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ASSOCIATED FUNCTIONS OF NON-SELFADJOINT STURM-LIOUVILLE OPERATOR WITH OPERATOR COEFFICIENT

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Abstract. Sturm-Liouville operator equation with selfadjoint operator coefficent has been studied in detail. In this paper, we consider the Sturm-Liouville operator equation with non-selfadjoint operator coefficent. Namely, we examine the non-selfadjoint Sturm-Liouville operator L which is generated in $L_2(\mathbb{R}_+, H)$ by the differential expression

$$L(Y) = -Y'' + Q(x)Y, \quad 0 < x < \infty,$$

with operator coefficient together with the boundary condition $Y(0) = 0$, where $Q(x)$ is a non-selfadjoint, completely continuous operator in a separable Hilbert space H for each $x \in (0, \infty)$. We find the associated functions corresponding to the eigenvalues and spectral singularities of L. Moreover, we prove that the associated functions corresponding to the eigenvalues belong to $L_2(\mathbb{R}_+, H)$ while the associated functions corresponding to the spectral singularities do not.

Keywords: Sturm-Liouville operator equation, associated functions, operator coefficient, non-selfadjoint operators.

AMS Subject Classification: 47A25, 47A50, 47A56, 47A62, 47A75.

1. Introduction

Spectral analysis of non-selfadjoint differential and difference operators have been studied intensively in last decades. In particular, non-selfadjoint operators with a continuous spectrum were first investigated by Naimark [22, 23]. He showed that the continuous spectrum of the non-selfadjoint Sturm-Liouville operator on the half line is $[0, \infty)$ and there are some points in the continuous spectrum called spectral singularities which are not the eigenvalues of the operator. He also obtained the sufficient conditions which guarantee the finiteness of the eigenvalues and spectral singularities. Lyance used the spectral singularities in the spectral expansion in term of the associated functions of this operator [19]. More information can be found in [21, 25] about non-selfadjoint differential operators and in [12, 20] about spectral singularities. Keldysh developed a new method for evaluating the resolvent of an abstract completely continuous non-selfadjoint operator of finite order and also proved the completeness of eigenfunctions for some classes of non-selfadjoint operators [13, 14].

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Spectral properties of the selfadjoint matrix differential and difference equations are studied in [7, 8, 11]. As for the non-selfadjoint case, discrete spectrum and the spectral singularities of the non-selfadjoint Sturm-Liouville operator with matrix coefficients were investigated in [4, 9, 24]. Further, in [2, 5, 6], the authors examined the spectral properties of a finite system of non-selfadjoint Sturm-Liouville differential operators.

For the Sturm-Liouville operator equations, first works were conducted in [10, 15, 16, 17, 18] in the selfadjoint case. More explicitly, the authors considered the following operator. Let H be a separable Hilbert space ($\dim H \leq \infty$) and $L_2(\mathbb{R}_+, H)$ denote the space of vector-valued functions $f(x)$ $(0 < x < \infty)$ which are strongly-integrable in each finite subinterval of $(0, \infty)$ and such that $\int_0^\infty \|f(x)\|^2 dx < \infty$. Consider the differential expression in $L_2(\mathbb{R}_+, H)$

$$l_0(y) = -y'' + Q(x)y, \quad 0 < x < \infty, \tag{1}$$

where $Q(x)$ is a selfadjoint, completely continuous operator in H for each $x \in (0, \infty)$. Expression (1) is called Sturm-Liouville operator equation. The discrete spectrum of the operator generated by (1) and the boundary condition $y(0) = 0$ has been studied in detail in [10, 15, 16, 17, 18].

In our previous paper [3], we investigated the spectral properties of the Sturm-Liouville operator equation on the half-line with non-selfadjoint operator coefficients, on the contrary to [10, 15, 16, 17, 18] and also generalized the results in [2, 4, 9, 24] to the operator coefficient case. More explicitly, we considered the following non-selfadjoint Sturm-Liouville operator equation. Let H be a separable Hilbert space ($\dim H \leq \infty$) and $H_1 := L_2(\mathbb{R}_+, H)$. Consider the differential expression in H_1

$$L(y) = -y'' + Q(x)y, \quad 0 < x < \infty, \tag{2}$$

where $Q(x)$ is a non-selfadjoint, completely continuous operator in H for each $x \in (0, \infty)$. We considered the operator L which is generated by (2) and the boundary condition $y(0) = 0$. We found the point spectrum and spectral singularities of L and showed that L has a finite number of eigenvalues and spectral singularities under the condition

$$\int_0^\infty e^{\epsilon t} \|Q(t)\| \, dt < \infty, \quad \epsilon > 0.$$

In this paper, we obtain the associated (principal) functions of L corresponding to the eigenvalues and the spectral singularities. In particular, we prove that the associated functions corresponding to the eigenvalues belong to $L_2(\mathbb{R}_+, H)$ whereas the associated functions corresponding to the spectral singularities do not.

