TWMS J. App. and Eng. Math. V.11, N.1, 2021, pp. 250-261

SHARP INEQUALITIES FOR UNIVALENCE OF MEROMORPHIC FUNCTIONS IN THE PUNCTURED UNIT DISK

OQLAH AL-REFAI, §

Abstract. A new class of meromorphic functions f that are univalent in the punctured unit disk $\mathbb{U}^* = \{z : \ 0 < |z| < 1\}$ is introduced. This class is denoted by $\mathcal{MU}$ and consisting of functions f defined by $|1 + f'(z)/f^2(z)| < 1$ and $zf(z) \neq 0$, whenever $z \in \mathbb{U} = \{z : \ |z| < 1\}$. For every $n \geq 2$, sharp bound for the nth derivative of $1/(zf(z))$ that implies univalency of f in $\mathbb{U}^*$ is established. In particular, the best improvements for known univalence criteria are obtained. Distortion and growth estimates are investigated. Further, various sufficient coefficient conditions and a necessary coefficient condition for f to be in $\mathcal{MU}$ are derived and best radii of univalence are obtained for certain cases.

Keywords: univalent functions, meromorphic functions, distortion theorem, coefficient bounds, area theorem.

AMS Subject Classification: 30C45, 30C55

1. Introduction and preliminaries

Let Σ_n denotes the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + a_n z^n + a_{n+1} z^{n+1} + \cdots, \quad (n \in \mathbb{N} \cup \{0\}),$$

which are analytic in the punctured unit disk $\mathbb{U}^* = \{z : \ 0 < |z| < 1\}$. For simplicity, write $\Sigma_0 := \Sigma$. In [1], Macgregor proved that a normalized analytic function g is univalent in $\mathbb{U} = \{z : \ |z| < 1\}$ if

$$|g'(z) - 1| < 1, \quad (z \in \mathbb{U}). \tag{1}$$

Earlier, Aksentév [2] proved for $F(\zeta) = \zeta + \sum_{n=0}^{\infty} a_n \zeta^{-n}$ that the condition $|F'(\zeta) - 1| < 1$, for $\zeta \in \Delta = \{\eta : \ 1 < |\eta| < \infty\}$ is sufficient for F to be univalent in Δ. Liu [3] proved for $f \in \Sigma_n$ with $zf(z) \neq 0$ in $\mathbb{U}^*$ that

$$\left| \left(\frac{1}{zf(z)} \right)^{(n+2)} \right| < \frac{1 - (n+2)|a_n|}{2}, \quad (z \in \mathbb{U}, \ n \in \mathbb{N} \cup \{0\}), \tag{2}$$

Department of Mathematics, Faculty of Science, Taibah University, P.O. Box 30097, Almadinah Almunawwarah 41477, Saudi Arabia.
e-mail: oso_alrefai@yahoo.com; ORCID: https://orcid.org/0000-0002-7617-8863.

§ Manuscript received: March 7, 2019; accepted: April 24, 2019.

is sufficient condition for f to be univalent in $\mathbb{U}^*$. Note that condition (2) is not sharp and same result was obtained in [4, Theorem 8], for the case $n = 0$. The interesting problems of finding sufficient and necessary conditions for meromorphic univalent functions have been extensively studied by many authors, see [5-10]. To fulfill this aim, in this paper, the following new subclasses of meromorphic functions are studied.

Definition 1.1. *Let $\mathcal{MU}$ denotes the subclass of meromorphic functions $f \in \Sigma$ defined by*

$$\left|\frac{f'(z)}{f^2(z)} + 1\right| < 1, \quad (z \in \mathbb{U}), \tag{3}$$

and $zf(z) \neq 0$ in $\mathbb{U}$.

Definition 1.2. *Let $\mathcal{MP}_n$ $(n \geq 2)$ denotes the subclass of meromorphic functions $f \in \Sigma$ defined by $zf(z) \neq 0$ in $\mathbb{U}$ and*

$$\left|\left(\frac{1}{zf(z)}\right)^{(n)}\right| \leq \beta_n, \quad (z \in \mathbb{U}), \tag{4}$$

where

$$\beta_n = \frac{n!}{n+1}\left(1 - \sum_{k=1}^{n-1} \frac{k+1}{k!}|\alpha_k|\right) \quad \text{and} \quad \alpha_k = \left(\frac{1}{zf(z)}\right)^{(k)} |_{z=0} \ .$$

In fact, Theorem 2.1 shows that the functions of $\mathcal{MU}$ are univalent in $\mathbb{U}^*$ and the bound 1 of condition (3) is best possible for univalence. Motivated by results due to Al-Refai and Darus [11] and Obradović and Ponnusamy [12], Theorem 2.3 proves for every $n \geq 2$ that $\mathcal{MP}_n \subseteq \mathcal{MU}$ and condition (4) is sharp for univalence. In particular, the best improvement of condition (2) due to Liu [3] is obtained, where the upper bound can not be replaced by a larger one. For $n = 2$, condition (4) reduces to

$$\left|\left(\frac{1}{zf(z)}\right)''\right| \leq \frac{2}{3}(1 - 2|a_0|),$$

which is the best improvement of [4, Theorem 8] and the bound is best possible for univalence. Moreover, for $f \in \Sigma$ with $zf(z) \neq 0$ in $\mathbb{U}$ and

$$\frac{1}{zf(z)} = 1 + b_1 z + b_2 z^2 + \cdots, \tag{5}$$

various sufficient coefficient conditions and a necessary coefficient condition for f to be in $\mathcal{MU}$ are derived.

