TWMS J. App. and Eng. Math. V.11, N.1, 2021, pp. 134-150

COMMON FIXED SOFT ELEMENT RESULTS IN SOFT COMPLEX VALUED b-METRIC SPACES

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Abstract. In this paper, firstly, on the base of the soft complex numbers suggested by Das and Samanta, we introduce the concept of a soft complex valued b-metric space and investigate some of its properties. Also, we compare it to a soft complex valued metric space and a soft topological space. Next, we establish some fixed soft element theorems in the context of soft complex valued b-metric spaces and give suitable examples to illustrate the usability of the obtained main results. These results extend and generalize the corresponding results given in the existing literature.

Keywords: Soft set, soft complex number, soft complex valued b-metric space, fixed soft element.

AMS Subject Classification: 06D72, 54A40, 54H25

1. Introduction

In 1999, Molodtsov [21] initiated the concept of a soft set theory as a new approach for coping with uncertainties and also presented the basic results of the new theory. In [21], Molodtsov successfully applied the soft set theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration and theory of measurement. After presentation of the operations of soft sets [20], the properties and applications of this theory have been studied increasingly [4, 16, 22, 28].

Shabir and Naz [32] initiated the study of soft topological spaces. Das and Samanta [10] presented the notion of a soft metric space by employing soft elements introduced in [9] and investigated some of its fundamental properties. Recently, many papers concerning the soft set theory have been published [5, 13, 19, 26, 35].

Fixed point theory plays a fundamental role in mathematics and applied sciences, such as optimization, mathematical models and economic theories. Also, this theory have been applied to show the existence and uniqueness of the solutions of differential equations, integral equations and many other branches of mathematics [14, 27, 33]. A basic result in fixed point theory is the Banach contraction principle. Since the appearance of this principle, there has been a lot of activity in this area.

In 2011, Azam et al. [6] defined the notion of a complex valued metric space which is

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§ Manuscript received: March 30, 2019; accepted: July 16, 2019.

more general than the well-known metric space and obtained some fixed point results for a pair of mappings satisfying a rational inequality. In this line, Rouzkard et al. [30] studied some common fixed point theorems in this space to generalize the result of [6]. Ahmad et al. [3] investigated some common fixed point results for the mappings satisfying rational expressions on a closed ball in such space. Later, Rao et al. [29] gave a common fixed point theorem in complex valued b-metric spaces, generalizing both the b-metric spaces introduced by Czerwik [7] and the complex valued metric spaces. After the establishment of this new idea, Mukhemier [23] presented common fixed point results of two self-mappings satisfying a rational inequality in complex valued b-metric spaces. Verma [36] studied a common fixed point theorem using property (CLCS) in these spaces. In recent years, there has been a considerable literature on fixed point theory in complex valued metric spaces [2, 15, 24, 25, 33, 34, 37].

Extensions of fixed point theorems to the soft sets have been studied by some authors. Wardowski [38] defined, in a different way than in the literature, a soft mapping and obtained fixed point results in the soft set theory. Das and Samanta [10] investigated Banach fixed point theorem in the soft setting. Then, Yazar et al. [39] proved some fixed point theorems in soft metric spaces. Abbas et al. [1] established soft metric versions of several important fixed point theorems for metric spaces. Guler et al. [17, 18] introduced soft G-metric spaces with the help of soft elements and proved a fixed point theorem on these spaces.

In this work, with the help of soft complex numbers, we define the notion of soft complex valued b-metric spaces and study some of their topological aspects. Next, we prove some fixed soft element theorems for various soft mappings on soft complex valued b-metric spaces. Moreover, we furnish some examples to demonstrate the validity of the obtained results.

2. Preliminaries

In this section, we recollect some basic notions regarding soft sets. Throughout this work, let X be an initial universe, $P(X)$ be the power set of X and E be a set of parameters for X.

Definition 2.1. ([21]) *A soft set F on the universe X with the set E of parameters is defined by the set of ordered pairs*

$$F = \{(e, F(e)) : e \in E, F(e) \in P(X)\}$$

where F is a mapping given by $F : E \to P(X)$.

Throughout this paper, the family of all soft sets over X is denoted by $S(X, E)$ [5].

Definition 2.2. ([4, 20, 28]) *Let $F, G \in S(X, E)$. Then,*

(i) If $F(e) = \varnothing$ for every $e \in E$,then F is called null soft set, denoted by $\widetilde{\varnothing}$.

(ii) If $F(e) = X$ for every $e \in E$, then F is called absolute soft set, denoted by $\widetilde{X}$.

(iii) F is a soft subset of G if $F(e) \subseteq G(e)$ for every $e \in E$. It is denoted by $F \sqsubseteq G$.

(iv) The complement of F is denoted by F^c, where $F^c : E \to P(X)$ is a mapping defined by $F^c(e) = X \backslash F(e)$ for every $e \in E$. Clearly, $(F^c)^c = F$.

(v) The union of F and G is a soft set H defined by $H(e) = F(e) \cup G(e)$ for every $e \in E$. H is denoted by $F \sqcup G$.

(vi) The intersection of F and G is a soft set H defined by $H(e) = F(e) \cap G(e)$ for every $e \in E$. H is denoted by $F \sqcap G$.

Definition 2.3. ([9]) *Let X be a non-empty set and E be a non-empty parameter set. Then a function $\tilde{x} : E \to X$ is said to be a soft element of X. A soft element $\tilde{x}$ of X is said to belong to a soft set F of X, denoted by $\tilde{x} \tilde{\in} F$, if $\tilde{x}(e) \in F(e)$ for every $e \in E$. Thus a soft set F can be expressed as $F(e) = \{\tilde{x}(e) : \tilde{x} \tilde{\in} F\}$ for every $e \in E$.*

Throughout this paper, the family of all soft elements in X with the set E of parameters is denoted by X^E.

Definition 2.4. ([9]) *Let $\tilde{x}, \tilde{y} \in X^E$. Then,*

$\tilde{x} = \tilde{y} \Leftrightarrow \tilde{x}(e) = \tilde{y}(e)$ *for every $e \in E$.*

Definition 2.5. ([9]) *Let $\tilde{x} \in X^E$ and let $F_i \in S(X,E)$ for all $i \in J$ where J is an index set. Then,*

(i) $\tilde{x} \tilde{\in} \bigsqcup_{i\in J} F_i \Leftrightarrow \tilde{x} \tilde{\in} F_{i_0}, \exists i_0 \in J$.

(ii) $\tilde{x} \tilde{\in} \bigsqcap_{i\in J} F_i \Leftrightarrow \tilde{x} \tilde{\in} F_i, \forall i \in J$.

Remark 2.1. ([9]) *It is to be noted that every singleton soft set (that is, for every $e \in E$, $F(e)$ is a singleton set) can be identified with a soft element by simply identifying the singleton set with the element that it contains for every $e \in E$.*

Definition 2.6. ([9]) *Let $\mathbb{R}$ be the set of real numbers, $\mathfrak{B}(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and E be a set of parameters. Then,*

$$F = \{(e, F(e)) : e \in E, F(e) \in \mathfrak{B}(\mathbb{R})\}$$

is called a soft real set.

If specifically F is a singleton soft set, then after identifying F with the corresponding soft element, it will be called a soft real number.

Definition 2.7. ([9]) *Let F, G be soft real numbers. Then,*

(i) *The sum is defined by $(F + G)(e) = F(e) + G(e)$ for every $e \in E$.*

(ii) *The difference is defined by $(F - G)(e) = F(e) - G(e)$ for every $e \in E$.*

(iii) *The product is defined by $(F.G)(e) = F(e).G(e)$ for every $e \in E$.*

(iv) *The division is defined by $\left(\frac{F}{G}\right)(e) = \frac{F(e)}{G(e)}$, provided $G(e) \neq 0$ for every $e \in E$.*

(v) *The modulus is defined by $|F|(e) = |F(e)|$ for every $e \in E$.*

(vi) *The scalar multiplication of F by k is defined by $(k.F)(e) = k.F(e)$ for every $e \in E$.*

From the above definition of soft real numbers it follows that $F + G, F - G$, $F.G$, $\frac{F}{G}$, $|F|$ and $k.F$ are soft real numbers.

We use notations $\tilde{r}$, $\tilde{s}$, $\tilde{b}$ to denote soft real numbers whereas $\bar{r}$, $\bar{s}$ will denote a particular type of soft real numbers such that $\bar{r}(e) = r$ for every $e \in E$ etc. For example, $\overline{0}$ is the soft real number, where $\bar{0}(e) = 0$ for every $e \in E$ [9].

Definition 2.8. ([9]) *Let $\tilde{r}$ and $\tilde{s}$ be two soft real numbers. Then,*

(i) $\tilde{r} \tilde{\leqslant} \tilde{s}$ if $\tilde{r}(e) \leqslant \tilde{s}(e)$ for every $e \in E$.

(ii) $\tilde{r} \tilde{\geqslant} \tilde{s}$ if $\tilde{r}(e) \geqslant \tilde{s}(e)$ for every $e \in E$.

(iii) $\tilde{r} \tilde{<} \tilde{s}$ if $\tilde{r}(e) < \tilde{s}(e)$ for every $e \in E$.

(iv) $\tilde{r} \tilde{>} \tilde{s}$ if $\tilde{r}(e) > \tilde{s}(e)$ for every $e \in E$.

Definition 2.9. ([9]) *A sequence $\{\widetilde{s_n}\}$ of soft real numbers is said to converge to $\tilde{s}$, and we write $lim_{n\to\infty}\widetilde{s_n} = \tilde{s}$, if for every $\tilde{\epsilon} \tilde{>} \overline{0}$, there exists an $n_0 \in \mathbb{N}$ such that $|\widetilde{s_n} - \tilde{s}| \tilde{<} \tilde{\epsilon}$ for all $n \geqslant n_0$.*

Theorem 2.1. ([9]) *A sequence* $\{\widetilde{s_n}\}$ *of soft real numbers is said to be a Cauchy sequence if for every* $\tilde{\epsilon} \tilde{>} \overline{0}$, *there exists an* $n_0 \in \mathbb{N}$ *such that* $|\widetilde{s_n} - \widetilde{s_m}| \tilde{<} \tilde{\epsilon}$ *for all* $m, n \geqslant n_0$.

Theorem 2.2. ([8]) *Every Cauchy sequence of soft real numbers with a finite set of parameters converges to a soft real number.*

Definition 2.10. ([11]) *Let* $\mathbb{C}$ *be the set of complex numbers,* $\mathfrak{B}(\mathbb{C})$ *be the collection of all non-empty bounded subsets of* $\mathbb{C}$ *and* E *be a set of parameters. Then,*

$$F = \{(e, F(e)) : e \in E, F(e) \in \mathfrak{B}(\mathbb{C})\}$$

is called a soft complex set.

