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RESULTS ON MAJORITY DOM-CHROMATIC SETS OF A GRAPH

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Abstract. A majority dominating set $S \subseteq V(G)$ is said to be majority dominating chromatic set if S satisfies the condition $\chi(\langle S\rangle) = \chi(G)$. The majority dom-chromatic number $\gamma_{M\chi}(G)$ is the minimum cardinality of majority dominating chromatic set. In this article we investigated some inequalities on Majority dominating chromatic sets of a connected and disconnected graph G. Also characterization theorems and some results on majority dom-chromatic number $\gamma_{M\chi}(G)$ for a vertex color critical graph and biparte graph are determined. we established the relationship between three parameters namely $\chi(G), \gamma_M(G)$ and $\gamma_{M\chi}(G)$ for some graphs.

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1. Introduction

All the graphs $G = (V, E)$ considered here are simple, finite and undirected. The concept of domination is early discussed by Ore and Berge in 1962. Then Haynes et.al [2] defined the domination number $\gamma(G)$ as the minimum cardinality of a minimal dominating set $D \subseteq V(G)$ such that each vertex of $(V - D)$ is adjacent to some vertex in D. The majority dominating number $\gamma_M(G)$ was introduced by Joseline Manora and Swaminathan [6] is the smallest cardinality of a minimal majority dominating set $S \subseteq V(G)$ of vertices and S satisfies $|N[S]| \geq \left|\left\lceil \frac{(V(G))}{2} \right\rceil\right|$.

Janakiraman and Poobalaranjani [3] defined the dom-chromatic set as a dominating set $S \subseteq V(G)$ such that the induced subgraph $\langle S\rangle$ satisfies the property $\chi(\langle S\rangle) = \chi(G)$. The minimum cardinality of a dom-chromatic S is called dom-chromatic number and it is denoted by $\gamma_{ch(G)}$ or $\gamma_{\chi(G)}$.

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Definition 1.1. *[4] A majority dom-chromatic number $\gamma_{M\chi}(G)$ is defined as the smallest cardinality of the majority dom-chromatic set (MDC set) S of $V(G)$ if S is a majority dominating set and it satisfies the property $\chi(\langle S\rangle) = \chi(G)$.*

Results 1.2.

(i) *[4] Let $G = mK_2, m \geq 1$ with $p = 2m$. Then $\gamma_{M\chi}(G) = \lceil \frac{p}{4} \rceil + 1, p \geq 2$.*

(ii) *[4] For any graph $G, \max\{\chi(G), \gamma_M(G)\} \leq \gamma_{M\chi}(G) \leq p$*
.

(iii) *[4] Let G be any graph of order p. Then $\gamma_{M\chi}(G) = p$ if and only if G is vertex χ - critical.*

(iv) *[6] For a cycle $C_p, \gamma_M(C_p) = \lceil \frac{p}{6} \rceil$.*

(v) *[6] For a path $P_p, \gamma_M(P_p) = \lceil \frac{p}{6} \rceil$.*

Definition 1.3. *[5] If a vertex with degree $d(u) \geq \lceil \frac{p}{2} \rceil - 1$ then u is called a majority dominating vertex. A full degree vertex is a majority dominating vertex but a majority dominating vertex is not a full degree vertex.*

2. Some Inequalities On Majority Dom-Chromatic Sets.

In this section, Inequality between the sum of the degrees of all vertices of a MDC set S of G and the complement of S i.e., $(V - S)$ in a graph G is discussed. We determine some inequalities such as

$|V - S| \leq \sum_{v_i \in S} deg\ (v_i)$ and $|V - S| \geq \sum_{v_i \in S} deg\ (v_i)$ with respect to the MDC set S of a connected graph G.

Theorem 2.1. *If S is a MDC set with two majority dominating vertices of a connected graph G then $|V - S| \leq \sum_{v_i \in S} deg(v_i)$.*

Proof: Let $v_i \in V(G)$ be a majority dominating vertex such that $d(v_i) \geq \lceil \frac{p}{2} \rceil - 1$ and $S = \{v_1, v_2\}$ be a MDC set with only two majority dominating vertices of G.

Case 1. *The graph G is a tree.*

Since $d(v_i) \geq \lceil \frac{p}{2} \rceil - 1, i = 1, 2,$ for all $v_i \in S$. It implies that $\chi(G) = 2, \gamma_M(G) = 1$

$$\text{then} \sum_{v_i \in S} deg(v_i) = d(v_1) + d(v_2) \geq \left\lceil \frac{p}{2} \right\rceil - 1 + \left\lceil \frac{p}{2} \right\rceil - 1$$

$$\sum_{v_i \in S} deg(v_i) = p - 2 \text{ or } p \text{ if } \ p \text{ is even or odd}$$

$$\text{Therefore } |V - S| = p - 2 \leq \sum_{v_i \in S} deg(v_i).$$

Case 2. *The graph G is not a tree and G contains two majority dominating vertices. Then G is not complete but G consists of triangles. It implies that $\chi(G) = 3, \gamma_M(G) = 1$.*

Then $S = \{v_1, v_2, v_3\}$ be a majority dominating chromatic set of G where v_3 is joined with a majority dominating vertex v_1 or v_2 of G.