2. Univalence Criteria

Considering $g(z) = 1/f(z)$ in condition (1), where $zf(z) \neq 0$ in $\mathbb{U}$, implies that $f \in \Sigma$ is meromorphic univalent in $\mathbb{U}^*$. For completeness,

Theorem 2.1. *The functions of $\mathcal{MU}$ are meromorphic univalent in $\mathbb{U}^*$ and the bound 1 of condition (3) is best possible for univalence.*

Proof. Let $f \in \mathcal{MU}$. Evidently, the function $h(z) = (1/f(z)) - z$ is analytic in $\mathbb{U}^*$ and for $z_1, z_2 \in \mathbb{U}^*$ with $z_1 \neq z_2$, it can be seen that

$$\left(\frac{1}{f(z_2)} - z_2\right) - \left(\frac{1}{f(z_1)} - z_1\right) = h(z_2) - h(z_1) = \int_{z_1}^{z_2} h'(z)dz \tag{6}$$

Putting $z = z_1 + t(z_2 - z_1)$, $(0 \le t \le 1)$ in (6) gives

$$\left(\frac{1}{f(z_2)} - \frac{1}{f(z_1)}\right) - (z_2 - z_1) = \int_0^1 (z_2 - z_1)h'(z)dt. \tag{7}$$

Therefore, from (7) and (3), it follows that

$$\begin{aligned} \left|\frac{\frac{1}{f(z_2)} - \frac{1}{f(z_1)}}{z_2 - z_1} - 1\right| &= \left|\int_0^1 h'(z)\,dt\right| \\ &\le \int_0^1 |h'(z)|\,dt < \int_0^1 dt = 1. \end{aligned}$$

This shows that $(1/f(z_2)) - (1/f(z_1)) \neq 0$ and hence f is meromorphic univalent in $\mathbb{U}^*$. To show that the bound 1 is best possible for univalence, notice that the functions

$$f(z) = \frac{1}{z \pm (1/n)z^n}, \quad (n = 2, 3, ...)$$

are satisfying (3). However, for every $\epsilon > 0$ and

$$f_\epsilon(z) = \frac{1}{z + \frac{1+\epsilon}{n} z^n},$$

it can be found that

$$\left|\frac{f_\epsilon'(z)}{f_\epsilon^2(z)} + 1\right| = |(1+\epsilon)z^{n-1}| < 1 + \epsilon$$

and there exists

$$z_0 = \left(\frac{-1}{1+\epsilon}\right)^{\frac{1}{n-1}} \in \mathbb{U}^*$$

such that $f_\epsilon'(z_0) = 0$. Thus f_ϵ is not univalent in $\mathbb{U}^*$ and the proof is complete. □

Setting $f(z) = F(1/z)$ in Theorem 2.1 leads to the following result

Corollary 2.1. *Let $F(\zeta) = \zeta + \sum_{n=0}^{\infty} a_n \zeta^{-n}$ with $F(\zeta)/\zeta \neq 0$ and*

$$\left|\frac{\zeta^2 F'(\zeta)}{F^2(\zeta)} - 1\right| < 1, \tag{8}$$

for $\zeta \in \Delta$. Then F is univalent in Δ. The bound 1 is best possible for univalence and the functions $F(\zeta) = \zeta/(1 \pm (1/n)\zeta^{1-n})$ are extreme for every $n = 2, 3, ...$.

Proof. Let $f(z) \in \mathcal{MU}$ be given by $f(z) = F(1/z)$. Then

$$\left|\frac{f'(z)}{f^2(z)} + 1\right| = \left|\frac{-(1/z^2)F'(1/z)}{F^2(1/z)} + 1\right| < 1,$$

which is equivalent to (8), for $\zeta = 1/z$. Therefore $F(1/z)$ is univalent for $z \in \mathbb{U}^*$, i.e $F(\zeta)$ is univalent in Δ. To prove the sharpness, let

$$F_\epsilon(\zeta) = \frac{\zeta}{1 + \frac{1+\epsilon}{n}\zeta^{1-n}}, \quad (\epsilon \ge 0).$$

A computation shows that

$$\left|\frac{\zeta^2 F_0'(\zeta)}{F_0^2(\zeta)} - 1\right| = |\zeta|^{1-n} < 1.$$

However, for every $\epsilon > 0$,

$$\left|\frac{\zeta^2 F_\epsilon'(\zeta)}{F_\epsilon^2(\zeta)} - 1\right| = (1+\epsilon)|\zeta|^{1-n} < 1 + \epsilon$$

and $F'_\epsilon\left((-1-\epsilon)^{1/(n-1)}\right)=0$. So F_ϵ is not univalent in Δ, for every $\epsilon>0$. □

In the following theorem, estimates for the bounds of functions in $\mathcal{MU}$ and their derivatives are found.