If specifically F *is a singleton soft set, then after identifying* F *with the corresponding soft element, it will be called a soft complex number.*

Definition 2.11. ([11]) *Let* F *be a soft complex set (number). Then the real and imaginary parts of* F *are denoted by* ReF *and* ImF *and are defined by*

$$ReF(e) = \{Re(z) : z \in F(e)\} \quad \big(ReF(e) = Re(F(e))\big)$$
$$ImF(e) = \{Im(z) : z \in F(e)\} \quad \big(ImF(e) = Im(F(e))\big)$$

for every $e \in E$. *By definition of soft real sets (numbers) it follows that* ReF *and* ImF *are soft real sets (numbers).*

Definition 2.12. ([11]) *Let* F, G *be soft complex numbers. Then,*

(i) *The sum is defined by* $(F + G)(e) = F(e) + G(e)$ *for every* $e \in E$.

(ii) *The difference is defined by* $(F - G)(e) = F(e) - G(e)$ *for every* $e \in E$.

(iii) *The product is defined by* $(F.G)(e) = F(e).G(e)$ *for every* $e \in E$.

(iv) *The division is defined by* $\left(\frac{F}{G}\right)(e) = \frac{F(e)}{G(e)}$, *provided* $G(e) \neq 0$ *for every* $e \in E$.

(v) *The scalar multiplication of* F *by* k *is defined by* $(k.F)(e) = k.F(e)$ *for every* $e \in E$.

From the above definition of soft complex numbers it follows that $F + G, F - G$, $F.G$, $\frac{F}{G}$ *and* $k.F$ *are soft complex numbers.*

We use notations as $\hat{z}, \hat{u}$ to denote soft complex numbers where $\hat{z} = \widetilde{z_1} + i\widetilde{z_2}$ and $\hat{u} = \widetilde{u_1} + i\widetilde{u_2}$ with $Re\hat{z} = \widetilde{z_1}$, $Im\hat{z} = \widetilde{z_2}$ and $Re\hat{u} = \widetilde{u_1}$, $Im\hat{u} = \widetilde{u_2}$. On the other hand, $\overset{\triangle}{v} = \overline{v} + i\overline{v}$ and $\overset{\triangle}{w} = \overline{w} + i\overline{w}$ will denote a particular type of soft complex numbers where $Re\overset{\triangle}{v} = Im\overset{\triangle}{v} = \overline{v}$ and $Re\overset{\triangle}{w} = Im\overset{\triangle}{w} = \overline{w}$. For example, $\overset{\triangle}{0} = \overline{0} + i\overline{0}$ is the soft complex number, where $\overset{\triangle}{0}(e) = \overline{0}(e) + i\overline{0}(e) = 0 + i0$ for every $e \in E$ [12].

Definition 2.13. ([11]) *Let* $\hat{z} = \widetilde{z_1} + i\widetilde{z_2}$ *be a soft complex number. Then the complex conjugate of* $\hat{z}$ *is denoted by* $\overline{\hat{z}}$ *and is defined by* $\overline{\hat{z}} = \widetilde{z_1} - i\widetilde{z_2}$.
By definition of soft complex numbers it follows that $\overline{\hat{z}}$ *is a soft complex number.*

Definition 2.14. ([11]) *Let* $\hat{z} = \widetilde{z_1} + i\widetilde{z_2}$ *be a soft complex number. Then, the modulus of* $\hat{z}$ *is denoted by* $|\hat{z}|$ *and is defined by* $|\hat{z}| = \sqrt{(\widetilde{z_1})^2 + (\widetilde{z_2})^2}$. *By definition of soft real numbers it follows that* $|\hat{z}|$ *is a nonnegative soft real number (that is,* $0 \leqslant |\hat{z}|(e) = \sqrt{(\widetilde{z_1}(e))^2 + (\widetilde{z_2}(e))^2}$ *for every* $e \in E$*).*

Theorem 2.3. ([11]) *Let* $\hat{z}$, $\hat{u} \in \mathbb{C}^E$ *where* $\hat{z} = \widetilde{z_1} + i\widetilde{z_2}$ *and* $\hat{u} = \widetilde{u_1} + i\widetilde{u_2}$. *Then, the following properties are satisfied.*

(i) $\overline{\hat{z} + \hat{u}} = \overline{\hat{z}} + \overline{\hat{u}}$.

(ii) $|\hat{z}| = |\overline{\hat{z}}|$.

(iii) $|\hat{z}|^2 = \hat{z}.\overline{\hat{z}}$.

(iv) $|\widetilde{z_1}| \widetilde{\leqslant} |\hat{z}|$ *and* $|\widetilde{z_2}| \widetilde{\leqslant} |\hat{z}|$.

(v) $|\hat{z}.\hat{u}| = |\hat{z}|.|\hat{u}|$.

(vi) $\left|\frac{\hat{z}}{\hat{u}}\right| = \frac{|\hat{z}|}{|\hat{u}|}$.

(vii) $|\hat{z} + \hat{u}| \widetilde{\leqslant} |\hat{z}| + |\hat{u}|$.

Definition 2.15. ([12]) *A sequence* $\{\widehat{z_n}\}$ *of soft complex numbers is said to converges to* $\hat{z}$*, and we write* $lim_{n\to\infty}\widehat{z_n} = \hat{z}$*, if for every* $\tilde{\epsilon} \widetilde{>} \overline{0}$*, there exists an* $n_0 \in \mathbb{N}$ *such that* $|\widehat{z_n} - \hat{z}| \widetilde{<} \tilde{\epsilon}$ *for all* $n \geqslant n_0$.

Theorem 2.4. ([12]) *Let* $\{\widehat{z_n}\}$ *be a sequence of soft complex numbers where* $\widehat{z_n} = \widetilde{z_{n_1}} + i\widetilde{z_{n_2}}$ *with* $\widetilde{z_{n_1}}, \widetilde{z_{n_2}} \in \mathbb{R}^E$*. Let* $\hat{z} = \widetilde{z_1} + i\widetilde{z_2}$ *such that* $\widetilde{z_1}, \widetilde{z_2} \in \mathbb{R}^E$*. Then,*

$$lim_{n\to\infty}\widehat{z_n} = \hat{z} \ \ if \ and \ only \ if \ \ lim_{n\to\infty}\widetilde{z_{n_1}} = \widetilde{z_1} \ \ and \ \ lim_{n\to\infty}\widetilde{z_{n_2}} = \widetilde{z_2}.$$

Definition 2.16. ([13]) *Let* X *be a set,* E *be a set of parameters and* $\mathfrak{R}$ *be a soft relation on* X^E*, i.e., a subset of* $X^E \times X^E$*. We say that* $\mathfrak{R}$ *is a soft order on* X^E *if it has the following properties (we write* $\tilde{x}\,\mathfrak{R}\,\tilde{y}$ *instead of* $(\tilde{x}, \tilde{y}) \in \mathfrak{R}$*):*

(so_1) For every $\tilde{x} \widetilde{\in} \widetilde{X}$*,* $\tilde{x}\,\mathfrak{R}\,\tilde{x}$.

(so_2) If $\tilde{x}\,\mathfrak{R}\,\tilde{y}$ *and* $\tilde{y}\,\mathfrak{R}\,\tilde{x}$*, then* $\tilde{x} = \tilde{y}$.

(so_3) If $\tilde{x}\,\mathfrak{R}\,\tilde{y}$ *and* $\tilde{y}\,\mathfrak{R}\,\tilde{z}$*, then* $\tilde{x}\,\mathfrak{R}\,\tilde{z}$.

A set X *together with a soft order* $\mathfrak{R}$ *on* X^E *is called a soft ordered set, denoted by* $(X, \mathfrak{R}, E)$*. For example,* $(\mathbb{R}, \widetilde{\leqslant}, E)$ *is a soft ordered set.*

Demir [12] introduced a soft order $\widetilde{\lesssim}$ on $\mathbb{C}^E$ for comparing two soft complex numbers as follows: Let $\hat{z}$, $\hat{u} \in \mathbb{C}^E$ where $\hat{z} = \widetilde{z_1} + i\widetilde{z_2}$ and $\hat{u} = \widetilde{u_1} + i\widetilde{u_2}$, then

$$\hat{z} \,\widetilde{\lesssim}\, \hat{u} \ \Leftrightarrow \ \widetilde{z_1} \,\widetilde{\leqslant}\, \widetilde{u_1} \ \ and \ \ \widetilde{z_2} \,\widetilde{\leqslant}\, \widetilde{u_2}.$$

Thus, $\hat{z} \,\widetilde{\lesssim}\, \hat{u}$ if one of the followings holds:

(1) $\widetilde{z_1} = \widetilde{u_1}$ and $\widetilde{z_2} = \widetilde{u_2}$,

(2) $\widetilde{z_1} \widetilde{<} \widetilde{u_1}$ and $\widetilde{z_2} = \widetilde{u_2}$,

(3) $\widetilde{z_1} = \widetilde{u_1}$ and $\widetilde{z_2} \widetilde{<} \widetilde{u_2}$,

(4) $\widetilde{z_1} \widetilde{<} \widetilde{u_1}$ and $\widetilde{z_2} \widetilde{<} \widetilde{u_2}$.

In particular, we will write $\hat{z} \,\widetilde{\lnsim}\, \hat{u}$ if $\hat{z} \lesssim \hat{u}$ and $\hat{z} \neq \hat{u}$ i.e., one of (2), (3) and (4) is satisfied and we will write $\hat{z} \,\widetilde{\prec}\, \hat{u}$ if only (4) is satisfied [12].

Lemma 2.1. ([12]) *Let* $m, n \in \mathbb{R}$ *and* $\hat{z}, \hat{u}, \hat{v}, \hat{w} \in \mathbb{C}^E$*. Then, the following properties hold.*

(i) If $m \leqslant n$*, then* $m.\hat{z} \,\widetilde{\lesssim}\, n.\hat{z}$.

(ii) If $\hat{z} \,\widetilde{\lesssim}\, \hat{u}$ *and* $\hat{u} \,\widetilde{\prec}\, \hat{v}$*, then* $\hat{z} \,\widetilde{\prec}\, \hat{v}$.

(iii) If $\overset{\triangle}{0} \,\widetilde{\lesssim}\, \hat{z} \,\widetilde{\lnsim}\, \hat{u}$ *, then* $|\hat{z}| \widetilde{<} |\hat{u}|$.

(iv) If $\hat{z} \,\widetilde{\lesssim}\, \hat{u}$*,* $\hat{v} \,\widetilde{\lesssim}\, \hat{w}$*, then* $\hat{z} + \hat{v} \,\widetilde{\lesssim}\, \hat{u} + \hat{w}$.

Definition 2.17. ([32]) *Let* τ *be a collection of soft sets over* X*. Then,* τ *is said to be a soft topology on* X *if*

(st_1) $\widetilde{\varnothing}, \widetilde{X}$ *belong to* τ,

(st_2) *the union of any number of soft sets in* τ *belongs to* τ,

(st_3) *the intersection of any two soft sets in* τ *belongs to* τ.