$$Therefore \sum_{v_i \in S} deg(v_i) = d(v_1) + d(v_2) + d(v_3) \geq \left\lceil \frac{p}{2} \right\rceil - 1 + \left\lceil \frac{p}{2} \right\rceil - 1 + 2$$
$$\geq p \ or \ p + 2$$
$$Hence \ |V - S| = p - 3 < \sum_{(v_i \in S)} deg(v_i).$$

In the above cases, we obtain $|V - S| \leq \sum_{(v_i \in S)} deg(v_i)$. □

Example 2.2. *Consider the following Hajos graph with $p = 10$.*

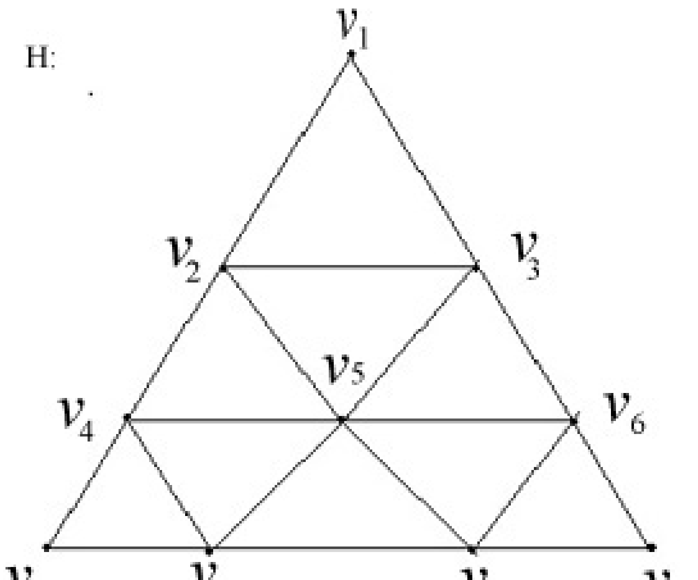

Fig. (2.1)

For the graph $H, \chi(H) = 3, \gamma_M(H) = 1$

Then $S = \{v_2, v_3, v_5\}$ is the MDC set of H.

$$\sum_{v_i \in S} deg(v_i) = d(v_2) + d(v_3) + d(v_5) = 14 \text{ and } |V - S| = 7 < \sum_{v_i \in S} deg(v_i).$$

□

Proposition 2.3. *Let G be a non-trivial connected graph with atleast one full degree vertex. If S is a majority dom-chromatic set of G then*

$$|V - S| < \sum_{u_i \in S} deg(u_i).$$

Proof: The graph G contains atleast one full degree vertex $u_1 \in V(G)$ then $d(u_1) = p - 1$.

Case 1. *The graph G is complete.*

Then the graph G contains all vertices are full degree vertices. Since $\chi(G) = p$, $S = \{u_1, u_2, \cdots, u_p\}$ is a MDC set of G.

$$\text{Therefore } |V-S| = 0 \text{ and } \sum_{u_i \in S} deg(u_i) = p(p-1) \Rightarrow |V-S| < \sum_{u_i \in S} deg(u_i).$$

Case 2. *The graph G is not complete.*

SubCase 1. *If G has only one full degree vertex u and it is not tree then G contains a triangle. Since $\chi(G) = 3, S = \{u, u_1, u_2\}$ is a MDC set of G. It implies that $|V-S| = p-3$.*

$$\sum_{u_i \in S} deg(u_i) = (p-1) + 3 + 3 = p + 5. \text{ Hence, } |V-S| < \sum_{u_i \in S} deg(u_i).$$

SubCase 2. *If G has only one full degree vertex and the graph G is a tree.*

Consider $S = \{u_1, u_2\}$ be the MDC set of G which contains a full degree vertex u_1. Then $\gamma_{M\chi}(G) = 2$. Hence $|V-S| \leq p-2$.

$$\text{Also } \sum_{u_i \in S} deg(u_i) = d(u_1) + d(u_2) \geq p - 1 + 1 = p. \quad \text{Hence, } |V-S| < \sum_{u_i \in S} deg(u_i).$$

SubCase 3. *Suppose the graph G has two full degree vertices u_1 and u_2, then G contains a triangle. Hence, $\chi(G) = 3$. Let $S = \{u_1, u_2, u_3\}$ be a majority dominating chromatic set of G. Then $|V-S| = p-3$.*

$$\text{Now, } \sum_{u_i \in S} deg(u_i) = (p-1) + (p-1) + 2 = 2p. \Rightarrow |V-S| < \sum_{u_i \in S} deg(u_i).$$

In all cases, the vertices of S majority dominates the graph G and also addition with its coloring number. Thus $|V-S| < \sum_{u_i \in S} deg(u_i)$. □

Corollary 2.4. *If the graph G is a vertex color critical and S is a MDC set of G then $|V-S| = 0$.*

Proof. Let G be a vertex color critical graph with p vertices. Then $S = \{v_1, v_2, \cdots, v_p\}$ is a MDC set for G. It implies that $\gamma_{M\chi}(G) = |S| = p$. Hence $|V-S| = 0$. □

Proposition 2.5. *If a connected graph G contains all vertices are majority dominating vertices then $|V-S| \leq \sum_{u_i \in S} deg(u_i)$, where S is the MDC set of G.*

Proof: Let G be a connected graph which contains only majority dominating vertices. Then $\gamma_M(G) = 1$ and $\chi(G) \geq 2$. Consider the set $S = \{u_1, u_2, \cdots, u_t\}$ be a MDC set of G. Then $|V-S| \leq p-2$. Since G contains only majority dominating vertices, $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1$,for each $u_i \in S$.