2. Some notions on the non-selfadjoint Sturm-Liouville operator equation

Let us recall some results obtained in [3] for the sake of completeness.
Let H be a separable Hilbert space and $H_1 = L_2(\mathbb{R}_+, H)$ denote the space of vector-valued funtions $f(x)$ $(0 < x < \infty)$ which are strongly-integrable in each finite subinterval of $(0, \infty)$ and such that $\int_0^\infty \|f(x)\|^2 dx < \infty$. Then, H_1 is a Hilbert space [26] with inner product

$$(f, g)_1 = \int_0^\infty (f(x), g(x))_H dx.$$

Consider the differential expression in H_1

$$l(y) = -y'' + Q(x)y, \quad 0 < x < \infty, \tag{3}$$

where $Q(x)$ is a non-selfadjoint, completely continuous operator in H for each $x \in (0,\infty)$. We consider the operator L which is generated by (3) and the boundary condition

$$y(0) = 0. \tag{4}$$

The domain $D(L)$ of L is the subspace of H_1, consisting all $y \in H_1$ such that;
(i) y is twice strongly-differentiable,
(ii) $L(y) \in H_1$,
(iii) $y(0) = 0$.
Consider the equations

$$-y'' + Q(x)y = \lambda^2 y, \quad 0 < x < \infty, \tag{5}$$

$$-Y'' + Q(x)Y = \lambda^2 Y, \quad 0 < x < \infty. \tag{6}$$

where $y(x)$ is a vector-valued function and $Y(x)$ is an operator-valued function i.e, $Y(x)$ is an operator in H for each $x \in (0,\infty)$.

Lemma 2.1. *Every sequence of solutions of Equation (5) can be represented as an operator-valued function which satisfies Equation (6). Conversely, one can construct a sequence of vector-valued functions which satisfy Equation (5) for a given operator-valued solution of the Equation (6).*

Proof. Since H is a separable Hilbert space, there exists an orthonormal basis $(u_n)_{n\in\mathbb{N}}$. Suppose vector-valued functions $(y_n(x))_{n\in\mathbb{N}}$ satisfy Equation (5). We can construct an operator-valued function $Y(x)$ such that $Y(x)u_n = y_n(x)$ for every $n \in \mathbb{N}$. It is obvious that $Y(x)$ satisfies Equation (6).
Conversely, suppose operator-valued function $Y(x)$ satisfies Equation (6). Let $y_n(x) = Y(x)u_n$ for every $n \in \mathbb{N}$. Then, it is clear that $(y_n(x))$ satisfies Equation (5) for every $n \in \mathbb{N}$. □

As a result of above lemma, there is a one to one correspondance between the solutions of (5) and (6). Therefore, it is enough to consider only one of the equations (5) and (6). We shall use the notations

$$\sigma(x) = \int_x^\infty \|Q(t)\|\, dt, \quad \sigma_1(x) = \int_x^\infty t\,\|Q(t)\|\, dt.$$

Let us assume

$$\int_0^\infty t\,\|Q(t)\|\, dt < \infty. \tag{7}$$

Under the condition (7), Equation (6) has a bounded solution $E(x,\lambda)$ satisfying the condition

$$\lim_{x\to\infty} e^{i\lambda x} E(x,\lambda) = I, \quad \operatorname{Im}(\lambda) \le 0. \tag{8}$$

$E(x,\lambda)$ is called the Jost solution of Equation (6) (see Theorem 2 in [3]). We have the representation

$$E(x,\lambda) = e^{-i\lambda x} I + \int_x^\infty e^{-i\lambda t} K(x,t)dt, \quad \operatorname{Im}(\lambda) \le 0. \tag{9}$$

where the operator kernel satisfies

$$\|K(x,t)\| \le c\sigma\left(\frac{x+t}{2}\right) \tag{10}$$

where $c > 0$ is constant (see Theorem 3 in [3]).
The point spectrum of L is (see Theorem 6 in [3])

$$\sigma_d(L) = \left\{\lambda^2 : \operatorname{Im}(\lambda) < 0,\ E(\lambda) := E(0,\lambda) \text{ is not invertible}\right\}.$$

Let us recall

$$E(\lambda) = I + \int_0^\infty e^{-i\lambda t} K(0,t)dt, \ \mathrm{Im}(\lambda) \leq 0.$$

Let

$$A(\lambda) := \int_0^\infty e^{-i\lambda t} K(0,t)dt.$$

Then, $A(\lambda) \in \sigma_\infty$ for $\mathrm{Im}(\lambda) \leq 0$ and $A(\lambda)$ is analytic operator function in $\mathrm{Im}(\lambda) < 0$.