The triplet (X, τ, E) is called a soft topological space. The members of τ are called soft open sets in X. A soft set F over X is called a soft closed in X if $F^c \in \tau$.

Definition 2.18. ([5, 26]) *Let (X, τ, E) be a soft topological space. A subcollection $\mathcal{B}$ of τ is called a base for τ if every member of τ can be expressed as the union of some members of $\mathcal{B}$.*

Definition 2.19. ([31]) *Let (X, τ, E) be a soft topological space. A subcollection δ of τ is called a subbase for τ if the collection of all finite intersections of members from δ is a base for τ.*

Theorem 2.5. ([31]) *A collection δ of soft sets over X is a subbase for some soft topology τ on X if and only if*

(i) $\widetilde{\varnothing} \in \delta$ or $\widetilde{\varnothing}$ is the intersection of a finite number of members of δ,

(ii) $\widetilde{X} = \bigsqcup \delta$.

Definition 2.20. ([12]) *Let X be a nonempty set and E be a set of parameters. A mapping $d : X^E \times X^E \to \mathbb{C}^E$ is called a soft complex valued metric on X^E if it satisfies the following axioms:*

(scm_1) $\overset{\triangle}{0} \widetilde{\precsim} d(\tilde{x}, \tilde{y})$, for all $\tilde{x}, \tilde{y} \in X^E$.

(scm_2) $d(\tilde{x}, \tilde{y}) = \overset{\triangle}{0}$ if and only if $\tilde{x} = \tilde{y}$, for all $\tilde{x}, \tilde{y} \in X^E$.

(scm_3) $d(\tilde{x}, \tilde{y}) = d(\tilde{y}, \tilde{x})$, for all $\tilde{x}, \tilde{y} \in X^E$.

(scm_4) $d(\tilde{x}, \tilde{y}) \widetilde{\precsim} d(\tilde{x}, \tilde{z}) + d(\tilde{z}, \tilde{y})$, for all $\tilde{x}, \tilde{y}, \tilde{z} \in X^E$.

The triplet (X, d, E) is called a soft complex valued metric space on X^E.

Definition 2.21. ([26]) *Let $S(X, E)$ and $S(Y, E)$ be the families of all soft sets over X and Y, respectively. Let $f : X \to Y$ be a mapping. Then, the mapping f_E is called a soft mapping from X to Y, denoted by $f_E : S(X, E) \to S(Y, E)$.*

(i) Let $F \in S(X, E)$. Then, $f_E(F)$ is the soft set over Y defined as follows:

$$f_E(F)(e) = f(F(e))$$

for all $e \in E$. $f_E(F)$ is called a soft image of a soft set F.

(ii) Let $G \in S(Y, E)$. Then, $f_E^{-1}(G)$ is the soft set over X defined as follows:

$$f_E^{-1}(G)(e) = f^{-1}(G(e))$$

for all $e \in E$. $f_E^{-1}(G)$ is called a soft inverse image of a soft set G.

In particular, if we take $S(X, E) = X^E$ and $S(Y, E) = Y^E$, respectively then we have

(i) If $\tilde{x} \in X^E$, then $f_E(\tilde{x})$ is the soft element over Y defined as follows:

$$f_E(\tilde{x})(e) = f(\tilde{x}(e))$$

for all $e \in E$.

(ii) If f is bijective and $\tilde{y} \in Y^E$, then $f_E^{-1}(\tilde{y})$ is the soft element over X defined as follows:

$$f_E^{-1}(\tilde{y})(e) = f^{-1}(\tilde{y}(e))$$

for all $e \in E$ [12].

Definition 2.22. ([12]) *Let (X, d, E) be a soft complex valued metric space and let $f_E, g_E : (X, d, E) \to (X, d, E)$ be two soft mappings.*

(i) A soft element $\tilde{x} \in X^E$ is called a fixed soft element of f_E if $f_E(\tilde{x}) = \tilde{x}$.

(ii) A soft element $\tilde{x} \in X^E$ is called a common fixed soft element of f_E and g_E if $f_E(\tilde{x}) = g_E(\tilde{x}) = \tilde{x}$.

3. On Soft Complex Valued b-Metric Spaces

In this section, we present the notion of a soft complex valued b-metric space and study some of its topological aspects which strengthen this concept.

Definition 3.1. *Let X be a nonempty set, E be a set of parameters and $\tilde{b} \gtrapprox \overline{1}$ be a soft real number. A mapping $d_b : X^E \times X^E \to \mathbb{C}^E$ is called a soft complex valued b-metric on X^E if it satisfies the following axioms:*

$(scbm_1)$ $\overset{\triangle}{0} \precsim d_b(\tilde{x}, \tilde{y})$, for all $\tilde{x}, \tilde{y} \in X^E$.

$(scbm_2)$ $d_b(\tilde{x}, \tilde{y}) = \overset{\triangle}{0}$ if and only if $\tilde{x} = \tilde{y}$, for all $\tilde{x}, \tilde{y} \in X^E$.

$(scbm_3)$ $d_b(\tilde{x}, \tilde{y}) = d_b(\tilde{y}, \tilde{x})$, for all $\tilde{x}, \tilde{y} \in X^E$.

$(scbm_4)$ $d_b(\tilde{x}, \tilde{y}) \precsim \tilde{b}\,[d_b(\tilde{x}, \tilde{z}) + d_b(\tilde{z}, \tilde{y})]$, for all $\tilde{x}, \tilde{y}, \tilde{z} \in X^E$.

The triplet (X, d_b, E) is called a soft complex valued b-metric space on X^E.

It is seen that the above definition coincides with that of the soft complex valued metric when $\tilde{b} = \overline{1}$. Thus, the class of the soft complex valued b-metric spaces is larger than that of the soft complex valued metric spaces, that is, every soft complex valued metric space is a soft complex valued b-metric space. But the converse is not true in general as seen in the following example.

Example 3.1. *Let $X = \{1 + i, 3 - i2\}$, $E = \{e_1, e_2\}$ and let $\widetilde{x_1}, \widetilde{x_2}, \widetilde{x_3}, \widetilde{x_4} \in X^E$ where*

$$\begin{array}{llll} \widetilde{x_1}(e_1) = 1 + i & \widetilde{x_2}(e_1) = 1 + i & \widetilde{x_3}(e_1) = 3 - i2 & \widetilde{x_4}(e_1) = 3 - i2 \\ \widetilde{x_1}(e_2) = 1 + i & \widetilde{x_2}(e_2) = 3 - i2 & \widetilde{x_3}(e_2) = 1 + i & \widetilde{x_4}(e_2) = 3 - i2. \end{array}$$

Let us consider a mapping $d_b : X^E \times X^E \to \mathbb{C}^E$ such that

$$\begin{array}{ll} d_b(\widetilde{x_1}, \widetilde{x_1}) = d_b(\widetilde{x_2}, \widetilde{x_2}) = \overset{\triangle}{0}, & d_b(\widetilde{x_3}, \widetilde{x_3}) = d_b(\widetilde{x_4}, \widetilde{x_4}) = \overset{\triangle}{0}, \\ d_b(\widetilde{x_1}, \widetilde{x_2}) = d_b(\widetilde{x_2}, \widetilde{x_1}) = \overline{2} + i\overline{3}, & d_b(\widetilde{x_1}, \widetilde{x_3}) = d_b(\widetilde{x_3}, \widetilde{x_1}) = \overline{2} - i\overline{1}, \\ d_b(\widetilde{x_1}, \widetilde{x_4}) = d_b(\widetilde{x_4}, \widetilde{x_1}) = \overline{5} + i\overline{3}, & d_b(\widetilde{x_2}, \widetilde{x_3}) = d_b(\widetilde{x_3}, \widetilde{x_2}) = i\overline{4}, \\ d_b(\widetilde{x_2}, \widetilde{x_4}) = d_b(\widetilde{x_4}, \widetilde{x_2}) = \overline{3} + i\overline{1}, & d_b(\widetilde{x_3}, \widetilde{x_4}) = d_b(\widetilde{x_4}, \widetilde{x_3}) = \hat{z} = \tilde{z_1} + i\tilde{z_2}, \end{array}$$

where $\tilde{z_1}(e_1) = 1, \tilde{z_1}(e_2) = 3$ and $\tilde{z_2}(e_1) = 2, \tilde{z_2}(e_2) = 4$. Then, (X, d_b, E) is a soft complex valued b-metric space with the constant $\tilde{b} = \overline{3}$. However, since

$$d_b(\widetilde{x_2}, \widetilde{x_3}) = i\overline{4} \text{ and } d_b(\widetilde{x_2}, \widetilde{x_1}) + d_b(\widetilde{x_1}, \widetilde{x_3}) = \overline{4} + i\overline{2}$$

it is not a soft complex valued metric space.

Definition 3.2. *Let (X, d_b, E) be a soft complex valued b-metric space.*

(i) A sequence $\{\widetilde{x_n}\}$ of soft elements in X is said to converge to $\tilde{x} \in X^E$ if for every $\hat{c} \in \mathbb{C}^E$ with $\overset{\triangle}{0} \precnsim \hat{c}$, there exists an $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$, $d_b(\widetilde{x_n}, \tilde{x}) \precnsim \hat{c}$. We denote this by $lim_{n\to\infty}\widetilde{x_n} = \tilde{x}$.

(ii) A sequence $\{\widetilde{x_n}\}$ of soft elements in X is said to be a Cauchy sequence in (X, d_b, E) if for every $\hat{c} \in \mathbb{C}^E$ with $\overset{\triangle}{0} \precnsim \hat{c}$, there exists an $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$, $d_b(\widetilde{x_n}, \widetilde{x_{n+m}}) \precnsim \hat{c}$, where $m \in \mathbb{N}$.

(iii) (X, d_b, E) is said to be a complete soft complex valued b-metric space if for every Cauchy sequence $\{\widetilde{x_n}\}$ in (X, d_b, E) there exists an $\tilde{x} \in X^E$ such that $lim_{n\to\infty}\widetilde{x_n} = \tilde{x}$.

Example 3.2. *Let $X = \mathbb{C}$ and E be a finite set of parameters. Define the mapping $d_b : \mathbb{C}^E \times \mathbb{C}^E \to \mathbb{C}^E$ by*

$$d_b(\hat{z}, \hat{u}) = |\tilde{z_1} - \widetilde{u_1}|^2 + i|\tilde{z_2} - \widetilde{u_2}|^2 \quad for\ every\ \hat{z}, \hat{u} \in \mathbb{C}^E.$$

Then, (X, d_b, E) is a complete soft complex valued b-metric space with $\tilde{b} = \overline{2}$.