Case 1. *The graph G has no triangles. Let $S = \{u_1, u_2\}$ be a majority dominating chromatic set of G.*

$$\text{Then } \sum_{u_i \in S} deg(u_i) = d(u_1) + d(u_2) \geq \left\lceil \frac{p}{2} \right\rceil - 1 + \left\lceil \frac{p}{2} \right\rceil - 1$$

$$\sum_{u_i \in S} deg(u_i) \geq p \text{ or } p-2 \text{ and } |V-S| = p-2. \text{ Hence } |V-S| \leq \sum_{u_i \in S} deg(u_i).$$

Case 2. *The graph G has triangles.*

Then $\gamma_M(G) = 1$ and $\chi(G) \geq 3$. It implies that $S = \{u_1, u_2, u_3\}$ is a MDC set of G. Hence $|V - S| = p - 3$.

$$Then \sum_{u_i \in S} deg(u_i) = 3\left(\left\lceil\frac{p}{2}\right\rceil - 1\right) \geq \frac{3p}{2} \; or \; \left(\frac{3p}{2} - 3\right). Hence \; |V - S| \leq \sum_{u_i \in S} deg(u_i).$$

□

Proposition 2.6. *If a connected graph G has no majority dominating vertices then $|V - S| \geq \sum_{u_i \in S} deg(u_i)$, where S is the MDC set of G.*

Proof: Let S be the MDC set of a connected graph G of p vertices and q edges. Since the graph G has no majority dominating vertices, it has no full degree vertex and it contains all vertices with degree of $d(u_i) < \lceil\frac{p}{2}\rceil - 1$. Assume that $S = \{u_1, u_2, \cdots\}$ be the MDC set of G. Then $|V - S| \leq p - 2, p > 6$.

$$Also, \sum_{u_i \in S} deg(u_i) = d(u_1) + d(u_2) + \cdots \leq \left\lceil\frac{p}{2}\right\rceil - 2 + \left\lceil\frac{p}{2}\right\rceil - 2 + \cdots \leq 2\left\lceil\frac{p}{2}\right\rceil - 4$$

$$\sum_{u_i \in S} deg(u_i) \leq (p - 2) \; or \; (p - 4), \; if \; p \; is \; odd \; or \; even.$$

$$Hence \; we \; obtain, \; |V - S| \geq \sum_{u_i \in S} deg(u_i).$$

□

Proposition 2.7. *If a MDC set S contains a majority dominating vertex v and other vertices u_i such that $d(u_i) \leq \lceil\frac{p}{2}\rceil - 3$ then*

$$|V - S| > \sum_{u_i \in S} deg(u_i).$$

Proof: Let u be the majority dominating vertex such that $d(u) = \lceil\frac{p}{2}\rceil - 1$ and other vertices u_i with degree $d(u_i) \leq \lceil\frac{p}{2}\rceil - 3$ in G. Then $\gamma_M(G) = |\{u\}| = 1$ and $\chi(G) = 2$. Therefore $S = \{u, u_1\}$ is a MDC set of G and $|V - S| \leq p - 2$.

$$Then \sum_{u_i \in S} deg(u_i) = d(u) + d(u_1) \leq \left\lceil\frac{p}{2}\right\rceil - 1 + \left\lceil\frac{p}{2}\right\rceil - 3$$
$$\leq \begin{cases} \frac{p}{2} - 1 + \frac{p}{2} - 3 = p - 4, & if \; p \; is \; even \\ \frac{p}{2} + \frac{p}{2} + 1 - 4 = p - 3, & if \; p \; is \; odd \end{cases}$$

$$Therefore \sum_{u_i \in S} deg(u_i) \leq (p - 4) \; or \; (p - 3). \; Hence \; |V - S| > \sum_{u_i \in S} deg(u_i).$$

□

Theorem 2.8. *Let G be a connected graph with exactly one vertex v such that $\lceil\frac{p}{2}\rceil - 1 \leq d(v) \leq \lceil\frac{p}{2}\rceil + 2$ and $d(u_i) \leq 3$, for all $u_i \in V(G)$. Then*

$$|V - S| > \sum_{v_i \in S} deg(v_i), \; where \; S \; is \; MDC \; set \; such \; that \; v \in S.$$

Proof: Let $v \in V(G)$ with the condition $\lceil\frac{p}{2}\rceil - 1 \leq d(v) \leq \lceil\frac{p}{2}\rceil + 2$. (1)

Case 1. *The graph G is a tree. Let $S = \{v, u_1\}$ be a MDC set in which u_1 is a pendant or $d(u_1) = 3$. Then by (1), $d(v) = \lceil \frac{p}{2} \rceil - 1$ and $|V - S| = p - 2$.*

$$Then \sum_{v_i \in S} deg(v_i) = d(v) + d(u_1) = \left\lceil \frac{p}{2} \right\rceil - 1 + 1 = \left\lceil \frac{p}{2} \right\rceil \text{ or } \left\lceil \frac{p}{2} \right\rceil + 1$$

$$It\ implies\ that\ |V - S| = p - 2 > \sum_{v_i \in S} deg(v_i).$$

Suppose $d(v) = \lceil \frac{p}{2} \rceil + 2$.