Definition 2.1. *An operator R is called the resolvent* [14] *of the operator A if*

$$(I + R)(I - A) = I.$$

Now, we apply [14] into our case. Let $R(\lambda)$ denote the resolvent of $-A(\lambda)$. We have $I + R(\lambda) = (I + A(\lambda))^{-1} = (E(\lambda))^{-1}$. According to [14], if $I + R(\lambda)$ exists for $\lambda = \lambda_0$, i.e., $E(\lambda)$ is invertible, then $I + R(\lambda)$ exists over $\mathbb{C}_-$ except for a set of isolated points, and is a meromorphic function of λ. Since $\sigma_d(L) \neq \mathbb{C}_-$, we have at least one $\lambda = \lambda_0$ such that $I + R(\lambda)$ exists. As a result, $I + R(\lambda)$ exists over $\mathbb{C}_-$ except for a set of isolated points which are the eigenvalues of L, and is a meromorphic function of λ. Hence, we can represent

$$(E(\lambda))^{-1} = I + R(\lambda) = \frac{S(\lambda)}{d(\lambda)}, \quad \lambda \in \mathbb{C}_-,$$

where $S(\lambda)$ is an analytic operator function and $d(\lambda)$ is an analytic scalar function in $\mathbb{C}_-$. Further, these isolated singular points are poles of $I + R(\lambda)$ and they are the zeros of the analytic function $d(\lambda)$. Therefore, we can rewrite the set

$$\begin{aligned} \sigma_d(L) &= \left\{\lambda^2 : \mathrm{Im}(\lambda) < 0, \ \lambda \text{ is a pole of } I + R(\lambda)\right\} \\ &= \left\{\lambda^2 : \mathrm{Im}(\lambda) < 0, \ d(\lambda) = 0\right\}. \end{aligned}$$

Theorem 2.1. *If $\int_0^\infty t \|Q(t)\| \, dt < \infty$ holds, then $\sigma_d(L)$ is bounded and countable. Moreover, the limit points (if exist)of it lie in a bounded subinterval of the real line (see Theorem 8 in* [3]*).*

Now let us assume

$$\int_0^\infty e^{\epsilon t} \|Q(t)\| \, dt < \infty, \quad \epsilon > 0. \tag{11}$$

Theorem 2.2. *Under the condition (11), L has a finite number of eigenvalues (see Theorem 9 in* [3]*).*

Theorem 2.3. *The continuous spectrum of L is $\sigma_c(L) = \mathbb{R}_+$ (see Theorem 13 in* [3]*)*

Now, we introduce the set of spectral singularities $\sigma_{ss}(L)$ of L according to [19, 21, 22].

$$\sigma_{ss}(L) = \left\{\lambda^2 : \ \lambda \in \mathbb{R} \setminus \{0\}, \ E(\lambda) := E(0,\lambda) \text{ is not invertible}\right\}. \tag{12}$$

Theorem 2.4. *Under the condition (11), L has a finite number of spectral singularities (see Theorem 15 in* [3]*).*

3. Associated Functions

Definition 3.1. *Let $B(\lambda)$ be an operator function defined on $D \subset \mathbb{C}$ such that $B(\lambda) \in \sigma_\infty$ for each $\lambda \in D$. If the equation $y = B(c)y$ has a non-trival solution in H then, y is an* *eigenelement and c is an characteristic value of $B(\lambda)$. We call y_k an associated element of order k to the eigenelement y if y_k is obtained as a result of solving the chain of equations*

$$\begin{aligned} y &= B(c)y, \\ y_1 &= B(c)y_1 + \frac{1}{1!}\frac{\partial B(c)}{\partial c}y, \\ &\vdots \\ y_k &= B(c)y_k + \frac{1}{1!}\frac{\partial B(c)}{\partial c}y_{k-1} + ... + \frac{1}{k!}\frac{\partial^k B(c)}{\partial c^k}y. \end{aligned}$$