Definition 3.3. *Let (X, d_b, E) be a soft complex valued b-metric space, $\tilde{x} \in X^E$ and $\overset{\triangle}{0} \precsim \hat{z}$.*

(*i*) *A soft open ball with centre $\tilde{x}$ and radius $\hat{z}$ is denoted by $B(\tilde{x}, \hat{z})$ and defined by*

$$B(\tilde{x}, \hat{z})(e) = \bigcup \{\tilde{y}(e) \in X : d_b(\tilde{x}, \tilde{y}) \mathrel{\tilde{\prec}} \hat{z}\}$$

for every $e \in E$.

(*ii*) *A soft closed ball with centre $\tilde{x}$ and radius $\hat{z}$ is denoted by $B[\tilde{x}, \hat{z}]$ and defined by*

$$B[\tilde{x}, \hat{z}](e) = \bigcup \{\tilde{y}(e) \in X : d_b(\tilde{x}, \tilde{y}) \mathrel{\tilde{\precsim}} \hat{z}\}$$

for every $e \in E$.

Example 3.3. *Let (X, d_b, E) be a soft complex valued b-metric space which is defined in Example 3.1. Then, for $\widetilde{x_2} \in X^E$ and $\hat{z} = \overline{3} + i\overline{4} \in \mathbb{C}^E$,*

$$B(\tilde{x_2}, \hat{z}) = \{(e_1, \{1 + i\}), (e_2, X)\} \ and\ B[\tilde{x_2}, \hat{z}] = \{(e_1, X), (e_2, X)\}.$$

Theorem 3.1. *Let (X, d_b, E) be a soft complex valued b-metric space. Then, a family $\delta = \{B(\tilde{x}, \hat{z}) : \tilde{x} \in X^E,\ \overset{\triangle}{0} \mathrel{\tilde{\prec}} \hat{z}\}$ is a subbase for a soft topology τ on X.*

Proof. It follows immediately from Theorem 2.5 and the definition of soft open ball. □

Definition 3.4. *Let (X, d_b, E) be a soft complex valued b-metric space. Then, (X, d_b, E) is called a soft Hausdorff space if for any two soft elements $\tilde{x}, \tilde{y} \in X^E$ such that $d_b(\tilde{x}, \tilde{y}) \mathrel{\tilde{\succ}} \overset{\triangle}{0}$, there exist two soft open balls $B(\tilde{x}, \hat{z})$ and $B(\tilde{y}, \hat{u})$ such that $B(\tilde{x}, \hat{z}) \sqcap B(\tilde{y}, \hat{u}) = \widetilde{\emptyset}$.*

Theorem 3.2. *Every soft complex valued b-metric space is a soft Hausdorff space.*

Proof. Let $\tilde{x}, \tilde{y} \in X^E$ with $d_b(\tilde{x}, \tilde{y}) \mathrel{\tilde{\succ}} \overset{\triangle}{0}$. Then, we have $|d_b(\tilde{x}, \tilde{y})| = \tilde{r} \mathrel{\tilde{>}} \overline{0}$. Take a soft complex number $\hat{z} \mathrel{\tilde{\succ}} \overset{\triangle}{0}$ such that $|\hat{z}| = \frac{\tilde{r}}{2\tilde{b}}$. It is clear that $\tilde{x} \mathrel{\tilde{\in}} B(\tilde{x}, \hat{z})$ and $\tilde{y} \mathrel{\tilde{\in}} B(\tilde{y}, \hat{z})$. Now, we shall show that $B(\tilde{x}, \hat{z}) \sqcap B(\tilde{y}, \hat{z}) = \widetilde{\emptyset}$. Suppose that there is an $\tilde{x^*} \mathrel{\tilde{\in}} B(\tilde{x}, \hat{z}) \sqcap B(\tilde{y}, \hat{z})$. Therefore,

$$d_b(\tilde{x}, \tilde{y}) \mathrel{\tilde{\precsim}} \tilde{b}\, d_b(\tilde{x}, \tilde{x^*}) + \tilde{b}\, d_b(\tilde{x^*}, \tilde{y}) \mathrel{\tilde{\prec}} 2\tilde{b}\, \hat{z}.$$

Hence, $|d_b(\tilde{x}, \tilde{y})| \mathrel{\tilde{<}} 2\tilde{b}\, |\hat{z}| = \tilde{r} = |d_b(\tilde{x}, \tilde{y})|$ and we get a contradiction. □

4. Main Results

In this section, we prove some fixed soft element theorems in soft complex valued b-metric spaces, which are the soft versions of fixed point theorems which is proposed by Dubey et al. [15] and Mukheimer [23]. We start with the following lemmas that will play a crucial role in the proofs of the main theorems.

Throughout this chapter, we will use the case of nonequality of two soft elements as follows:

$\tilde{x} \neq \tilde{y} \Leftrightarrow \tilde{x}(e) \neq \tilde{y}(e)$ *for every* $e \in E$.

Lemma 4.1. *Let (X, d_b, E) be a soft complex valued b-metric space and $\{\widetilde{x_n}\}$ be a sequence of soft elements in X. Then, $\{\widetilde{x_n}\}$ converges to $\tilde{x}$ if and only if $lim_{n\to\infty}|d_b(\widetilde{x_n}, \tilde{x})| = \overline{0}$.*

Proof. Let $\{\widetilde{x_n}\}$ be converges to $\tilde{x}$ in (X, d_b, E). Let us consider $\hat{c} = \frac{\tilde{\epsilon}}{\sqrt{2}} + i\frac{\tilde{\epsilon}}{\sqrt{2}}$, for a soft real number $\tilde{\epsilon} \tilde{>} \overline{0}$. From the fact that $\overset{\triangle}{0} \tilde{\precsim} \hat{c} \in \mathbb{C}^E$ it follows that there exists a natural number n_0 such that $d_b(\widetilde{x_n}, \tilde{x}) \tilde{\precsim} \hat{c}$ for all $n \geqslant n_o$. Since

$$|d_b(\widetilde{x_n}, \tilde{x})| \tilde{<} |\hat{c}| = \tilde{\epsilon} \quad for\ all\ n \geqslant n_o$$

we have $lim_{n\to\infty}|d_b(\widetilde{x_n}, \tilde{x})| = \overline{0}$.

To prove the sufficiency, let us take a $\hat{c} \in \mathbb{C}^E$ with $\hat{c} \tilde{\succ} \overset{\triangle}{0}$. Then, there exists a soft real number $\tilde{r} \tilde{>} \overline{0}$ such that for $\hat{z} \in \mathbb{C}^E$

$$if\ |\hat{z}| \tilde{<} \tilde{r}\ \ then\ \ \hat{z} \tilde{\precsim} \hat{c}.$$

Hence, by $\tilde{r} \tilde{>} \overline{0}$, there is a natural number n_0 such that $|d_b(\widetilde{x_n}, \tilde{x})| \tilde{<} \tilde{r}$ for all $n \geqslant n_0$. Thus, we obtain $d_b(\widetilde{x_n}, \tilde{x}) \tilde{\precsim} \hat{c}$ for all $n \geqslant n_0$ and the proof is concluded. □

Lemma 4.2. *Let (X, d_b, E) be a soft complex valued b-metric space and $\{\widetilde{x_n}\}$ be a sequence of soft elements in X. Then, $\{\widetilde{x_n}\}$ is a Cauchy sequence in (X, d_b, E) if and only if $lim_{n\to\infty}|d_b(\widetilde{x_n}, \widetilde{x_{n+m}})| = \overline{0}$, where $m \in \mathbb{N}$.*

Proof. Let $\{\widetilde{x_n}\}$ be a Cauchy sequence in (X, d_b, E). For a soft real number $\tilde{\epsilon} \tilde{>} \overline{0}$, let $\hat{c} = \frac{\tilde{\epsilon}}{\sqrt{2}} + i\frac{\tilde{\epsilon}}{\sqrt{2}}$. Since $\overset{\triangle}{0} \tilde{\precsim} \hat{c} \in \mathbb{C}^E$, there exists a natural number n_0 such that, for all $n \geqslant n_o$, $d_b(\widetilde{x_n}, \widetilde{x_{n+m}}) \tilde{\precsim} \hat{c}$, where $m \in \mathbb{N}$. From the fact that

$$|d_b(\widetilde{x_n}, \widetilde{x_{n+m}})| \tilde{<} |\hat{c}| = \tilde{\epsilon} \quad for\ all\ n \geqslant n_o$$

it follows that $lim_{n\to\infty}|d_b(\widetilde{x_n}, \widetilde{x_{n+m}})| = \overline{0}$. For the converse, let us take a $\hat{c} \in \mathbb{C}^E$ with $\hat{c} \tilde{\succ} \overset{\triangle}{0}$. Then, there exists a soft real number $\tilde{r} \tilde{>} \overline{0}$ such that for $\hat{z} \in \mathbb{C}^E$

$$if\ |\hat{z}| \tilde{<} \tilde{r}\ \ then\ \ \hat{z} \tilde{\precsim} \hat{c}.$$

Since $\tilde{r} \tilde{>} \overline{0}$, by hypothesis, there exists a natural number n_0 such that $|d_b(\widetilde{x_n}, \widetilde{x_{n+m}})| \tilde{<} \tilde{r}$ for all $n \geqslant n_0$. Hence, we get $d_b(\widetilde{x_n}, \widetilde{x_{n+m}}) \tilde{\precsim} \hat{c}$ for all $n \geqslant n_0$, which completes the proof. □

Theorem 4.1. *Let (X, d_b, E) be a complete soft complex valued b-metric space with E a finite set and let $f_E, g_E : (X, d_b, E) \to (X, d_b, E)$ be soft mappings satisfying*

$$d_b(f_E(\tilde{x}), g_E(\tilde{y})) \tilde{\precsim} \tilde{r} d_b(\tilde{x}, \tilde{y}) + \frac{\tilde{s} d_b(\tilde{x}, f_E(\tilde{x}))\, d_b(\tilde{y}, g_E(\tilde{y}))}{\overset{\triangle}{1} + d_b(\tilde{x}, \tilde{y})}$$

for all $\tilde{x}, \tilde{y} \in X^E$, where $\tilde{r}, \tilde{s}$ are nonnegative soft real numbers with $\tilde{b}\tilde{r} + \tilde{s} \tilde{<} \overline{1}$. Then, f_E and g_E have a unique common fixed soft element.