$$Then, \sum_{v_i \in S} deg(v_i) = d(v) + d(u_1) = \left\lceil \frac{p}{2} \right\rceil + 2 + 1 = \left\lceil \frac{p}{2} \right\rceil + 3 \text{ or } \left\lceil \frac{p}{2} \right\rceil + 4.$$

$$Therefore\ by\ (1), \sum_{v_i \in S} deg(v_i)\ takes\ the\ value\ from\ \left\lceil \frac{p}{2} \right\rceil\ to \left\lceil \frac{p}{2} \right\rceil + 4.$$

$$Hence\ |V - S| > \sum_{v_i \in S} deg(v_i).$$

Case 2. *The graph G is not a tree.*

Let S be a MDC set of G and $S = \{v, v_1\}$ where v is a majority dominating vertex and v_1 is not a pendant of G. Then $|V - S| \leq p - 2$.

$$Then \sum_{v_i \in S} deg(v_i) = d(v) + d(v_1) \geq \left\lceil \frac{p}{2} \right\rceil - 1 + 3$$

$$Therefore \sum_{v_i \in S} deg(v_i) = \left\lceil \frac{p}{2} \right\rceil + 2,\ if\ d(v) \geq \left\lceil \frac{p}{2} \right\rceil - 1\ and$$

$$\sum_{v_i \in S} deg(v_i) = \left\lceil \frac{p}{2} \right\rceil + 5,\ if\ d(v) \leq \left\lceil \frac{p}{2} \right\rceil + 2$$

$$Hence,\ |V - S| = p - 2 > \sum_{v_i \in S} deg(v_i).$$

□

3. Results on $\gamma_{M\chi}(G)$

Proposition 3.1. *Let G be any bipartite graph with a majority dominating vertex. Then $\gamma_{M\chi}(G) = 2$ and $\gamma_M(G) < \gamma_{M\chi}(G)$.*

Proof: Let $G = K_{m,n}, m \leq n$, be a complete bipartite graph.

Case 1. *Since G has a majority dominating vertex, $\gamma_M(G) = 1$ and $\chi(G) = 2$. Then $S = \{u_1, v_1\}$ is a MDC set of G, where $u_1 \in V_1(G)$ and $v_1 \in V_2(G)$.*

$$\Rightarrow \gamma_{M\chi}(G) = 2 \text{ and } \gamma_M(G) < \gamma_{M\chi}(G).$$

Case 2. *If G is not a complete bipartite graph then G may contains pendants. Since G has a majority dominating vertex $u_1 \in V(G), S = \{u_1, u_2\}$ is a MDC set of G where $u_1 \in V_1(G)$ and $v_1 \in V_2(G)$.*

$\Rightarrow \gamma_{M\chi}(G) = 2$ *and* $\gamma_M(G) = 1$. *Hence* $\gamma_M(G) < \gamma_{M\chi}(G)$. □

The following theorem gives the characterization of $\gamma_{M\chi}(G) = p - q$, where G is any graph with p vertices and q edges.

Theorem 3.2. *Let G be any graph with p vertices and q edges. Then $\gamma_{M\chi}(G) = p - q$ if and only if $G = K_p, p = 1$.*

Proof: Let $\gamma_{M\chi}(G) = p - q$. Since $\gamma_{M\chi}(G) \geq 1, (p - q) \geq 1$. (1)

Case 1. *The graph G is connected.*

Then $q \geq p - 1 \Rightarrow (p - q) \leq 1$. Hence by (1) we obtain $p - q = 1 = \gamma_{M\chi}(G)$. (2)
It implies that G is a tree. If G is a tree then $\chi(G) = 2$ and for any connected graph, $1 \leq \gamma_M(G) \leq \lceil \frac{p}{6} \rceil$.

By (2), since $p - q = 1 = \gamma_{M\chi}(G)$, the two numbers $\gamma(G)$ and $\gamma_M(G)$ must be one. In a tree, suppose $\chi(G) = 2$ and $\gamma_M(G) = 1$, then the graph becomes $G = K_2$. By the result (ii) of (1.2), $\gamma_{M\chi}(G) \geq \max\{\chi(G), \gamma_M(G)\}$. We have $\gamma_{M\chi}(G) = 2$. But it is contradiction to the result (2). Hence $G \neq K_2$ and $G = K_2$.

Case 2. *Suppose G is disconnected. If G is disconnected with isolates and without isolates. Then by the result (i) of (1.2), $\lceil \frac{p}{4} \rceil + 1 \leq \gamma_{M\chi}(G) \leq \lceil \frac{p}{2} \rceil$. The lower bound is attained for $G = mK_2$. If $m = 1, \gamma_{M\chi}(K_2) = 2 \neq p - q = 1$. Also the upper bound is attained for $G = \overline{k_p}$, when $p = 2$ then $\gamma_{M\chi}(\overline{K_2}) = 1 \neq p - q = 2$. Hence $G \neq \overline{K_2}$ or K_2. It follows that the graph must be $G = K_1$. The converse is obvious.*

□

Next result is the characterization of $|V - S| = 0$, where S is a MDC set of vertex color critical graph G.