Theorem 2.2 (Growth and Distortion)**.** *Let $f \in \mathcal{MU}$ and $z \in \mathbb{U}^*$. Then*

$$\frac{1}{|z|+\frac{1}{2}|z|^2} \le |f(z)| \le \frac{1}{|z|-\frac{1}{2}|z|^2}, \tag{9}$$

$$\frac{1-|z|}{|z|^2\left(1+\frac{1}{2}|z|\right)^2} \le |f'(z)| \le \frac{1+|z|}{|z|^2\left(1-\frac{1}{2}|z|\right)^2}. \tag{10}$$

The estimates (9) and (10) are precise for the functions

$$f(z)=\frac{1}{z+az^2}, \quad |a|=\frac{1}{2}.$$

Proof. Let $f \in \mathcal{MU}$ and $z \in \mathbb{U}^*$. Then, by Schwarz's lemma,

$$\left|\left(\frac{1}{f(z)}\right)'-1\right| \le |z|. \tag{11}$$

Therefore,

$$\begin{aligned}\left|\frac{1}{f(z)}-z\right| &= \left|\int_0^z\left(\left(\frac{1}{f(u)}\right)'-1\right)\,du\right| \\ &\le |z|\int_0^1\left|\left(\frac{1}{f(zt)}\right)'-1\right|\,dt \le |z|\int_0^1 |z|t\,dt=\frac{1}{2}|z|^2.\end{aligned}$$

Hence,

$$|z|-\frac{1}{2}|z|^2 \le \left|\frac{1}{f(z)}\right| \le |z|+\frac{1}{2}|z|^2$$

and (9) follows. From (11), it can be observed that

$$1-|z| \le \left|\left(\frac{1}{f(z)}\right)'\right| \le 1+|z|$$

and so

$$(1-|z|)|f(z)|^2 \le \left|f'(z)\right| \le (1+|z|)|f(z)|^2. \tag{12}$$

Therefore, (9) and (12) yield (10). By an application of the triangle inequality, it is clear that the estimates (9) and (10) are precise for the functions

$$f(z)=\frac{1}{z+az^2}, \quad |a|=\frac{1}{2}.$$

□

The estimate (9) gives

Corollary 2.2. *The range of any $f \in \mathcal{MU}$ must cover the punctured disk whose radius is 2/3. That is*

$$\left\{w: 0<|w|<\frac{2}{3}\right\} \subseteq f(\mathbb{U}^*).$$

The following result proves that the functions of $\mathcal{MP}_n$ are univalent in $\mathbb{U}^*$. Indeed, they are included in $\mathcal{MU}$.

Theorem 2.3 (Inclusion)**.** *For every $n \geq 2$, $\mathcal{MP}_n \subseteq \mathcal{MU}$. Moreover, inequality (4) is sharp for univalence, where equality attained for the functions*

$$f_n(z) = \frac{1}{z \pm (1/n)z^n} \quad \textit{and} \quad f_{n+1}(z) = \frac{1}{z \pm (1/(n+1))z^{n+1}}.$$

Proof. Let $f \in \mathcal{MP}_n$, $(n \geq 2)$. Then, for $n = 2$, condition (4) is equivalent to

$$\left(\frac{1}{zf(z)}\right)'' = \beta_2 \phi_1(z), \tag{13}$$

where ϕ_1 is analytic in $\mathbb{U}$ and $|\phi_1(z)| \leq 1$ in $\mathbb{U}$. It is easy to see that "if $\omega(z)$ is analytic in $\mathbb{U}$ and $|\omega(z)| \leq 1$ in $\mathbb{U}$, then for each $m \geq 1$, the function $\Phi_m(z)$ defined by

$$\Phi_m(z) = \int_0^z m u^{m-1} \omega(u) du = z^m \int_0^1 m t^{m-1} \omega(tz) dt = z^m \Psi_m(z) \tag{14}$$

is clearly analytic in $\mathbb{U}$ and moreover, $\Psi_m(z)$ is analytic in $\mathbb{U}$ such that $|\Psi(z)| \leq 1$ in $\mathbb{U}$". This fact will be used in the sequel. By integrating (13) from 0 to z and making use of (14), it can be seen that

$$\left(\frac{1}{zf(z)}\right)' = \alpha_1 + \beta_2 z \int_0^1 \phi_1(tz)\, dt := \alpha_1 + \beta_2 z \phi_2(z). \tag{15}$$