Proof. Let $\widetilde{x_0} \in X^E$. Construct a sequence $\{\widetilde{x_n}\}$ of soft elements in X as follows:

$$f_E(\widetilde{x_{2n}}) = \widetilde{x_{2n+1}}\ \ and\ \ g_E(\widetilde{x_{2n+1}}) = \widetilde{x_{2n+2}}\ \ for\ n \in \{0, 1, ...\}.$$

Then, we obtain

$$\begin{aligned} d_b(\widetilde{x_{2n+1}}, \widetilde{x_{2n+2}}) &= d_b(f_E(\widetilde{x_{2n}}), g_E(\widetilde{x_{2n+1}})) \\ &\tilde{\precsim} \tilde{r} d_b(\widetilde{x_{2n}}, \widetilde{x_{2n+1}}) + \frac{\tilde{s} d_b(\widetilde{x_{2n+1}}, g_E(\widetilde{x_{2n+1}}))\, d_b(\widetilde{x_{2n}}, f_E(\widetilde{x_{2n}}))}{\overset{\triangle}{1} + d_b(\widetilde{x_{2n}}, \widetilde{x_{2n+1}})} \\ &\tilde{\precsim} \tilde{r} d_b(\widetilde{x_{2n}}, \widetilde{x_{2n+1}}) + \frac{\tilde{s} d_b(\widetilde{x_{2n+1}}, \widetilde{x_{2n+2}})\, d_b(\widetilde{x_{2n}}, \widetilde{x_{2n+1}})}{\overset{\triangle}{1} + d_b(\widetilde{x_{2n}}, \widetilde{x_{2n+1}})}. \end{aligned}$$

Since

$$d_b(\widetilde{x_{2n}}, \widetilde{x_{2n+1}}) \precsim \overset{\triangle}{1} + d_b(\widetilde{x_{2n}}, \widetilde{x_{2n+1}})$$

we have

$$d_b(\widetilde{x_{2n+1}}, \widetilde{x_{2n+2}}) \precsim \frac{\tilde{r}}{\overline{1} - \tilde{s}} d_b(\widetilde{x_{2n}}, \widetilde{x_{2n+1}}).$$

Similarly, by putting $\tilde{x} = \widetilde{x_{2n+2}}$ and $\tilde{y} = \widetilde{x_{2n+1}}$ in the contractive condition, we obtain

$$\begin{aligned} d_b(\widetilde{x_{2n+2}}, \widetilde{x_{2n+3}}) &= d_b(f_E(\widetilde{x_{2n+2}}), g_E(\widetilde{x_{2n+1}})) \\ &\precsim \tilde{r} d_b(\widetilde{x_{2n+2}}, \widetilde{x_{2n+1}}) + \frac{\tilde{s} d_b(\widetilde{x_{2n+1}}, g_E(\widetilde{x_{2n+1}}))\, d_b(\widetilde{x_{2n+2}}, f_E(\widetilde{x_{2n+2}}))}{\overset{\triangle}{1} + d_b(\widetilde{x_{2n+2}}, \widetilde{x_{2n+1}})} \\ &\precsim \tilde{r} d_b(\widetilde{x_{2n+2}}, \widetilde{x_{2n+1}}) + \frac{\tilde{s} d_b(\widetilde{x_{2n+1}}, \widetilde{x_{2n+2}})\, d_b(\widetilde{x_{2n+2}}, \widetilde{x_{2n+3}})}{\overset{\triangle}{1} + d_b(\widetilde{x_{2n+1}}, \widetilde{x_{2n+2}})}. \end{aligned}$$

Because

$$d_b(\widetilde{x_{2n+1}}, \widetilde{x_{2n+2}}) \precsim \overset{\triangle}{1} + d_b(\widetilde{x_{2n+1}}, \widetilde{x_{2n+2}})$$

we get

$$d_b(\widetilde{x_{2n+2}}, \widetilde{x_{2n+3}}) \precsim \frac{\tilde{r}}{\overline{1} - \tilde{s}} d_b(\widetilde{x_{2n+2}}, \widetilde{x_{2n+1}}).$$

By $\tilde{b}.\tilde{r} + \tilde{s} \lessapprox \overline{1}$ and $\tilde{b} \geqslant \overline{1}$, it follows that $\tilde{r} + \tilde{s} \lessapprox \overline{1}$. Let us take $\tilde{h} = \frac{\tilde{r}}{\overline{1} - \tilde{s}} \lessapprox \overline{1}$. Therefore, for all $n \in \mathbb{N}$, we have

$$d_b(\widetilde{x_{n+1}}, \widetilde{x_{n+2}}) \precsim \tilde{h}\, d_b(\widetilde{x_n}, \widetilde{x_{n+1}}) \precsim \ldots \precsim (\tilde{h})^{n+1}\, d_b(\widetilde{x_0}, \widetilde{x_1}).$$

Now, we shall show that $\{\widetilde{x_n}\}$ is a Cauchy sequence in (X, d_b, E). For any $m > n$,

$$\begin{aligned} d_b(\widetilde{x_n}, \widetilde{x_m}) &\precsim \tilde{b}\, d_b(\widetilde{x_n}, \widetilde{x_{n+1}}) + \tilde{b}\, d_b(\widetilde{x_{n+1}}, \widetilde{x_m}) \\ &\precsim \tilde{b}\, d_b(\widetilde{x_n}, \widetilde{x_{n+1}}) + (\tilde{b})^2 d_b(\widetilde{x_{n+1}}, \widetilde{x_{n+2}}) + (\tilde{b})^2 d_b(\widetilde{x_{n+2}}, \widetilde{x_m}) \\ &\precsim \tilde{b}\, d_b(\widetilde{x_n}, \widetilde{x_{n+1}}) + (\tilde{b})^2 d_b(\widetilde{x_{n+1}}, \widetilde{x_{n+2}}) + (\tilde{b})^3 d_b(\widetilde{x_{n+2}}, \widetilde{x_{n+3}}) + (\tilde{b})^3 d_b(\widetilde{x_{n+3}}, \widetilde{x_m}) \\ &\vdots \\ &\precsim \tilde{b}\, d_b(\widetilde{x_n}, \widetilde{x_{n+1}}) + (\tilde{b})^2 d_b(\widetilde{x_{n+1}}, \widetilde{x_{n+2}}) + (\tilde{b})^3 d_b(\widetilde{x_{n+2}}, \widetilde{x_{n+3}}) \\ &\quad + \ldots + (\tilde{b})^{m-n-1} d_b(\widetilde{x_{m-2}}, \widetilde{x_{m-1}}) + (\tilde{b})^{m-n} d_b(\widetilde{x_{m-1}}, \widetilde{x_m}). \end{aligned}$$

So, we get

$$\begin{aligned} d_b(\widetilde{x_n}, \widetilde{x_m}) &\precsim \tilde{b}\, (\tilde{h})^n\, d_b(\widetilde{x_0}, \widetilde{x_1}) + (\tilde{b})^2\, (\tilde{h})^{n+1}\, d_b(\widetilde{x_0}, \widetilde{x_1}) + (\tilde{b})^3\, (\tilde{h})^{n+2}\, d_b(\widetilde{x_0}, \widetilde{x_1}) \\ &\quad + \ldots + (\tilde{b})^{m-n-1}\, (\tilde{h})^{m-2}\, d_b(\widetilde{x_0}, \widetilde{x_1}) + (\tilde{b})^{m-n}\, (\tilde{h})^{m-1}\, d_b(\widetilde{x_0}, \widetilde{x_1}) \\ &\precsim (\tilde{b}\tilde{h})^n\, d_b(\widetilde{x_0}, \widetilde{x_1}) + (\tilde{b}\tilde{h})^{n+1}\, d_b(\widetilde{x_0}, \widetilde{x_1}) + (\tilde{b}\tilde{h})^{n+2}\, d_b(\widetilde{x_0}, \widetilde{x_1}) \\ &\quad + \ldots + (\tilde{b}\tilde{h})^{m-2}\, d_b(\widetilde{x_0}, \widetilde{x_1}) + (\tilde{b}\tilde{h})^{m-1}\, d_b(\widetilde{x_0}, \widetilde{x_1}) \\ &= \left((\tilde{b}\tilde{h})^n + (\tilde{b}\tilde{h})^{n+1} + (\tilde{b}\tilde{h})^{n+2} + \ldots + (\tilde{b}\tilde{h})^{m-2} + (\tilde{b}\tilde{h})^{m-1}\right) d_b(\widetilde{x_0}, \widetilde{x_1}) \\ &\precsim \frac{(\tilde{b}\tilde{h})^n}{\overline{1} - \tilde{b}\tilde{h}}\, d_b(\widetilde{x_0}, \widetilde{x_1}) \end{aligned}$$

Hence, we have $|d_b(\widetilde{x_n}, \widetilde{x_m})| \lessapprox \frac{(\tilde{b}\tilde{h})^n}{\overline{1} - \tilde{b}\tilde{h}} |d_b(\widetilde{x_0}, \widetilde{x_1})|$. Because E is a finite set and $\tilde{b}\tilde{h} \lessapprox \overline{1}$, $lim_{n\to\infty}(\tilde{b}\tilde{h})^n = \overline{0}$ and therefore, by Lemma 4.2, $\{\widetilde{x_n}\}$ is a Cauchy sequence in (X, d_b, E). Since (X, d_b, E) is a complete space, there exists an $\tilde{x} \in X^E$ such that $lim_{n\to\infty}\widetilde{x_n} = \tilde{x}$.

Next, we shall show that $\tilde{x}$ is a fixed soft element of f_E. For this, let $d_b(\tilde{x}, f_E(\tilde{x})) = \hat{z}$. By using the triangular inequality, we have

$$\begin{aligned} \hat{z} &\precsim \tilde{b}\, d_b(\tilde{x}, \widetilde{x_{2n+2}}) + \tilde{b}\, d_b(\widetilde{x_{2n+2}}, f_E(\tilde{x})) \\ &= \tilde{b}\, d_b(\tilde{x}, \widetilde{x_{2n+2}}) + \tilde{b}\, d_b(g_E(\widetilde{x_{2n+1}}), f_E(\tilde{x})) \end{aligned}$$

$$\precsim \tilde{b}\, d_b(\tilde{x}, \widetilde{x_{2n+2}}) + \tilde{b}\, \tilde{r}\, d_b(\widetilde{x_{2n+1}}, \tilde{x}) + \frac{\tilde{b}\, \tilde{s}\, d_b(\tilde{x}, f_E(\tilde{x}))\, d_b(\widetilde{x_{2n+1}}, g_E(\widetilde{x_{2n+1}}))}{\overset{\triangle}{1} + d_b(\tilde{x}, \widetilde{x_{2n+1}})}$$
$$= \tilde{b}\, d_b(\tilde{x}, \widetilde{x_{2n+2}}) + \tilde{b}\, \tilde{r}\, d_b(\widetilde{x_{2n+1}}, \tilde{x}) + \frac{\tilde{b}\, \tilde{s}\, d_b(\widetilde{x_{2n+1}}, \widetilde{x_{2n+2}})\, \hat{z}}{\overset{\triangle}{1} + d_b(\tilde{x}, \widetilde{x_{2n+1}})}.$$

So, by Theorem 2.3, we obtain

$$|\hat{z}| \lessapprox \tilde{b}\, |d_b(\tilde{x}, \widetilde{x_{2n+2}})| + \tilde{b}\, \tilde{r}\, |d_b(\widetilde{x_{2n+1}}, \tilde{x})| + \frac{\tilde{b}\, \tilde{s}\, |d_b(\widetilde{x_{2n+1}}, \widetilde{x_{2n+2}})|\, |\hat{z}|}{|\overset{\triangle}{1} + d_b(\tilde{x}, \widetilde{x_{2n+1}})|}.$$

Taking the limit of this inequality as $n \to \infty$ proves that $|\hat{z}| = \overline{0}$. Thus, we get $f_E(\tilde{x}) = \tilde{x}$, that is, $\tilde{x}$ is a fixed soft element of f_E. Similarly, we obtain $g_E(\tilde{x}) = \tilde{x}$.