In this case, we say that $y, y_1, ..., y_k$ form a chain of associated elements. We denote the maximum order of elements associated to y by m. The number $m+1$ is defined as the multiplicity of the eigenelement y. We call $y^k, y_1^k, ..., y_{m_k}^k$ $(k = 1, 2, ...)$ a canonical system of eigenelements and associated elements for $\lambda = c$ if;

i) $y^1, y^2, ..., y^k$ form a basis of the subspace of eigenelements corresponding to $\lambda = c$,

ii) y^1 is an eigenelement whose multiplicity attains the possible maximum $m_1 + 1$,

iii) y^k is an eigenelement, not expressible as a linear combination of $y^1, y^2, ..., y^{k-1}$ whose multiplicity attains the possible maximum $m_k + 1$,

iv) $y^k, y_1^k, ..., y_{m_k}^k$ form a chain of associated elements.

Note that the numbers $m_1, m_2, ..., m_k$ do not depend on the choice of the canonical system. The number $N = m_1 + 1 + m_2 + 1 + ... + m_k + 1$ is defined as the multiplicity of the characteristic value $\lambda = c$ (see [14]*).*

It is obvious that c^2 is an eigenvalue or a spectral singularity of L iff c is a characteristic value of $A(\lambda)$. We define the multiplicity of the eigenvalue or the spectral singularity $\lambda = c^2$ as the multiplicity of the characteristic value $\lambda = c$.

Theorem 3.1. *Let c^2 be an eigenvalue of L. Then, the multiplicity of the eigenvalue $\lambda = c^2$ is finite.*

Proof. It is well known for a completely continuous operator $A(\lambda)$ and a given characteristic value that the number of linearly independent eigenelements is finite. Also, the order of the associated elements for an eigenelement doesn't exceed the order of the pole of the resolvent $I + R(\lambda)$ at $\lambda = c$ [14]. Hence, each canonical system has a finite number of elements and c has finite multiplicity. □

Corollary 3.1. *Under the condition*

$$\int_0^\infty e^{\epsilon t} \|Q(t)\| \, dt < \infty, \quad \epsilon > 0,$$

L has a finite number eigenvalues and spectral singularities with finite multiplicity.

Proof. The result is a combination of Theorems 2.2, 2.4 and 3.1. □

Let $\lambda_1^2, \lambda_2^2, ..., \lambda_j^2$ and $\lambda_{j+1}^2, \lambda_{j+2}^2, ..., \lambda_v^2$ denote the eigenvalues and spectral singularities with multiplicities $m_1, m_2, ..., m_j$ and $m_{j+1}, m_{j+2}, ..., m_v$, respectively. We define the operator functions

$$\begin{aligned} u_{n,k}(x) &= \frac{1}{n!}\left\{\frac{\partial^n E(x,\lambda)}{\partial \lambda^n}\right\}_{\lambda=\lambda_k}, \; n = 0, 1, ..., \; m_k - 1, \; k = 1, 2, ..., j \\ v_{n,k}(x) &= \frac{1}{n!}\left\{\frac{\partial^n E(x,\lambda)}{\partial \lambda^n}\right\}_{\lambda=\lambda_k}, \; n = 0, 1, ..., \; m_k - 1, \; k = j+1, j+2, ..., v \end{aligned}$$

Then, for $\lambda=\lambda_k \quad (k=1,2,...,j)$ we have

$$\begin{aligned} l\left(u_{0,k}\right) &= 0, \\ l\left(u_{1,k}\right) &+\frac{1}{1!}\frac{\partial}{\partial\lambda} l\left(u_{0,k}\right)=0, \\ &\vdots \\ l\left(u_{n,k}\right) &+\frac{1}{1!}\frac{\partial}{\partial\lambda} l\left(u_{n-1,k}\right)+\frac{1}{2!}\frac{\partial^2}{\partial\lambda^2} l\left(u_{n-2,k}\right)=0, \quad n=2,3,...,\ m_k-1 \end{aligned}$$

where

$$l\left(u\right)=-u^{"}+Q(x)u-\lambda^2 u,$$

and $\frac{\partial^m}{\partial\lambda^m} l\left(u\right)$ denotes the differential expressions whose coefficients are the m-th derivatives with respect to λ of the corresponding coefficients of the differential expressions $l\left(u\right)$.