The relation (15), by integration and then multiplying by z, gives

$$\frac{1}{f(z)} - z = \alpha_1 z^2 + \beta_2 z \int_0^z u \phi_2(u)\, du. \tag{16}$$

By differentiating both sides of (16) and making use of (14),

$$\begin{aligned} \left(\frac{1}{f(z)}\right)' - 1 &= 2\alpha_1 z + \beta_2 \left(z^2 \phi_2(z) + \int_0^z u \phi_2(u)\, du\right) \\ &= 2\alpha_1 z + \beta_2 \left(z^2 \phi_2(z) + \frac{1}{2} z^2 \int_0^1 2t \phi_2(tz)\, dt\right). \end{aligned}$$

Therefore,

$$\left|\frac{f'(z)}{f^2(z)} + 1\right| = \left|\left(\frac{1}{f(z)}\right)' - 1\right| < 2|\alpha_1| + \frac{3}{2}|\beta_2| = 1,$$

and hence $f \in \mathcal{MU}$. For $n = 3$,

$$\left(\frac{1}{zf(z)}\right)''' = \beta_3 \psi_1(z), \tag{17}$$

where ψ_1 is analytic in $\mathbb{U}$ and $|\psi_1(z)| \leq 1$ in $\mathbb{U}$. By integration (17) from 0 to z,

$$\left(\frac{1}{zf(z)}\right)'' = \alpha_2 + \beta_3 z \int_0^1 \psi_1(tz)\, dt := \alpha_2 + \beta_3 z \psi_2(z). \tag{18}$$

The relation (18), by integration, gives

$$\left(\frac{1}{zf(z)}\right)' = \alpha_1 + \alpha_2 z + \beta_3 \frac{z^2}{2} \int_0^1 2t \psi_2(tz)\, dt := \alpha_1 + \alpha_2 z + \frac{\beta_3}{2} z^2 \psi_3(z). \tag{19}$$

Integrating (19) and then multiplying by z, gives

$$\frac{1}{f(z)} - z = \alpha_1 z^2 + \frac{1}{2}\alpha_2 z^3 + \frac{\beta_3}{2} z \int_0^z u^2 \psi_3(u)\, du. \tag{20}$$

By differentiating both sides of (20) and making use of (14),

$$\begin{aligned}\left(\frac{1}{f(z)}\right)' - 1 &= 2\alpha_1 z + \frac{3}{2}\alpha_2 z^2 + \frac{\beta_3}{2}\left(z^3\psi_3(z) + \int_0^z u^2\psi_3(u)\, du\right) \\ &= 2\alpha_1 z + \frac{3}{2}\alpha_2 z^2 + \frac{\beta_3}{2}\left(z^3\psi_3(z) + \frac{1}{3}z^3\int_0^1 3t^2\psi_3(tz)\, dt\right).\end{aligned}$$

Therefore,

$$\left|\left(\frac{1}{f(z)}\right)' - 1\right| < 2|\alpha_1| + \frac{3}{2}|\alpha_2| + \frac{2}{3}|\beta_3| = 1,$$

and hence $f \in \mathcal{MU}$. In general, if $f \in \mathcal{MP}_n$, then

$$\left(\frac{1}{f(z)}\right)' - 1 = \sum_{k=1}^{n-1}\frac{k+1}{k!}\alpha_k z^k + \frac{1}{(n-1)!}\beta_n\left(z^n\varphi_n(z) + \frac{z^n}{n}\int_0^1 nt^{n-1}\varphi_n(tz)\, dt\right).$$

Therefore,

$$\left|\left(\frac{1}{f(z)}\right)' - 1\right| < \sum_{k=1}^{n-1}\frac{k+1}{k!}|\alpha_k| + \frac{1}{(n-1)!}\left(1+\frac{1}{n}\right)|\beta_n| = 1,$$

and hence $f \in \mathcal{MU}$. To show that the result is sharp for $n \geq 2$, consider

$$f_\epsilon(z) = \frac{1}{z + \frac{1+\epsilon}{n+1}z^{n+1}}, \quad (\epsilon \geq 0). \tag{21}$$

A computation shows that

$$\left(\frac{1}{zf_\epsilon(z)}\right)^{(k)} = (1+\epsilon)(n+1)^{-1}n(n-1)\cdots(n+1-k)z^{n-k}, \quad \text{for } 1 \leq k \leq n.$$

Therefore,

$$\left(\frac{1}{zf_\epsilon(z)}\right)^{(n)} = \frac{n!}{n+1}(1+\epsilon) \text{ and } |\alpha_k| = 0, \quad \text{for } 1 \leq k \leq n-1. \tag{22}$$

Letting $\epsilon = 0$ in (22) implies that the equality in (4) holds. However, for every $\epsilon > 0$, it can be seen that

$$f'_\epsilon\left(\left(\frac{-1}{1+\epsilon}\right)^{\frac{1}{n}}\right) = 0.$$

Hence f_ϵ is not univalent in $\mathbb{U}^*$, for every $\epsilon > 0$ and the result is sharp. Note that the functions $f_n(z) = 1/(z \pm (1/n)z^n)$ are also satisfying the equality in (4), where both sides will be zeros. This completes the proof of Theorem 2.3. $\square$

Setting $n = 2$ in Theorem 2.3 yields the following result which is the best improvement of [4, Theorem 8].