Now, we check the uniqueness of the common fixed soft element of f_E and g_E. To show this, assume that $\tilde{x^*}$ is another common fixed soft element of f_E and g_E. Therefore,

$$d_b(\tilde{x}, \tilde{x^*}) = d_b(f_E(\tilde{x}), g_E(\tilde{x^*})) \precsim \tilde{r} d_b(\tilde{x}, \tilde{x^*}) + \frac{\tilde{s} d_b(\tilde{x}, f_E(\tilde{x}))\, d_b(\tilde{x}, g_E(\tilde{x^*}))}{\overset{\triangle}{1} + d_b(\tilde{x}, \tilde{x^*})} = \tilde{r} d_b(\tilde{x}, \tilde{x^*}) \precnsim d_b(\tilde{x}, \tilde{x^*}),$$

which is a contradiction. Thus, we get $\tilde{x} = \tilde{x^*}$ and so $\tilde{x}$ is a unique common fixed soft element of f_E and g_E. This completes the proof. □

Corollary 4.1. *Let (X, d_b, E) be a complete soft complex valued b-metric space with E a finite set and let $f_E : (X, d_b, E) \to (X, d_b, E)$ be a soft mapping satisfying*

$$d_b(f_E(\tilde{x}), f_E(\tilde{y})) \precsim \tilde{r} d_b(\tilde{x}, \tilde{y}) + \frac{\tilde{s} d_b(\tilde{x}, f_E(\tilde{x}))\, d_b(\tilde{y}, f_E(\tilde{y}))}{\overset{\triangle}{1} + d_b(\tilde{x}, \tilde{y})}$$

for all $\tilde{x}, \tilde{y} \in X^E$, where $\tilde{r}, \tilde{s}$ are nonnegative soft real numbers with $\tilde{b}\tilde{r} + \tilde{s} \precsim \overline{1}$. Then, f_E has a unique fixed soft element.

Proof. It is immediate from setting $f_E = g_E$ in Theorem 4.1. □

Corollary 4.2. *Let (X, d_b, E) be a complete soft complex valued b-metric space with E a finite set and let $f_E : (X, d_b, E) \to (X, d_b, E)$ be a soft mapping satisfying*

$$d_b(f_E^n(\tilde{x}), f_E^n(\tilde{y})) \precsim \tilde{r} d_b(\tilde{x}, \tilde{y}) + \frac{\tilde{s} d_b(\tilde{x}, f_E^n(\tilde{x}))\, d_b(\tilde{y}, f_E^n(\tilde{y}))}{\overset{\triangle}{1} + d_b(\tilde{x}, \tilde{y})}$$

for all $\tilde{x}, \tilde{y} \in X^E$, where $\tilde{r}, \tilde{s}$ are nonnegative soft real numbers with $\tilde{b}\tilde{r} + \tilde{s} \precsim \overline{1}$. Then, f_E has a unique fixed soft element (Here, f_E^n is the nth iterate of f_E).

Proof. From Corollary 4.1, we obtain $\tilde{x} \in X^E$ such that $f_E^n(\tilde{x}) = \tilde{x}$. Now, suppose that $f_E(\tilde{x}) \neq \tilde{x}$. From the condition $(scbm_2)$ of Definition 3.1, it follows that

$$d_b(f_E(\tilde{x}), \tilde{x})(e) \neq \overset{\triangle}{0}(e) = \overline{0}(e) + i\overline{0}(e) = 0 + i0 \ \ for\ every\ e \in E.$$

Then, we have

$$\begin{aligned} d_b(f_E(\tilde{x}), \tilde{x}) &= d_b(f_E(f_E^n(\tilde{x})), f_E^n(\tilde{x})) = d_b(f_E^n(f_E(\tilde{x})), f_E^n(\tilde{x})) \\ &\precsim \tilde{r} d_b(f_E(\tilde{x}), \tilde{x}) + \frac{\tilde{s} d_b(f_E(\tilde{x}), f_E^n(f_E(\tilde{x})))\, d_b(\tilde{x}, f_E^n(\tilde{x}))}{\overset{\triangle}{1} + d_b(f_E(\tilde{x}), \tilde{x})} \\ &= \tilde{r} d_b(f_E(\tilde{x}), \tilde{x}) + \frac{\tilde{s} d_b(f_E(\tilde{x}), f_E(f_E^n(\tilde{x})))\, d_b(\tilde{x}, \tilde{x})}{\overset{\triangle}{1} + d_b(f_E(\tilde{x}), \tilde{x})} \\ &= \tilde{r} d_b(f_E(\tilde{x}), \tilde{x}) \\ &\precnsim d_b(f_E(\tilde{x}), \tilde{x}), \end{aligned}$$

which yields a contradiction. Therefore, we get $f_E(\tilde{x}) = \tilde{x}$. Thus, from the fact that

$$f_E(\tilde{x}) = f_E^n(\tilde{x}) = \tilde{x}$$

it follows that the fixed soft element of f_E is unique. □

We construct the following example to show the validity of the hypotheses of Corollary 4.2.

Example 4.1. *Let (X, d_b, E) be a complete soft complex valued b-metric space defined as in Example 3.2. Take a mapping $f : \mathbb{C} \to \mathbb{C}$ by*

$$f(z) = \begin{cases} 0, & \text{if } x, y \in \mathbb{Q}, \\ 5, & \text{if } x \in \mathbb{Q}^c, y \in \mathbb{Q}, \\ i5, & \text{if } x \in \mathbb{Q}^c, y \in \mathbb{Q}^c, \\ 5 + i5, & \text{if } x \in \mathbb{Q}, y \in \mathbb{Q}^c, \end{cases}$$

where $z = x + iy \in \mathbb{C}$. Now, for $\hat{z} = \overset{\triangle}{\left(\frac{1}{\sqrt{5}}\right)}$ and $\hat{u} = \overset{\triangle}{0}$, we obtain

$$d_b(f_E(\hat{z}), f_E(\hat{u})) = \overline{0} + i\overline{25} \precsim \tilde{r} d_b(\hat{z}, \hat{u}) + \frac{\tilde{s} d_b(\hat{z}, f_E(\hat{z}))\, d_b(\hat{u}, f_E(\hat{u}))}{\overset{\triangle}{1} + d_b(\hat{z}, \hat{u})} = \tilde{r}\overset{\triangle}{\left(\tfrac{1}{5}\right)} + \overset{\triangle}{0} = \tilde{r}\overset{\triangle}{\left(\tfrac{1}{5}\right)}.$$

This is a contradiction for every choice of $\tilde{r}$ satisfying $\overline{0} \lessapprox \tilde{r} \precsim \overline{1}$. But, one can readily verify that $f_E^n(\hat{z}) = \overset{\triangle}{0}$ for every $\hat{z} \in \mathbb{C}^E$ and $n > 1$. Therefore,

$$\overset{\triangle}{0} = d_b(f_E^n(\hat{z}), f_E^n(\hat{u})) \precsim \tilde{r} d_b(\hat{z}, \hat{u}) + \frac{\tilde{s} d_b(\hat{z}, f_E^n(\hat{z})) d_b(\hat{u}, f_E^n(\hat{u}))}{\overset{\triangle}{1} + d_b(\hat{z}, \hat{u})}$$

for every $\hat{z}, \hat{u} \in \mathbb{C}^E$ and $\tilde{r}, \tilde{s} \gtrapprox \overline{0}$ with $\overline{2}\tilde{r} + \tilde{s} \precsim \overline{1}$. Thus, all conditions of Corollary 4.2 are satisfied and it is seen that $\overset{\triangle}{0}$ is the unique fixed soft element of f_E.

Theorem 4.2. *Let (X, d_b, E) be a complete soft complex valued b-metric space with E a finite set and let $f_E, g_E : (X, d_b, E) \to (X, d_b, E)$ be soft mappings satisfying*

$$d_b(f_E(\tilde{x}), g_E(\tilde{y})) \precsim \tilde{r} d_b(\tilde{x}, \tilde{y}) + \frac{\tilde{s} d_b(\tilde{x}, f_E(\tilde{x}))\, d_b(\tilde{y}, g_E(\tilde{y}))}{d_b(\tilde{x}, g_E(\tilde{y})) + d_b(\tilde{y}, f_E(\tilde{x})) + d_b(\tilde{x}, \tilde{y})}$$

for all $\tilde{x}, \tilde{y} \in X^E$ such that $d_b(\tilde{x}, g_E(\tilde{y})) + d_b(\tilde{y}, f_E(\tilde{x})) + d_b(\tilde{x}, \tilde{y}) \neq \overset{\triangle}{0}$ where $\tilde{r}, \tilde{s}$ are non-negative soft real numbers with $\tilde{b}(\tilde{r} + \tilde{b}\tilde{s}) \precsim \overline{1}$ or $d_b(f_E(\tilde{x}), g_E(\tilde{y})) = \overset{\triangle}{0}$ if $d_b(\tilde{x}, g_E(\tilde{y})) + d_b(\tilde{y}, f_E(\tilde{x})) + d_b(\tilde{x}, \tilde{y}) = \overset{\triangle}{0}$. Then, f_E and g_E have a unique common fixed soft element.