Then, $u_{0,k}(x)$ is the eigenfunction corresponding to the eigenvalue $\lambda=\lambda_k^2$ and $u_{1,k}(x)$, $u_{2,k}(x),...,u_{m_k-1,k}(x)$ are the associated elements of $u_{0,k}(x)$ for $k=1,2,...,j$ (see [14]). $u_{0,k}(x),u_{1,k}(x),...,u_{m_k-1,k}(x)$ are called the principal functions corresponding to the eigenvalue $\lambda=\lambda_k^2$ for $k=1,2,...,j$. Similarly, the principal functions corresponding to the spectral singularities $\lambda=\lambda_k^2$ are $v_{0,k}(x)\ ,v_{1,k}(x)\ ,...,\ v_{m_k-1,k}(x)$ for $k=j+1,j+2,...,v$.

Theorem 3.2. *$u_{n,k}\in L_2\left(\mathbb{R}_+,H\right),\ n=0,1,2,...,\ m_k-1,\ k=1,2,...,j$ and $v_{n,k}\ \notin\ L_2\left(\mathbb{R}_+,H\right),\ n=0,1,2,...,\ m_k-1,\ k=j+1,j+2,...,v$.*

Proof. From (9) we have for $k=1,2,...,j$ that

$$u_{n,k}(x)=\frac{1}{n!}\left(-1\right)^n\left(ix\right)^n e^{-i\lambda_k x}I+\frac{1}{n!}\int_x^\infty\left(-1\right)^n\left(it\right)^n e^{-i\lambda_k t}K(x,t)dt.$$

Since $\operatorname{Im}(\lambda)<0$ we have for $k=1,2,...,j$ that

$$\begin{aligned} \int_0^\infty\left\|\left(-1\right)^n\left(ix\right)^n e^{-i\lambda_k x}I\right\|^2 dx &= \int_0^\infty x^{2n}e^{2x\operatorname{Im}(\lambda_k)}dx \\ &= \frac{-1}{\left(2\operatorname{Im}(\lambda_k)\right)^{2n+1}}\Gamma\left(2n+1\right) \\ &< \infty. \end{aligned}$$

For $k=1,2,...,j$ we define

$$g_k(x)=\frac{1}{n!}\int_x^\infty\left(-1\right)^n\left(it\right)^n e^{-i\lambda_k t}K(x,t)dt.$$

We have from (10) and (11) that

$$\begin{aligned} \left\|K(x,t)\right\| &\le c\sigma(\frac{x+t}{2}) \\ &= c\int_{\frac{x+t}{2}}^\infty e^{\epsilon s}e^{-\epsilon s}\left\|Q(s)\right\|ds \\ &\le ce^{-\epsilon\left(\frac{x+t}{2}\right)}\int_{\frac{x+t}{2}}^\infty e^{\epsilon s}\left\|Q(s)\right\|ds \\ &\le ce^{-\epsilon\left(\frac{x+t}{2}\right)}\int_0^\infty e^{\epsilon s}\left\|Q(s)\right\|ds \\ &\le Ce^{-\epsilon\left(\frac{x+t}{2}\right)} \end{aligned} \tag{13}$$

where

$$c = \frac{1}{2} e^{\sigma_1(x)}, \quad C = c \int_0^\infty e^{\epsilon s} \|Q(s)\| \, ds$$

are positive constants. From (13) we obtain

$$\begin{aligned} \|g_k(x)\| &\leq \frac{C}{n!} \int_x^\infty t^n exp\left(t \operatorname{Im}(\lambda_k) - \epsilon \left(\frac{x+t}{2} \right) \right) dt \\ &\leq \frac{C}{n!} e^{x \operatorname{Im}(\lambda_k)} \int_0^\infty t^n e^{-\epsilon \left(\frac{x+t}{2} \right)} dt \\ &\leq C_0 e^{x \operatorname{Im}(\lambda_k)}, \end{aligned}$$

where

$$C_0 = \frac{C}{n!} \int_0^\infty t^n e^{-\epsilon \frac{t}{2}} dt$$

is a positive constant. Therefore, we have

$$\int_0^\infty \|g_k(x)\|^2 \, dx < C_0^2 \int_0^\infty e^{2x \operatorname{Im}(\lambda_k)} dx < \infty,$$

and hence $u_{n,k} \in L_2(\mathbb{R}_+, H)$, $n = 0, 1, 2, ..., m_k - 1$, $k = 1, 2, ..., j$.
From (9) we have for $k = j+1, j+2, ..., v$ that

$$v_{n,k}(x) = \frac{1}{n!} (-1)^n (ix)^n e^{-i\lambda_k x} I + \frac{1}{n!} \int_x^\infty (-1)^n (it)^n e^{-i\lambda_k t} K(x,t) dt. \tag{14}$$