Proposition 3.3. *A MDC set S belongs to a vertex color critical graph if and only if $|V - S| = 0$.*

Proof: Suppose $|V - S| = 0. \Rightarrow |V(G)| = |S| = p$. Then the set $S = \{u_1, u_2 \cdots, u_p\}$ is a MDC set for G. Suppose we remove one vertex from S then S may not be a MDC set of G. Hence G is vertex color critical graph.

Conversely by the definition (iv) in (1.1), if G is vertex color critical graph with p vertices then $\gamma_{M\chi}(G) = p$. Hence $|V - S| = 0$. □

Proposition 3.4. *Let G be any graph with p vertices. Then $\gamma_{M\chi}(G) \leq \gamma_\chi(G)$, where $\gamma_\chi(G)$ is the dom-chromatic number of G.*

Proof: Let $\gamma_{M\chi}(G)$ be the majority dom-chromatic number of graph G. Since every dom-chromatic set of a graph G is a majority dom-chromatic set of a graph G, $\gamma_{M\chi}(G) \leq \gamma_\chi(G)$.

Case 1. *When G is vertex color critical graph.*
By the known results (3.2.6) of [3] and (ii) of (1.2), $\gamma_\chi(G) = p = \gamma_{M\chi}(G)$.

Case 2. *The graph G is a tree.*
If diam $(G) \leq 3$, then $\gamma_\chi(G) = \gamma_{M\chi}(G) = 2$.
Suppose diam$(G) \geq 4$, then the graph structures like $P_p, p \geq 5$, Caterpillar, etc. By the known results, $\gamma_\chi(G) \leq \frac{p+3}{3}$ and $\gamma_{M\chi}(G) \leq \lceil \frac{p}{6} \rceil + 1$.
Hence $\gamma_{M\chi}(G) < \gamma_\chi(G)$.

Case 3. *When the graph G is not a tree and not a vertex color critical graph.*
Then the graph structures like C_p (cycle, p is even), F_p (Fan), W_p (wheel), etc. By the known results, $\gamma_{M\chi}(G) \leq \lceil \frac{p}{6} \rceil + 1$ and $\gamma_\chi(G) \leq \frac{p+4}{3}$.
Hence $\gamma_{M\chi}(G) \leq \gamma_\chi(G)$. □

Corollary 3.5.

(i) *If the graph G is a sub division of a star, then* $\gamma_{M\chi} < \lceil \frac{\gamma_\chi}{2} \rceil$.

(ii) *If G is a path or cycle then,*

a) $\gamma_{M\chi} \leq \lceil \frac{\gamma_\chi(G)}{2} \rceil; p \equiv 0, 1, 2, 5(\mod 6)$,

b) $\gamma_{M\chi} \leq \lceil \frac{\gamma_\chi(G)}{2} \rceil + 1; p \equiv 3, 4(\mod 6)$.

Example 3.6.

(i) *Let P_p be a path with $p \equiv 0(mod6)$. Consider $G = P_{18}$ then $\gamma_{M\chi}(G) = 4$ and $\gamma_\chi(G) = 7$. Now, $\left\lceil \frac{\gamma_\chi(G)}{2} \right\rceil = \lceil \frac{7}{2} \rceil = 4$. Hence $\gamma_{M\chi}(G) = \left\lceil \frac{\gamma_\chi(G)}{2} \right\rceil$.*

(ii) *Let $G = S(K_{1,t})$. Then $S_1 = \{u, u_1, u_2, \cdots, u_t\}$ is a dom-chromatic set which contains a central vertex u of G. $\Rightarrow \gamma_\chi(G) = |S_1| = t + 1$ and $S_2 = \{u, u_1\}$ is a MDC set of G. $\Rightarrow \gamma_{M\chi}(G) = 2$. Hence $\gamma_{M\chi}(G) < \left\lceil \frac{\gamma_\chi(G)}{2} \right\rceil$.*

Construction 3.7. *For every integer $k \geq 0$, there exist a graph G such that $\left\lceil \frac{\gamma_\chi(G)}{2} \right\rceil - \gamma_{M\chi}(G) = k$.*

Proof. Let G be the subdivision of a star $K_{1,2k+2}$ by dividing each edge exactly once. Then $|V(G)| = 2(2k + 2) + 1, \gamma_\chi(G) = 2k + 2 + 1$ and $\gamma_{M\chi}(G) = 2$.
Then $\left\lceil \frac{\gamma_\chi(G)}{2} \right\rceil - \gamma_{M\chi}(G) = k + 2 - 2 = k$. □

Observation 3.8. *Let G be any connected graph with p vertices. Let $\chi(G), \gamma_M(G)$ and $\gamma_{M\chi}(G)$ be the chromatic number, majority domination number and majority dom-chromatic number respectively. Then $\chi(G)$ and $\gamma_M(G)$ are not comparable.*
i.e., $\gamma_M(G) < \chi(G) < \gamma_{M\chi}(G)$ and $\chi(G) < \gamma_M(G) < \gamma_{M\chi}(G)$.

For Example:-

(i) Let $G = C_p, p \leq 11$ and p is odd. Since C_p is vertex χ-critical, by the result (iv) of (1.2), $\gamma_M(G) = \lceil \frac{p}{6} \rceil, \chi(G) = 3$ and $\gamma_{M\chi}(G) = 5$.
Hence, $\gamma_M(G) < \chi(G) < \gamma_{M\chi}(G)$.