Corollary 2.3. *Let $f \in \Sigma$ with $zf(z) \neq 0$ and*

$$\left|\left(\frac{1}{zf(z)}\right)''\right| \leq \frac{2}{3}\left(1 - 2|a_0|\right), \quad (z \in \mathbb{U}).$$

Then f is meromorphic univalent in $\mathbb{U}^*$. The result is sharp, where equality attained for the functions

$$f_2(z) = \frac{1}{z \pm (1/2)z^2} \;\; \textit{and} \;\; f_3(z) = \frac{1}{z \pm (1/3)z^3}.$$

Setting $n = 3$ in Theorem 2.3 gives the following result.

Corollary 2.4. *Let $f \in \Sigma$ with $zf(z) \neq 0$ and*

$$\left|\left(\frac{1}{zf(z)}\right)'''\right| \leq \frac{3}{2}\left(1-2|a_0|-3|a_1-a_0^2|\right), \quad (z \in \mathbb{U}).$$

Then f is meromorphic univalent in $\mathbb{U}^*$. The result is sharp, where equality attained for the functions

$$f_3(z) = \frac{1}{z \pm (1/3)z^3} \quad \textit{and} \quad f_4(z) = \frac{1}{z \pm (1/4)z^4}.$$

The following corollary establishes the best improvement of (2) due to Liu [3] and the sharp bound for univalence of $f \in \Sigma_n$.

Corollary 2.5. *For $n \geq 0$, let $f \in \Sigma_n$ with $zf(z) \neq 0$ and*

$$\left|\left(\frac{1}{zf(z)}\right)^{(n+2)}\right| \leq \frac{(n+2)!}{n+3}(1-(n+2)|a_n|), \quad (z \in \mathbb{U}).$$

Then $f \in \mathcal{MP}_{n+2}$ and hence it is meromorphic univalent in $\mathbb{U}^*$. The result is sharp, where equality attained for the functions

$$f_n(z) = \frac{1}{z \pm \frac{1}{n+2}z^{n+2}} \quad \textit{and} \quad f_{n+1}(z) = \frac{1}{z \pm \frac{1}{n+3}z^{n+3}}.$$

Proof. Let $g(z) = zf(z) = 1 + a_n z^{n+1} + a_{n+1}z^{n+2} + \cdots$. Then

$$g(z) \cdot \frac{1}{zf(z)} = 1. \tag{23}$$

Differentiating (23) gives

$$g(z)\left(\frac{1}{zf(z)}\right)' + g'(z)\left(\frac{1}{zf(z)}\right) = 0.$$

One can easily observe, for $n = 0$, that $|\alpha_1| = |g'(0)| = |a_0|$. Differentiating (23) twice gives

$$g(z)\left(\frac{1}{zf(z)}\right)'' + 2g'(z)\left(\frac{1}{zf(z)}\right)' + g''(z)\left(\frac{1}{zf(z)}\right) = 0.$$

Therefore, for $n = 1$, $\alpha_1 = g'(0) = 0$ and $|\alpha_2| = |g''(0)| = 2|a_1|$. In general, differentiating (23) $(n+1)$-times gives

$$\sum_{k=0}^{n+1}\binom{n+1}{k}g^{(k)}(z)\left(\frac{1}{zf(z)}\right)^{(n+1-k)} = 0.$$

Therefore, $|\alpha_k| = |g^{(k)}(0)| = 0$, for $k = 1,..,n$ and $|\alpha_{n+1}| = |g^{(n+1)}(0)| = (n+1)!\ |a_n|$. Hence,

$$\beta_{n+2} = \frac{(n+2)!}{n+3}(1-(n+2)|a_n|)$$

and the result follows. □

From Theorem 2.3, one can derive the following

Corollary 2.6. *Let $f \in \Sigma$ be of the form (5) with $zf(z) \neq 0$ in $\mathbb{U}$ and*

$$\sum_{k=1}^{n-1}(k+1)|b_k| + (n+1)\sum_{k=n}^{\infty}\binom{k}{n}|b_k| \leq 1, \tag{24}$$

for some $n \geq 2$. Then $f \in \mathcal{MP}_n$.

