alpha,[\psi]^\alpha) = \Delta_\alpha(\rho_0). \end{aligned} \tag{15}$$

By regarding to above relation, we will have $\Delta_\alpha(\rho_0)=0$. The proof is completed. □

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