Proof. Let $\widetilde{x_0} \in X^E$ and define a sequence $\{\widetilde{x_n}\}$ of soft elements in X by

$$f_E(\widetilde{x_{2n}}) = \widetilde{x_{2n+1}} \ \ and \ \ g_E(\widetilde{x_{2n+1}}) = \widetilde{x_{2n+2}} \ \ for\ n \in \{0, 1, ...\}.$$

Then, we obtain

$$\begin{aligned} d_b(\widetilde{x_{2n+1}}, \widetilde{x_{2n+2}}) &= d_b(f_E(\widetilde{x_{2n}}), g_E(\widetilde{x_{2n+1}})) \\ &\precsim \tilde{r} d_b(\widetilde{x_{2n}}, \widetilde{x_{2n+1}}) + \frac{\tilde{s} d_b(\widetilde{x_{2n+1}}, g_E(\widetilde{x_{2n+1}}))\, d_b(\widetilde{x_{2n}}, f_E(\widetilde{x_{2n}}))}{d_b(\widetilde{x_{2n}}, g_E(\widetilde{x_{2n+1}})) + d_b(\widetilde{x_{2n+1}}, f_E(\widetilde{x_{2n}})) + d_b(\widetilde{x_{2n}}, \widetilde{x_{2n+1}})} \\ &= \tilde{r} d_b(\widetilde{x_{2n}}, \widetilde{x_{2n+1}}) + \frac{\tilde{s} d_b(\widetilde{x_{2n}}, \widetilde{x_{2n+1}})\, d_b(\widetilde{x_{2n+1}}, \widetilde{x_{2n+2}})}{d_b(\widetilde{x_{2n}}, \widetilde{x_{2n+2}}) + d_b(\widetilde{x_{2n+1}}, \widetilde{x_{2n+1}}) + d_b(\widetilde{x_{2n}}, \widetilde{x_{2n+1}})}. \end{aligned}$$

Since

$$d_b(\widetilde{x_{2n+1}}, \widetilde{x_{2n+2}}) \precsim \tilde{b}\, d_b(\widetilde{x_{2n+1}}, \widetilde{x_{2n}}) + \tilde{b}\, d_b(\widetilde{x_{2n}}, \widetilde{x_{2n+2}})$$

we get

$$d_b(\widetilde{x_{2n+1}},\widetilde{x_{2n+2}}) \precsim \tilde{r}d_b(\widetilde{x_{2n}},\widetilde{x_{2n+1}}) + \tilde{b}\tilde{s}d_b(\widetilde{x_{2n}},\widetilde{x_{2n+1}}) = (\tilde{r}+\tilde{b}\tilde{s})d_b(\widetilde{x_{2n}},\widetilde{x_{2n+1}}).$$

Similarly, replacing $\tilde{x}$ by $\widetilde{x_{2n+2}}$ and $\tilde{y}$ by $\widetilde{x_{2n+1}}$ in the contractive condition, we have

$$\begin{aligned} d_b(\widetilde{x_{2n+2}},\widetilde{x_{2n+3}}) &= d_b(f_E(\widetilde{x_{2n+2}}), g_E(\widetilde{x_{2n+1}})) \\ &\precsim \tilde{r}d_b(\widetilde{x_{2n+2}},\widetilde{x_{2n+1}}) + \tfrac{\tilde{s}d_b(\widetilde{x_{2n+2}},f_E(\widetilde{x_{2n+2}}))\,d_b(\widetilde{x_{2n+1}},g_E(\widetilde{x_{2n+1}}))}{d_b(\widetilde{x_{2n+1}},f_E(\widetilde{x_{2n+2}}))+d_b(\widetilde{x_{2n+2}},g_E(\widetilde{x_{2n+1}}))+d_b(\widetilde{x_{2n+2}},\widetilde{x_{2n+1}})} \\ &= \tilde{r}d_b(\widetilde{x_{2n+2}},\widetilde{x_{2n+1}}) + \tfrac{\tilde{s}d_b(\widetilde{x_{2n+2}},\widetilde{x_{2n+3}})\,d_b(\widetilde{x_{2n+1}},\widetilde{x_{2n+2}})}{d_b(\widetilde{x_{2n+1}},\widetilde{x_{2n+3}})+d_b(\widetilde{x_{2n+2}},\widetilde{x_{2n+2}})+d_b(\widetilde{x_{2n+2}},\widetilde{x_{2n+1}})}. \end{aligned}$$

From the fact that

$$d_b(\widetilde{x_{2n+2}},\widetilde{x_{2n+3}}) \precsim \tilde{b}\,d_b(\widetilde{x_{2n+2}},\widetilde{x_{2n+1}}) + \tilde{b}\,d_b(\widetilde{x_{2n+1}},\widetilde{x_{2n+3}})$$

it follows that

$$d_b(\widetilde{x_{2n+2}},\widetilde{x_{2n+3}}) \precsim \tilde{r}d_b(\widetilde{x_{2n+2}},\widetilde{x_{2n+1}}) + \tilde{b}\tilde{s}d_b(\widetilde{x_{2n+1}},\widetilde{x_{2n+2}}) = (\tilde{r}+\tilde{b}\tilde{s})d_b(\widetilde{x_{2n+1}},\widetilde{x_{2n+2}}).$$

By $\tilde{b}\big(\tilde{r}+\tilde{b}\tilde{s}\big) \precsim \overline{1}$ and $\tilde{b} \succcurlyeq \overline{1}$, we obtain $\tilde{h} = (\tilde{r}+\tilde{b}\tilde{s}) \precsim \overline{1}$. Therefore, for all $n \in \mathbb{N}$, we have

$$d_b(\widetilde{x_{n+1}},\widetilde{x_{n+2}}) \precsim \tilde{h}\,d_b(\widetilde{x_n},\widetilde{x_{n+1}}) \precsim \ldots \precsim (\tilde{h})^{n+1}\,d_b(\widetilde{x_0},\widetilde{x_1}).$$

So, for any $m > n$,

$$\begin{aligned} d_b(\widetilde{x_n},\widetilde{x_m}) &\precsim \tilde{b}\,d_b(\widetilde{x_n},\widetilde{x_{n+1}}) + \tilde{b}\,d_b(\widetilde{x_{n+1}},\widetilde{x_m}) \\ &\precsim \tilde{b}\,d_b(\widetilde{x_n},\widetilde{x_{n+1}}) + (\tilde{b})^2 d_b(\widetilde{x_{n+1}},\widetilde{x_{n+2}}) + (\tilde{b})^2 d_b(\widetilde{x_{n+2}},\widetilde{x_m}) \\ &\precsim \tilde{b}\,d_b(\widetilde{x_n},\widetilde{x_{n+1}}) + (\tilde{b})^2 d_b(\widetilde{x_{n+1}},\widetilde{x_{n+2}}) + (\tilde{b})^3 d_b(\widetilde{x_{n+2}},\widetilde{x_{n+3}}) + (\tilde{b})^3 d_b(\widetilde{x_{n+3}},\widetilde{x_m}) \\ &\vdots \\ &\precsim \tilde{b}\,d_b(\widetilde{x_n},\widetilde{x_{n+1}}) + (\tilde{b})^2 d_b(\widetilde{x_{n+1}},\widetilde{x_{n+2}}) + (\tilde{b})^3 d_b(\widetilde{x_{n+2}},\widetilde{x_{n+3}}) \\ &\quad +\ldots + (\tilde{b})^{m-n-1} d_b(\widetilde{x_{m-2}},\widetilde{x_{m-1}}) + (\tilde{b})^{m-n} d_b(\widetilde{x_{m-1}},\widetilde{x_m}). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} d_b(\widetilde{x_n},\widetilde{x_m}) &\precsim \tilde{b}\,(\tilde{h})^n\,d_b(\widetilde{x_0},\widetilde{x_1}) + (\tilde{b})^2\,(\tilde{h})^{n+1}\,d_b(\widetilde{x_0},\widetilde{x_1}) + (\tilde{b})^3\,(\tilde{h})^{n+2}\,d_b(\widetilde{x_0},\widetilde{x_1}) \\ &\quad +\ldots + (\tilde{b})^{m-n-1}\,(\tilde{h})^{m-2}\,d_b(\widetilde{x_0},\widetilde{x_1}) + (\tilde{b})^{m-n}\,(\tilde{h})^{m-1}\,d_b(\widetilde{x_0},\widetilde{x_1}) \\ &\precsim (\tilde{b}\tilde{h})^n\,d_b(\widetilde{x_0},\widetilde{x_1}) + (\tilde{b}\tilde{h})^{n+1}\,d_b(\widetilde{x_0},\widetilde{x_1}) + (\tilde{b}\tilde{h})^{n+2}\,d_b(\widetilde{x_0},\widetilde{x_1}) \\ &\quad +\ldots + (\tilde{b}\tilde{h})^{m-2}\,d_b(\widetilde{x_0},\widetilde{x_1}) + (\tilde{b}\tilde{h})^{m-1}\,d_b(\widetilde{x_0},\widetilde{x_1}) \\ &= \big((\tilde{b}\tilde{h})^n + (\tilde{b}\tilde{h})^{n+1} + (\tilde{b}\tilde{h})^{n+2} + \ldots + (\tilde{b}\tilde{h})^{m-2} + (\tilde{b}\tilde{h})^{m-1}\big)\,d_b(\widetilde{x_0},\widetilde{x_1}) \\ &\precsim \tfrac{(\tilde{b}\tilde{h})^n}{\overline{1}-\tilde{b}\tilde{h}}\,d_b(\widetilde{x_0},\widetilde{x_1}). \end{aligned}$$

Therefore, we get $|d_b(\widetilde{x_n},\widetilde{x_m})| \precsim \frac{(\tilde{b}\tilde{h})^n}{\overline{1}-\tilde{b}\tilde{h}}|d_b(\widetilde{x_0},\widetilde{x_1})|$. Since E is a finite set and $\tilde{b}\tilde{h} \precsim \overline{1}$, it follows that $lim_{n\to\infty}(\tilde{b}\tilde{h})^n = \overline{0}$. By Lemma 4.2, $\{\widetilde{x_n}\}$ is a Cauchy sequence in (X, d_b, E). Because (X, d_b, E) is a complete space, there exists an $\tilde{x} \in X^E$ such that $lim_{n\to\infty}\widetilde{x_n} = \tilde{x}$. Now, let $d_b(\tilde{x}, f_E(\tilde{x})) = \hat{z}$. Then, we have

$$\begin{aligned} \hat{z} &\precsim \tilde{b}\,d_b(\tilde{x},\widetilde{x_{2n+2}}) + \tilde{b}\,d_b(\widetilde{x_{2n+2}}, f_E(\tilde{x})) \\ &= \tilde{b}\,d_b(\tilde{x},\widetilde{x_{2n+2}}) + \tilde{b}\,d_b(g_E(\widetilde{x_{2n+1}}), f_E(\tilde{x})) \\ &\precsim \tilde{b}\,d_b(\tilde{x},\widetilde{x_{2n+2}}) + \tilde{b}\,\tilde{r}\,d_b(\widetilde{x_{2n+1}},\tilde{x}) + \tfrac{\tilde{b}\,\tilde{s}\,d_b(\tilde{x},f_E(\tilde{x}))\,d_b(\widetilde{x_{2n+1}},g_E(\widetilde{x_{2n+1}}))}{d_b(\tilde{x},g_E(\widetilde{x_{2n+1}}))+d_b(\widetilde{x_{2n+1}},f_E(\tilde{x}))+d_b(\tilde{x},\widetilde{x_{2n+1}})} \\ &= \tilde{b}\,d_b(\tilde{x},\widetilde{x_{2n+2}}) + \tilde{b}\,\tilde{r}\,d_b(\widetilde{x_{2n+1}},\tilde{x}) + \tfrac{\tilde{b}\,\tilde{s}\,d_b(\widetilde{x_{2n+1}},\widetilde{x_{2n+2}})\,\hat{z}}{d_b(\tilde{x},\widetilde{x_{2n+2}})+d_b(\widetilde{x_{2n+1}},f_E(\tilde{x}))+d_b(\tilde{x},\widetilde{x_{2n+1}})}. \end{aligned}$$

Hence, using Theorem 2.3, we get

$$|\hat{z}| \precsim \tilde{b}\,|d_b(\tilde{x},\widetilde{x_{2n+2}})| + \tilde{b}\,\tilde{r}\,|d_b(\widetilde{x_{2n+1}},\tilde{x})| + \frac{\tilde{b}\,\tilde{s}\,|d_b(\widetilde{x_{2n+1}},\widetilde{x_{2n+2}})|\,|\hat{z}|}{|d_b(\tilde{x},\widetilde{x_{2n+2}}) + d_b(\widetilde{x_{2n+1}},f_E(\tilde{x})) + d_b(\tilde{x},\widetilde{x_{2n+1}})|}.$$

Taking the limit of this inequality as $n \to \infty$ gives $|\hat{z}| = \overline{0}$. Thus, we have $f_E(\tilde{x}) = \tilde{x}$, that is, $\tilde{x}$ is a fixed soft element of f_E. Arguing the same, one can show that $\tilde{x}$ is a fixed soft element of g_E.