Since $\operatorname{Im}(\lambda_k) = 0$ for $k = j+1, j+2, ..., v$, Equation (14) implies

$$\begin{aligned} \int_0^\infty \left\| (-1)^n (ix)^n e^{-i\lambda_k x} I \right\|^2 dx &= \int_0^\infty x^{2n} e^{2x \operatorname{Im}(\lambda_k)} dx \\ &= \int_0^\infty x^{2n} dx \\ &= \infty. \end{aligned}$$

Hence $v_{n,k} \notin L_2(\mathbb{R}_+, H)$, $n = 0, 1, 2, ..., m_k - 1$, $k = j+1, j+2, ..., v$. $\square$

Let us define the Hilbert space of vector-valued functions taking values in H, by

$$\begin{aligned} H_n &= \left\{ f : \int_0^\infty (1+x)^{2n} \|f(x)\|^2 \, dx \right\}, \quad n = 1, 2, ... \\ H_{-n} &= \left\{ g : \int_0^\infty (1+x)^{-2n} \|g(x)\|^2 \, dx \right\}, \quad n = 1, 2, ... \end{aligned}$$

Then, it follows $H_{n+1} \subsetneqq H_n \subsetneqq L_2(\mathbb{R}_+, H) \subsetneqq H_{-n} \subsetneqq H_{-(n+1)}$, $\quad n = 1, 2, ...$

Theorem 3.3. $v_{n,k} \in H_{-(n+1)}$, $\quad n = 0, 1, 2, ..., m_k - 1$, $k = j+1, j+2, ..., v$.

Proof. Since $\operatorname{Im}(\lambda_k) = 0$ for $k = j+1, j+2, ..., v$, Equation (14) implies

$$\begin{aligned} \int_0^\infty (1+x)^{-2(n+1)} \left\| \frac{1}{n!} (-1)^n (ix)^n e^{-i\lambda_k x} I \right\|^2 dx &= \frac{1}{n!} \int_0^\infty (1+x)^{-2(n+1)} x^{2n} dx \\ &< \infty, \end{aligned} \tag{15}$$

and also

$$\begin{aligned}
&\int_0^\infty (1+x)^{-2(n+1)} \left\| \frac{1}{n!} \int_x^\infty (-1)^n (it)^n e^{-i\lambda_k t} K(x,t) dt \right\|^2 dx \\
&\leq \int_0^\infty (1+x)^{-2(n+1)} \frac{1}{n!} \int_x^\infty t^{2n} \left\| K(x,t) \right\|^2 dt dx \\
&\leq \int_0^\infty (1+x)^{-2(n+1)} \frac{1}{n!} \int_0^\infty t^{2n} C^2 e^{-\epsilon(x+t)} dt dx \\
&= C_1 \int_0^\infty (1+x)^{-2(n+1)} e^{-\epsilon x} dx \\
&< \infty,
\end{aligned} \tag{16}$$

where

$$C = c \int_0^\infty e^{\epsilon s} \left\| Q(s) \right\| ds$$

and

$$C_1 = \frac{1}{n!} \int_0^\infty t^{2n} C^2 e^{-\epsilon t} dt = \frac{1}{n!} \frac{1}{\varepsilon^{2n+1}} \Gamma(2n+1)$$

are positive constants. Equations (15) and (16) imply $v_{n,k} \in H_{-(n+1)}, \quad n = 0, 1\ 2, ..., m_k - 1,\ k = j+1,\ j+2, ..., v.$ □

Corollary 3.2. $v_{n,k} \in H_{-m}$, *where* $m = \max\{m_{j+1}, m_{j+2}, ..., m_v\}, \quad n = 0, 1, 2, ..., m_k - 1,\ k = j+1, j+2, ..., v.$

Proof. The proof easily follows from Theorem 3.3 and $H_{-n} \subset H_{-(n+1)}$. □

4. Conclusions

Sturm-Liouville equations have a fundamental importance in mathematical physics, especially in quantum mechanics. As a result, the research on these equations are widespread. In particular, there have been many studies about Sturm-Liouville equations with selfadjoint operator coefficients. However, there are not enough results for the non-selfadjoint case. In this study, we obtain some important results about the principal functions of the non-selfadjoint Sturm-Liouville operator with operator coefficient with the aim of contributing the studies on Sturm-Liouville operator equation with non-selfadjoint operator coefficients.

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