(ii) Let $G = C_p, p$ is odd and $p \geq 19$. By the result (iv) of (1.2), $\gamma_M(G) = \lceil \frac{p}{6} \rceil, \chi(G) = 3$ and $\gamma_{M\chi}(G) = p$. Hence, $\chi(G) < \gamma_M(G) < \gamma_{M\chi}(G)$.

(iii) If $p = 13, 15, 17$ for $G = C_p$ then $\chi(G) = \gamma_M(G) < \gamma_{M\chi}(G)$.

4. Results of $\gamma_{M\chi}(G)$ for a Disconnected Graph

Theorem 4.1. *Let G be a disconnected graph then $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$ if and only if $G = \overline{K_p}$ or $G = g_t \cup \overline{K_{p-t}}$, $p \geq 2$, where g_t is a vertex color critical component with $|t| \leq \lceil \frac{p}{2} \rceil$.*

Proof: Let G be a disconnected graph with p vertices.

Assume that, $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$. (1)

Case 1. *Suppose $G \neq \overline{K_p}, p \geq 2$ then G has atleast one edge between a pair of vertices. It implies that G is a disconnected graph without isolates or $G = K_2 \cup \overline{K_{p-2}}$. By known result (i) of (1.2),$\gamma_{M\chi}(G) = \lceil \frac{p}{4} \rceil + 1$ or $\gamma_{M\chi}(G) = \lceil \frac{p}{4} \rceil - 1$. But it is a contradiction to (1). Therefore $G = \overline{K_p}, p \geq 2$.*

Case 2. *Suppose $G = g_t \cup \overline{K_{p-t}}$, where g_t is not a vertex color critical component with $|t| \leq \lceil \frac{p}{2} \rceil$. Then the graph G contains a path, an even cycle or any other component g_t with $|t| \leq \lceil \frac{p}{2} \rceil$. Since $\chi(g_t) \geq 2$ and $\gamma_M(g_t) \geq \lceil \frac{p}{6} \rceil$,*

SubCase 1. *Suppose* $|t| = \lceil \frac{p}{2} \rceil$. *Then* $S = \left\{u_1, u_2, \ldots, u_{\lceil \frac{p}{6} \rceil}\right\}$, *is a MDC set of* G, *where* $u_i \in V(g_t)$. *It implies that* $\gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil$, *it condradicts the condition (1).*

SubCase 2. *Suppose* $|t| < \lceil \frac{p}{2} \rceil$. *Then* $S = \{u_1, u_2, \left(\lceil \frac{p}{2} \rceil - t\right) K_1\}$ *is a MDC set of* G *where* $u_i \in V(g_t)$.

Therefore $\gamma_{M\chi}(G) = |S| = \lceil \frac{p}{2} \rceil - |t| + 2 = \lceil \frac{p}{2} \rceil - \lceil \frac{p}{2} \rceil + 1 + 3$ $(if\ |t| = \lceil \frac{p}{2} \rceil - 1)$.

$\Rightarrow \gamma_{M\chi}(G) = 4 < \lceil \frac{p}{2} \rceil$. *It is a contradiction to (1). Hence* g_t *is a vertex color critical component in* G *with* $|t| \leq \lceil \frac{p}{2} \rceil$.

Case 3. *Suppose* g_t *with* $|t| > \lceil \frac{p}{2} \rceil$. *Since* g_t *is a vertex color critical component of* G, g_t *is a complete graph or an odd cycle. If* g_t *is an odd cycle with* $|t| = \lceil \frac{p}{2} \rceil + 1$ *then* $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil + 1$. *It contradicts our assumption.*

If g_t *is a complete graph with* $|t| = \lceil \frac{p}{2} \rceil + 1$ *then* $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil + 1$, *it is a contradiction to (1). Hence,* g_t *is a vertex color critical component of* G *with* $|t| \leq \lceil \frac{p}{2} \rceil$. *Therefore* G *must be* $\overline{K_p}$ *or* $(g_t \cup \overline{K_{p-t}})$ *with* $|t| \leq \lceil \frac{p}{2} \rceil$. *In all the three cases if* $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$, *then* $G = \overline{K_p}$ *or* $(g_t \cup \overline{K_{p-t}})$.

Conversely, let $G = \overline{K_p}$ *or* $(g_t \cup \overline{K_{p-t}})$. *Suppose* $G = \overline{K_p}$ *then* $\gamma_M(G) = \lceil \frac{p}{2} \rceil$ *and* $\chi(G) = 1 \Rightarrow \gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$.*Suppose* $G = (g_t \cup \overline{K_{p-t}})$. *Since* g_t *is a vertex critical component with* $|t| = \lceil \frac{p}{2} \rceil, \chi(g_t) = \lceil \frac{p}{2} \rceil$ *and* $\gamma_M(g_t) \geq 1$. *It implies that* $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$.*Suppose* g_t *is a vertex critical component with* $|t| < \lceil \frac{p}{2} \rceil$. *Then* $S = \{u_1, u_2, ..., u_t, v_1, v_2, ..., v_{\lceil \frac{p}{2} \rceil - t}\}$ *is a MDC set of* G *where* $u_i \in V(g_t)$ *and* $v_i \in V(\overline{K_{p-t}})$. *Now,* $|S| = t + \lceil \frac{p}{2} \rceil - t = \lceil \frac{p}{2} \rceil$. *Hence* $\gamma_{M\chi}(G) = |S| = \lceil \frac{p}{2} \rceil$.