Proof. In view of (5) and by a simple computation,

$$\left(\frac{1}{zf(z)}\right)^{(n)} = n!\,b_n + \sum_{k=n+1}^{\infty} \frac{k!}{(k-n)!} b_k z^{k-n},$$

and so $\alpha_m = m!\,b_m$, for $1 \le m \le n-1$. It follows that

$$\left|\left(\frac{1}{zf(z)}\right)^{(n)}\right| \le \sum_{k=n}^{\infty} \frac{k!}{(k-n)!}|b_k|. \tag{25}$$

From (25) and (24), it can be seen that the assumption of Theorem 2.3 holds and the proof is complete. □

It is worth to state, for $n = 2$, that condition (24) reduces to

$$\sum_{k=2}^{\infty} k(k-1)|b_k| \le \frac{2}{3}(1 - 2|a_0(f)|).$$

Example 2.1. *From Corollary 2.6, it is easily to check that the functions*

$$f(z) = \frac{1}{z + \sum_{k=1}^{n} b_k z^{k+1}}$$

where $zf(z) \neq 0$ *in* $\mathbb{U}$ *and* $\sum_{k=1}^{n}(k+1)|b_k| \le 1$, *are meromorphic univalent in* $\mathbb{U}^*$.

3. Coefficient conditions

Motivated by the Gronwall Area Theorem which states that $\sum_{n=1}^{\infty} n|a_n|^2 \le 1$ is a necessary condition for $f \in \Sigma$ to be univalent in $\mathbb{U}^*$, (see [14, p.29] and [15, Theorem 1]), sufficient coefficient conditions and a necessary coefficient condition are derived for functions of the form (5) to be in $\mathcal{MU}$. In addition, best radii of univalence are investigated for certain cases.

Theorem 3.1. *Let* f *have the representation (5) with* $zf(z) \neq 0$ *in* $\mathbb{U}$ *and let*

$$\sum_{n=1}^{\infty}(n+1)|b_n| \le 1. \tag{26}$$

Then $f \in \mathcal{MU}$. *The constant* 1 *is best possible for univalence.*

Proof. Evidently, using the representation (5) and the coefficients condition (26), it can be seen that

$$\left|\frac{f'(z)}{f^2(z)} + 1\right| = \left|z\left(\frac{1}{zf(z)}\right)' + \frac{1}{zf(z)} - 1\right| = \left|\sum_{n=1}^{\infty}(n+1)b_n z^n\right| < \sum_{n=1}^{\infty}(n+1)|b_n| \le 1.$$

Therefore, $f \in \mathcal{MU}$. In order to prove the sharpness, consider the function

$$f_\epsilon(z) = \frac{1}{z + \frac{1+\epsilon}{k}z^k}, \quad (k \ge 2,\ \epsilon \ge 0).$$

Letting $\epsilon = 0$ yields that $f_0(z)$ satisfies the equality in (26). However, for f_ϵ with $\epsilon > 0$, a computation shows that

$$\sum_{n=1}^{\infty}(n+1)|b_n| = k|b_{k-1}| = k\left(\frac{1+\epsilon}{k}\right) = 1 + \epsilon$$

and

$$f_\epsilon'\left(\left(\frac{-1}{1+\epsilon}\right)^{\frac{1}{n-1}}\right)=0.$$

Hence, f_ϵ is not meromorphic univalent in $\mathbb{U}^*$, for every $\epsilon > 0$. The proof is complete. $\square$

The condition (26) is sufficient but not necessary for the function f, which has the representation (5), to be in $\mathcal{MU}$. For instance, consider the function f given by

$$\frac{1}{zf(z)}=1+\frac{1}{9}z^2+\frac{\sqrt{5}}{12}iz^3+\frac{1}{15}z^4.$$

It can be observed that

$$\left|\frac{1}{zf(z)}\right|\geq 1-\frac{1}{3}|z|^2\left|\frac{1}{3}+\frac{\sqrt{5}}{4}iz+\frac{1}{5}z^2\right|\geq 1-\frac{1}{3}\left(\frac{1}{3}+\frac{\sqrt{5}}{4}+\frac{1}{5}\right)>0$$

and so $1/(zf(z))$ is non-vanishing in the unit disk $\mathbb{U}$. Also

$$\left|\frac{f'(z)}{f^2(z)}+1\right|=\frac{1}{3}|z|^2|1+\sqrt{5}iz+z^2|.$$

Next, the function $\psi(z)=1+\sqrt{5}iz+z^2$ is univalent in $\mathbb{U}$ with $\psi(0)=1$. It was shown in [13] that $\max_{|z|=1}|\psi(z)|=3$, indeed

$$\max_{|z|=1}|\psi(z)|=\max_{0\leq\theta\leq 2\pi}|2\cos\theta+\sqrt{5}i|=\max_{0\leq\theta\leq 2\pi}\sqrt{4\cos^2\theta+5}=3.$$

This shows that $f\in\mathcal{MU}$. On the other hand,

$$\sum_{n=1}^{\infty}(n+1)|b_n|=\frac{1}{3}+\frac{\sqrt{5}}{3}+\frac{1}{3}>1.$$

The following theorem introduces a necessary condition for functions in $\mathcal{MU}$.