To investigate the uniqueness of the common fixed soft element of f_E and g_E, assume that $\widetilde{x^*}$ is another common fixed soft element of f_E and g_E. Therefore,

$$\begin{aligned} d_b(\tilde{x},\widetilde{x^*}) &= d_b(f_E(\tilde{x}),g_E(\widetilde{x^*})) \\ &\precsim \tilde{r}d_b(\tilde{x},\widetilde{x^*}) + \frac{\tilde{s}d_b(\tilde{x},f_E(\tilde{x}))\,d_b(\tilde{x},g_E(\widetilde{x^*}))}{d_b(\tilde{x},g_E(\widetilde{x^*}))+d_b(\widetilde{x^*},f_E(\tilde{x}))+d_b(\tilde{x},\widetilde{x^*})} \\ &= \tilde{r}d_b(\tilde{x},\widetilde{x^*}) \\ &\precnsim d_b(\tilde{x},\widetilde{x^*}). \end{aligned}$$

Thus, we arrive at a contradiction, so that $\tilde{x} = \widetilde{x^*}$, which proves the uniqueness of common fixed soft element in (X,d_b,E).

For the second case, if $d_b(\widetilde{x_{2n}},g_E(\widetilde{x_{2n+1}})) + d_b(\widetilde{x_{2n+1}},f_E(\widetilde{x_{2n}})) + d_b(\widetilde{x_{2n}},\widetilde{x_{2n+1}}) = \overset{\triangle}{0}$ for any $n \in \mathbb{N}$, then we have $d_b(f_E(\widetilde{x_{2n}}),g_E(\widetilde{x_{2n+1}})) = \overset{\triangle}{0}$. Therefore, we get

$$\widetilde{x_{2n}} = f_E(\widetilde{x_{2n}}) = \widetilde{x_{2n+1}} = g_E(\widetilde{x_{2n+1}}) = \widetilde{x_{2n+2}}.$$

Since $\widetilde{x_{2n}} = f_E(\widetilde{x_{2n}}) = \widetilde{x_{2n+1}}$, there exist two soft elements $\tilde{r_1}$ and $\tilde{s_1}$ such that $\tilde{r_1} = f_E(\tilde{r_1}) = \tilde{s_1}$. By repeating the same arguments, one can also show that there exist two soft elements $\tilde{r_2}$ and $\tilde{s_2}$ such that $\tilde{r_2} = g_E(\tilde{r_2}) = \tilde{s_2}$. From the fact that

$$d_b(\tilde{r_1},g_E(\tilde{r_2})) + d_b(\tilde{r_2},f_E(\tilde{r_1})) + d_b(\tilde{r_1},\tilde{r_2}) = \overset{\triangle}{0}$$

it follows that $\tilde{s_1} = f_E(\tilde{r_1}) = g_E(\tilde{r_2}) = \tilde{s_2}$, which deduce the equalities $\tilde{s_1} = f_E(\tilde{r_1}) = f_E(\tilde{s_1})$ and $\tilde{s_2} = g_E(\tilde{r_2}) = g_E(\tilde{s_2})$. Because $\tilde{s_1} = \tilde{s_2}$, we obtain $f_E(\tilde{s_1}) = g_E(\tilde{s_1}) = \tilde{s_1}$. Thus, $\tilde{s_1} = \tilde{s_2}$ is common fixed soft element of f_E and g_E.

For uniqueness of common fixed soft element, let $\widetilde{s_1^*} \in X^E$ be another common fixed soft element of f_E and g_E, i.e. $f_E(\widetilde{s_1^*}) = g_E(\widetilde{s_1^*}) = \widetilde{s_1^*}$. Since

$$d_b(\tilde{s_1},g_E(\widetilde{s_1^*})) + d_b(\widetilde{s_1^*},f_E(\tilde{s_1})) + d_b(\tilde{s_1},\widetilde{s_1^*}) = \overset{\triangle}{0}$$

we have $\widetilde{s_1^*} = g_E(\widetilde{s_1^*}) = f_E(\tilde{s_1}) = \tilde{s_1}$, which shows that $\widetilde{s_1^*} = \tilde{s_1}$. This completes the proof. □

Corollary 4.3. *Let (X,d_b,E) be a complete soft complex valued b-metric space with E a finite set and let $f_E : (X,d_b,E) \to (X,d_b,E)$ be a soft mapping satisfying*

$$d_b(f_E(\tilde{x}),f_E(\tilde{y})) \precsim \tilde{r}d_b(\tilde{x},\tilde{y}) + \frac{\tilde{s}d_b(\tilde{x},f_E(\tilde{x}))\,d_b(\tilde{y},f_E(\tilde{y}))}{d_b(\tilde{x},f_E(\tilde{y})) + d_b(\tilde{y},f_E(\tilde{x})) + d_b(\tilde{x},\tilde{y})}$$

for all $\tilde{x},\tilde{y} \in X^E$ such that $d_b(\tilde{x},f_E(\tilde{y})) + d_b(\tilde{y},f_E(\tilde{x})) + d_b(\tilde{x},\tilde{y}) \neq \overset{\triangle}{0}$ where $\tilde{r},\tilde{s}$ are non-negative soft real numbers with $\tilde{b}\big(\tilde{r} + \tilde{b}\tilde{s}\big) \precsim \overline{1}$ or $d_b(f_E(\tilde{x}),f_E(\tilde{y})) = \overset{\triangle}{0}$ if $d_b(\tilde{x},f_E(\tilde{y})) + d_b(\tilde{y},f_E(\tilde{x})) + d_b(\tilde{x},\tilde{y}) = \overset{\triangle}{0}$. Then, f_E has a unique fixed soft element.

Proof. We can prove this result by applying Theorem 4.2 with $f_E = g_E$. □

Now, we give an example to demonstrate the validity of the hypotheses of Corollary 4.3.

Example 4.2. *Let* $X = \mathbb{C}$ *and* E *be a finite set of parameters. Let us define a mapping* $d_b : \mathbb{C}^E \times \mathbb{C}^E \to \mathbb{C}^E$ *as follows:*

$$d_b(\hat{z}, \hat{u}) = |\hat{z} - \hat{u}|^2 + i|\hat{z} - \hat{u}|^2$$

where $\hat{z}, \hat{u} \in \mathbb{C}^E$ *with* $\hat{z} = \widetilde{z_1} + i\widetilde{z_2}$ *and* $\hat{u} = \widetilde{u_1} + i\widetilde{u_2}$*. Clearly,* (X, d_b, E) *is a complete soft complex valued b-metric space with* $\tilde{b} = \overline{2}$*. Now, take a mapping* $f : \mathbb{C} \to \mathbb{C}$ *such that*

$$f(z) = \frac{z}{3}, \quad for\ every\ z \in \mathbb{C}.$$

Therefore, the contractive condition in Corollary 4.3 holds for all $\hat{z}, \hat{u} \in \mathbb{C}^E$ *with* $\tilde{r} = \overline{\left(\frac{1}{9}\right)}$ *and* $\tilde{s} = \overline{\left(\frac{1}{6}\right)}$*. Also,*

$$\tilde{b}\big(\tilde{r} + \tilde{b}\tilde{s}\big) = \overline{2}\left(\overline{\left(\tfrac{1}{9}\right)} + \overline{2}\overline{\left(\tfrac{1}{6}\right)}\right) = \overline{\left(\tfrac{8}{9}\right)} \precsim \overline{1}.$$

Thus, all the conditions of Corollary 4.3 are satisfied and it is seen that $\overset{\triangle}{0} \in \mathbb{C}^E$ *is a unique fixed soft element of* f_E*.*

Corollary 4.4. *Let* (X, d_b, E) *be a complete soft complex valued b-metric space with* E *a finite set and let* $f_E : (X, d_b, E) \to (X, d_b, E)$ *be a soft mapping satisfying*

$$d_b(f_E^n(\tilde{x}), f_E^n(\tilde{y})) \,\widetilde{\precsim}\, \tilde{r} d_b(\tilde{x}, \tilde{y}) + \frac{\tilde{s} d_b(\tilde{x}, f_E^n(\tilde{x}))\, d_b(\tilde{y}, f_E^n(\tilde{y}))}{d_b(\tilde{x}, f_E^n(\tilde{y})) + d_b(\tilde{y}, f_E^n(\tilde{x})) + d_b(\tilde{x}, \tilde{y})}$$

for all $\tilde{x}, \tilde{y} \in X^E$ *such that* $d_b(\tilde{x}, f_E^n(\tilde{y})) + d_b(\tilde{y}, f_E^n(\tilde{x})) + d_b(\tilde{x}, \tilde{y}) \neq \overset{\triangle}{0}$ *where* $\tilde{r}, \tilde{s}$ *are non-negative soft real numbers with* $\tilde{b}\big(\tilde{r} + \tilde{b}\tilde{s}\big) \precsim \overline{1}$ *or* $d_b(f_E^n(\tilde{x}), f_E^n(\tilde{y})) = \overset{\triangle}{0}$ *if* $d_b(\tilde{x}, f_E^n(\tilde{y})) + d_b(\tilde{y}, f_E^n(\tilde{x})) + d_b(\tilde{x}, \tilde{y}) = \overset{\triangle}{0}$*. Then,* f_E *has a unique fixed soft element.*

Proof. The proof follows the same lines as that of Corollary 4.2 and is therefore omitted to avoid repetition. □

Acknowledgement. The author would like to thank the reviewers for their valuable comments.

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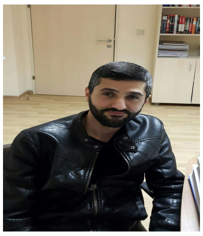

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