□

Observation 4.2. *(i) For a disconnected graph* $G, \chi(G) < \gamma_M(G) < \gamma_{M\chi}(G)$.

Example: Consider the disconnected graph with isolates with $p = 16$.

Let $G = P_{11} \cup \overline{K_5}$. Let $|V(G)| = |\{v_1, v_2, \cdots, v_{11}, u_1, \cdots, u_5\}| = 16$. Then $\gamma_M(G) = |\{v_2, v_5, v_7\}| = 3$ and $\gamma_{M\chi}(G) = |\{v_2, v_5, v_7, v_8\}| = 4$. Since P_{11} is a tree, $\chi(G) = 2$. Therefore $\chi(G) < \gamma_M(G) < \gamma_{M\chi}(G)$.

(ii) For a disconnected graph G with isolates, $\gamma_M(G) < \chi(G) < \gamma_{M\chi}(G)$.

Example: Let $G = C_3 \cup \overline{K_5}$ and $V(G) = \{v_1, v_2, v_3, u_1, \cdots, u_5\}$. Since C_3 is an odd cycle, $\chi(G) = 3$ and $\gamma_M(G) = |\{v_1, u_1\}| = 2$. Then $S = \{v_1, v_2, v_3, u_1\}$ be the MDC set of G where $v_i \in V(C_3)$ and $u_i \in V(\overline{K_5})$. $\Rightarrow \gamma_{M\chi}(G) = |S| = 4$. Therefore $\gamma_M(G) < \chi(G) < \gamma_{M\chi}(G)$.

(iii) Let G be a disconnected graph without isolates. Then $\chi(G) < \gamma_M(G) < \gamma_{M\chi}(G)$.

Example: Consider the graph $G = P_7 \cup C_6 \cup K_{1,3}$. For a tree with $p = 17$ and an even cycle, $\chi(G) = 2$.

$V(G) = \{u_1, \cdots, u_7, v_1, \cdots, v_6, w, w_1, w_2, w_3\}$. Then $\gamma_M(G) = |\{w, u_2, u_4\}| = 3$ and $\gamma_{M\chi}(G) = |\{w, u_2, u_4, u_5\}| = 4$. Hence $\chi(G) < \gamma_M(G) < \gamma_{M\chi}(G)$.

(iv) For a disconnected graph G with vertex color critical component, $\chi(G) < \gamma_M(G) < \gamma_{M\chi}(G)$.

Example: Let $G = C_{13} \cup \overline{K_6}$ be a graph with $p = 19$.

And $V(G) = \{u_1, \cdots, u_{13}, v_1, \cdots, v_6\}$. Since C_{13} is an odd cycle, $\chi(G) = 3$. The set $\{u_2, u_5, u_8\}$ be the γ_M-set of G and $\gamma_M(G) = 3$. By the result (iii) of (1.2), C_{13} is a vertex color critical component , $\gamma_{M\chi}(G) = 13$. Therefore $\chi(G) \leq \gamma_M(G) < \gamma_{M\chi}(G)$.

Proposition 4.3. *G be a disconnected graph with any vertex color critical*

$$\text{component then } |V - S| < \sum_{u_i \in S} deg(u_i).$$

Proof: Let $G = G_t \cup G_r$ be a disconnected graph with p vertices . Since G has a vertex color critical component , $\chi(G) \geq 3$. Consider $S = \{G_t, u_1, \cdots\}$ be the MDC set of G, where G_t is the vertex color critical component, such that $|t| \geq 3$ and $u_1 \in G_r$. If $|N[G_t]| = \lceil \frac{p}{2} \rceil$ then $|S| \geq 3$. If $|N[G_t]| < \lceil \frac{p}{2} \rceil$ then $|S| \geq 4$. It implies that $|S| = 3$ or 4 and $|V - S| \leq p - 3$ or $p - 4$. Let $V(G_t) = \{u_1, u_2, \cdots, u_t\}$, then

$$\sum_{u_i \in S} deg(u_i) = d(u_1) + d(u_2) \cdots \geq 3(t-2) + 1 \geq 3t - 5, \; if \; |t| \geq 3.$$

$$\text{Then, certainly we get } |V - S| < \sum_{u_i \in S} deg(u_i)$$

□

Proposition 4.4. *For a disconnected graph G without any vertex critical component, $|V - S| > \sum_{u_i \in S} deg(u_i)$.*

Proof: Let G be a disconnected graph with not vertex color critical component. Let S be a MDC set of G.

Case 1. *The graph G is totally disconnected.*

Then $S = \{u_1, u_2, \cdots, u_{\lceil \frac{p}{2} \rceil}\}$ be the MDC set of G and $deg(u_i) = 0$, for each $u_i \in S$. It implies that $\sum_{u_i \in S} deg(u_i) = 0$. Hence, $|V - S| > \sum_{u_i \in S} deg(u_i)$.

Case 2. *The graph G is disconnected with isolates.*

Then G contains some connected component 'g' along with isolates.