Theorem 3.2 (Necessary condition)**.** *Let $f\in\mathcal{MU}$ have the form (5). Then*

$$\sum_{n=1}^{\infty}(n+1)^2|b_n|^2\leq 1.$$

Proof. The power series representation of f yields that

$$\left|\frac{f'(z)}{f^2(z)}+1\right|=\left|\sum_{n=1}^{\infty}(n+1)b_nz^n\right|<1,\quad (z\in\mathbb{U}).$$

Letting $z=re^{i\theta}$ for $r\in(0,1)$ and $0\leq\theta\leq 2\pi$, the last inequality gives

$$\sum_{n=1}^{\infty}(n+1)^2|b_n|^2r^{2n}=\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{n=1}^{\infty}(n+1)b_nz^n\right|^2 d\theta<1.$$

The desired result follows by letting $r\to 1^-$. $\square$

Theorem 3.3. *Let f have the representation (5) with $zf(z)\neq 0$ in $\mathbb{U}$ and let*

$$\sum_{n=1}^{\infty}(n+1)|b_n|^2\leq 1. \tag{27}$$

Then f is meromorphic univalent in the disk $0<|z|<r_0=\sqrt{1-\frac{1}{\sqrt{2}}}\approx 0.541196$ and the radius is the best possible.

Proof. Consider the function $g(z) = rf(rz)$ where $0 < r \leq 1$. Then

$$\frac{1}{zg(z)} = 1 + \sum_{n=1}^{\infty} b_n r^n z^n.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{n=1}^{\infty}(n+1)|b_n|r^n &\leq \left(\sum_{n=1}^{\infty}(n+1)|b_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}(n+1)r^{2n}\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{n=1}^{\infty}(n+1)r^{2n}\right)^{\frac{1}{2}} = \frac{r\sqrt{2-r^2}}{1-r^2} \leq 1 \end{aligned} \tag{28}$$

whenever $0 < r \leq r_0 = \sqrt{1 - \frac{1}{\sqrt{2}}}$. From Theorem 3.1, (28) yields $g \in \mathcal{MU}$, whenever $0 < r \leq r_0$. Therefore, f is meromorphic univalent in $0 < |z| \leq r_0$. To prove the sharpness, consider the function $f_0(z)$ given by

$$f_0(z) = \frac{1 - r_0 z}{z - 2r_0 z^2}, \quad \left(r_0 = \sqrt{1 - \frac{1}{\sqrt{2}}}\right).$$

Therefore,

$$\frac{1}{zf_0(z)} = \frac{1 - 2r_0 z}{1 - r_0 z} = 1 - \sum_{n=1}^{\infty} r_0^n z^n$$

and

$$\sum_{n=1}^{\infty}(n+1)|b_n|^2 = \sum_{n=1}^{\infty}(n+1)r_0^{2n} = 1.$$

Hence, f_0 is meromorphic univalent in the disk $0 < |z| < r_0$, but not in a larger one, because

$$f_0'(z) = \frac{4r_0 z - 2r_0^2 z^2 - 1}{(z - 2r_0 z^2)^2}$$

yields $f_0'(r_0) = 0$ and this completes the proof of Theorem 3.3.

□

Theorem 3.4. *Let f have the representation (5) with $zf(z) \neq 0$ in $\mathbb{U}$ and let*

$$\sum_{n=1}^{\infty}(n+1)^2|b_n|^2 \leq 1. \tag{29}$$

Then f is meromorphic univalent in the disk $0 < |z| < r_0 = 1/\sqrt{2} \approx 0.707107$ and the radius r_0 is the best possible.

Proof. Consider the function $g(z) = rf(rz)$ where $0 < r \leq 1$. Then

$$\frac{1}{zg(z)} = 1 + \sum_{n=1}^{\infty} b_n r^n z^n.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{n=1}^{\infty}(n+1)|b_n|r^n &\leq \left(\sum_{n=1}^{\infty}(n+1)^2|b_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}r^{2n}\right)^{\frac{1}{2}}\\ &\leq \left(\sum_{n=1}^{\infty}r^{2n}\right)^{\frac{1}{2}} = \frac{r}{\sqrt{1-r^2}} \leq 1\end{aligned}$$

whenever $0 < r \leq r_0 = 1/\sqrt{2}$. It follows that Theorem 3.1 yields $g \in \mathcal{MU}$, whenever $0 < r \leq r_0$, and so f is meromorphic univalent in the disk $0 < |z| < r_0$. To prove the sharpness, consider the function $f_0(z)$ defined by

$$\frac{1}{zf_0(z)} = 1 + \sum_{n=1}^{\infty}\frac{r_0^n}{n+1}z^n = 2 + \frac{1}{r_0 z}\ln(1-r_0 z).$$

Now,

$$\sum_{n=1}^{\infty}(n+1)^2|b_n|^2 = \sum_{n=1}^{\infty}r^{2n} = \frac{r_0^2}{1-r_0^2} = 1.$$

For $0 < |z| < r_0$, it can be seen that

$$\left|\frac{f_0'(z)}{f_0^2(z)} + 1\right| = \left|\sum_{n=1}^{\infty}r_0^n z^n\right| = \left|\frac{r_0 z}{1-r_0 z}\right| < \frac{r_0^2}{1-r_0^2} = 1,$$

while for $r_0 < z = r < 1$,

$$\left|\frac{f_0'(r)}{f_0^2(r)} + 1\right| = \left|\sum_{n=1}^{\infty}r_0^n r^n\right| = \left|\frac{r_0 r}{1-r_0 r}\right| > 1.$$

It follows that $g_0(z) = rf_0(rz)$ belongs to $\mathcal{MU}$, for $0 < r \leq r_0$ and so f_0 is meromorphic univalent in $0 < |z| < r_0$, but not in a larger disk, because

$$f_0'(z) = \frac{(1-r_0 z)^{-1} - 2}{\left(2z + \frac{1}{r_0}\ln(1-r_0 z)\right)^2}$$

implies that $f_0'(r_0) = 0$ and the proof is complete. $\square$

4. Conclusion

Finding sharp sufficient and necessary conditions for univalent functions plays a main role in the geometric function theory. In this paper, various sharp univalence criteria for meromorphic functions in the punctured unit disk are derived. Some results establish the best improvements of known sufficient conditions for univalence. A necessary coefficient condition as well as sufficient coefficient conditions for functions in the class $\mathcal{MU}$ are derived and best radii of univalance are obtained for certain cases. For future research, the classes $\mathcal{MU}$ and $\mathcal{MP}_n$ can be studied for further properties.

References

[1] MacGregor, T. H., (1964), A class of univalent functions, Proc. Amer. Math. Soc., 15, pp. 311-317.

[2] Aksentév, LA., (1958), Sufficient conditions for univalence of regular functions, Izv. Vysš. Učeb. Zaved. Mathematica, 1958, pp. 3-7. (Russian).

[3] Liu, J., (2001), One criterion for meromorphic univalent functions, Soochow J. Math., 27, pp. 113-115.

[4] Xu, Y. and Liu, J., (2015), New applications of Nunokawa's lemmas, J. of Function Spaces, doi: 10.1155/2015/241264.

[5] Ganigi, M.D. and Uralegaddi, B.A., (1989), New criteria for meromorphic univalent functions, Bull. Math. Soc. Sci. Math. R.S. Roumanie (N.S.), 33, pp. 9-13.

[6] Aouf, M.K. and Hossen, H.M., (1994), New criterion for meromorphic univalent functions of order α, Nihonkai Math. J., 5, pp. 1-11.

[7] Pommerenke, Ch., (1963), On meromorphic starlike functions, Pacific J. Math., 13, pp. 221-235.

[8] Clunie, J., (1959), On meromorphic schlicht functions, J. London Math. Soc., 34, pp. 215-216.

[9] Libera, R. J., (1965), Meromorphic close-to-convex functions, Duck Math. J., 32, pp. 121-128.

[10] Frasin, B. and Ali, E., (2016), An application for certain subclasses of p-valent meromorphic functions associated with the generalized hypergeometric function, Theor. Appl. Math. Comp. Sci, 6(1), pp. 60-68.

[11] Al-Refai, O. and Darus, M., (2012), General univalence criterion associated with the nth derivative, Abstract and Applied Analysis,, doi: 10.1155/2012/307526.

[12] Obradović, M. and Ponnusamy, S., (2013), Criteria for univalent functions in the unit disk, Arch. Math. (Basel), 100(2), pp. 149-157.

[13] Ali, R., Obradović, M. and Ponnusamy, S., (2013), Necessary and sufficient conditions for univalent functions, Complex Var. Elliptic Equ., 58(5), pp. 611-620.

[14] Duren, P. L., Univalent Functions, (1983), Springer-Verlag: New York, Berlin, Heidelberg, Tokyo, MR708494 (85j:30034).

[15] Pavlović, M. and Angel Peláez, J., (2011), Remarks on the area theorem in the theory of univalent functions, Proc. Amer. Math. Soc., 139(3), pp. 909-916.

Oqlah Al-Refai was born in Jordan in 1981. He received his Ph.D. in mathematics from The National University of Malaysia in 2010. After graduation, he worked as an assistant professor in the Department of Mathematics, Faculty of Science, Zarqa University, Jordan. Since 2013, he has been working in the Department of Mathematics, Faculty of Science, Taibah University, Saudi Arabia. His area of research interest is complex analysis, geometric functions theory, univalent functions theory and subordinations.