SubCase 1. *If the component 'g' such that $|N[g]| \geq \lceil \frac{p}{2} \rceil$ then S is a MDC set of G with $1 \leq |S| = \lceil \frac{p}{6} \rceil$. Suppose $|S| = 1 \Rightarrow S = \{u\}$ such that $|N[S]| = \lceil \frac{p}{2} \rceil - 1$.*

Then $|V - S| = p - 1 > \sum_{u_i \in S} deg(u_i) = \lceil \frac{p}{2} \rceil - 1$. Suppose $|S| = \lceil \frac{p}{6} \rceil$.

Then $d(u_i) \leq 2$, for all $u_i \in V(g)$. Now, $\sum_{u_i \in S} deg(u_i) = 2\lceil \frac{p}{6} \rceil = \frac{p}{3}$ or $\frac{p}{3} + 2$ and $|V - S| = p - \lceil \frac{p}{6} \rceil = \frac{5p}{6} - 1$.

Therefore, $|V - S| > \sum_{u_i \in S} deg(u_i)$.

SubCase 2. *If the component 'g' such that $|N[S]| < \lceil \frac{p}{2} \rceil$ then S is a MDC set with isolates.*

$\Rightarrow \sum_{u_i \in S} deg(u_i) \leq \frac{p}{3}$. *Since S contains more isolates, the value $\sum_{u_i \in S} deg(u_i)$ will be reduced. Then $|V - S| > \sum_{u_i \in S} deg(u_i)$.*

Case 3. *G is a disconnected graph without isolates.*

Then G contains only connected components . Suppose $G = mK_2$. Then by the result (i) of (1.2), $\gamma_{M\chi}(G) = |S| = \lceil \frac{p}{4} \rceil + 1$. It implies that

$$\sum_{u_i \in S} deg(u_i) = \left\lceil \frac{p}{4} \right\rceil + 1. \ But \ |V - S| = \left| p - \left(\left\lceil \frac{p}{4} \right\rceil + 1 \right) \right| = \frac{3p}{4} - 1$$

If the size of the component g increases such as $G = mC_4, mK_{1,t}, \cdots$ then $|S|$ will be decreased. i.e.,

$$|S| < \left\lceil \frac{p}{4} \right\rceil + 1 \ and \ \sum_{u_i \in S} deg(u_i) > \left\lceil \frac{p}{4} \right\rceil + 1. \ But \ in \ all \ structures,$$

$$We \ obtain, \ |V - S| > \sum_{u_i \in S} deg(u_i).$$

□

Proposition 4.5. *Let G be a disconnected graph without any vertex color critical component then $|V - S| = \lfloor \frac{p}{2} \rfloor$ if and only if $G = \overline{K_p}$.*

Proof: Let G has no vertex color critical subgraph. Let $G = \overline{K_p}$, p is odd. Then $S = \{u_1, u_2, \cdots, u_{\lceil \frac{p}{2} \rceil}\}$ is a MDC set of G and $\gamma_{M\chi}(G) = |S| = \lceil \frac{p}{2} \rceil$. Hence $|V - S| = \lfloor \frac{p}{2} \rfloor$, if p is odd. When p is even, $S = \{u_1, u_2, \cdots, u_{\frac{p}{2}}\}$ is the MDC set and $\gamma_{M\chi}(G) = |S| = \frac{p}{2}$ and $|V - S| = \frac{p}{2}$. Hence $|V - S| = \lfloor \frac{p}{2} \rfloor$.

Conversely , suppose $G \neq \overline{K_p}$. Then either G is disconnected graph without isolates or G contains atleast one component which is not a vertex color critical with some isolates. Let $|V - S| = \lfloor \frac{p}{2} \rfloor$. (1)

Case 1. *If G has components which is not vertex color critical with no isolates then the structure like $G = mK_2$. By the result (i) of (1.2), we have $\gamma_{M\chi}(G) = |S| = \lceil \frac{p}{4} \rceil + 1$. If $|S| = \lceil \frac{p}{4} \rceil + 1 \Rightarrow |V - S| = |p - \lceil \frac{p}{4} \rceil + 1| > \lfloor \frac{p}{2} \rfloor$. It is a contradiction to (1).*

Case 2. *Suppose $G = C_6 \cup \overline{K_{P-6}}$, where C_6 is not a vertex color critical. Then $S = \{u_2, u_5, (\lceil \frac{p}{2} \rceil - 6)K_1\}$, where $u_2, u_5 \in V(C_6)$.*

$\Rightarrow |S| = \lceil \frac{p}{2} \rceil - 6 + 2 = \lceil \frac{p}{2} \rceil - 4.$

Therefore $|V - S| = |p - \lceil \frac{p}{2} \rceil + 4| = \lfloor \frac{p}{2} \rfloor + 4 > \lfloor \frac{p}{2} \rfloor$. It is a contradiction to (1).

Hence $G = \overline{K_p}$ if and only if $|V - S| = \lfloor \frac{p}{2} \rfloor$.

□

5. Conclusion

In this article, we have discussed the inequality between the sum of the degrees of the vertices of majority dominating chromatic set S and its complement $(V - S)$ of a graph. The comparison between the domination parameters $\gamma_M(G), \chi(G)$ and $\gamma_{M\chi}(G)$ are discussed. Also some results of $\gamma_{M\chi}(G)$ of a disconnected graph with isolates and without isolates are studied.

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