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A NEW APPROACH TO FIND APPROXIMATE SOLUTIONS OF BURGER'S AND COUPLED BURGER'S EQUATIONS OF FRACTIONAL ORDER

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ABSTRACT. The paper presents a new technique called homotopy perturbation Sumudu transform Method (HPSTM), which is a combination of the Sumudu transform (ST) and homotopy perturbation method (HPM) for solving the fractional Burger's and coupled fractional Burger's equations with time fractional derivative operators. The fractional derivative is described in the Caputo sense. The method in general is easy to implement and yields good results. Illustrative examples are included to demonstrate the validity and applicability of the new technique. The approximate solutions obtained are compared with the results obtained by variational iteration method (VIM) and homotopy perturbation method (HPM).

Keywords: Homotopy perturbation method, Sumudu transform, Burger's equation, Caputoi fractional derivative.

AMS Subject Classification: 35M31, 35R11

1. Introduction

The homotopy perturbation method (HPM), which is introduced by He [1], has been widely used to obtain approximate solutions of linear and non-linear problems arising as ordinary or partial differential equations of integer or fractional order in science and engineering. In the recent years, fractional homotopy perturbation method, which is modified to improve the efficiency and accuracy of homotopy perturbation method, is proposed and successful results have been achieved [2].

The name fractional calculus stems from the fact that the order of derivatives and integrals are fractions rather than integers. Early work on fractional calculus dates back to the early nineteenth century. Integration and differentiation with arbitrary order is called fractional calculus, and it is the general expansion of integer order calculus to arbitrary order. Recently, fractional calculus has become a powerful tool because of its favorable properties such as analyticity, linearity, and nonlocality. With the fast growth of digital computer knowledge, many authors have started to work on the theory and applications of fractional calculus to present their viewpoints [3]. In general, the better performance of the fractional calculus becomes evident based on lower error levels produced during

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an estimation process. Computational methods and numerical simulations successfully applied to several works, such as those found in [4]–[16].

2. Preliminaries

Some fractional calculus definitions and notation needed [17, 18] in the course of this work are discussed in this section

Definition 2.1. *A real function $\varphi(\mu), \mu > 0$, is said to be in the space $C_\vartheta, \vartheta \in R$ if there exists a real number $q, (q > \vartheta)$, such that $\varphi(\mu) = \mu^q \varphi_1(\mu)$, where $\varphi_1(\mu) \in C[0, \infty)$, and it is said to be in the space C_ϑ^m if $\varphi^{(m)} \in C_\vartheta, m \in N$.*

Definition 2.2. *The Riemann Liouville fractional integral operator of order $\delta \geq 0$, of a function $\varphi(\mu) \in C_\vartheta, \vartheta \geq -1$ is defined as*

$$I^\delta \varphi(\mu) = \begin{cases} \frac{1}{\Gamma(\delta)} \int_0^\mu (\mu - \tau)^{\delta-1} \varphi(\tau) d\tau, & \delta > 0, \mu > 0, \\ \phi(\mu), & \delta \geq 0, \end{cases} \tag{1}$$

where $\Gamma(\cdot)$ is the well-known Gamma function.

Properties of the operator I^δ, which we will use here, are as follows:
For $\varphi \in C_\vartheta, \vartheta \geq -1, \delta, \gamma \geq -1$, then

(1) $I^\delta I^\gamma \varphi(\mu) = I^{\delta+\gamma} \varphi(\mu)$.

(2) $I^\delta I^\gamma \varphi(\mu) = I^\gamma I^\delta \varphi(\mu)$.

(3) $I^\delta \mu^m = \frac{\Gamma(m+1)}{\Gamma(\delta + m + 1} \mu^{\delta+m}$.

Definition 2.3. *The fractional derivative of $\varphi(\mu)$ in the Caputo sense is defined as*

$$\begin{aligned} D^\delta \varphi(\mu) &= I^{m-\delta} D^m \varphi(\mu) \\ &= \frac{1}{\Gamma(m-\delta)} \int_0^\mu (\mu - \tau)^{m-\delta-1} \varphi^{(m)}(\tau) d\tau, \end{aligned} \tag{2}$$

for $m - 1 < \delta \leq m, m \in N, \mu > 0, \varphi \in C_{-1}^m$.

The following are the basic properties of the operator D^δ:

(1) $D^\delta I^\delta \varphi(\mu) = \varphi(\mu)$.

(2) $D^\delta I^\delta \varphi(\mu) = \varphi(\mu) - \sum_{k=0}^{m-1} \varphi^{(k)}(0) \frac{\mu^k}{k!}$.

Definition 2.4. *The Mittag-Leffler function E_δ with $\delta > 0$ is defined as*

$$E_\delta(z) = \sum_{m=0}^{\infty} \frac{z^\delta}{\Gamma(m\delta + 1)}. \tag{3}$$

Definition 2.5. *The Sumudu transform is defined over the set of function*

$$A = \left\{ \varphi(\tau) / \exists M, \omega_1, \omega_2 > 0, |\varphi(\tau)| < M e^{\frac{|\tau|}{\omega_j}}, if \tau \in (-1)^j \times [0, \infty) \right\},$$

by the following formula

$$S[\varphi(\tau)] = \int_0^\infty e^{-\tau} \varphi(\omega\tau) d\tau, \omega \in (-\omega_1, \omega_2). \tag{4}$$

Definition 2.6. *The Sumudu transform of the Caputo fractional derivative is defined as*

$$S[D_\tau^{m\delta}\varphi(\mu,\tau)] = \omega^{-m\delta}S[\varphi(\mu,\tau)] - \sum_{k=0}^{m-1}\omega^{-m\delta+k}\varphi^{(k)}(\mu,0), m-1 < m\delta < m. \tag{5}$$

3. Homotopy Perturbation Sumudu Transform Method (HPSTM)

Let us consider a general fractional non-linear partial differential equation of the form:

$$D_\tau^\delta\varphi(\mu,\tau) + R\left[\varphi(\mu,\tau)\right] + N\left[\varphi(\mu,\tau)\right] = g(\mu,\tau) \tag{6}$$

with the initial condition

$$\varphi(\mu,0) = f(\mu), \tag{7}$$

where $D_\tau^\delta\varphi(\mu,\tau)$ is the Caputo fractional derivative of the function $\varphi(\mu,\tau)$ defined as:

$$D_\tau^\delta\varphi(\mu,\tau) = \frac{\partial^\delta\varphi(\mu,\tau)}{\partial\tau^\delta}$$
$$= \begin{cases} \dfrac{1}{\Gamma(m-\delta)}\int_0^\tau(\tau-\omega)^{m-\delta-1}\dfrac{\partial^m\varphi(\mu,\omega)}{\partial\tau^m}d\omega, & m-1<\delta<m, \\ f_1(-\infty)\sim C(-\xi)^\alpha,\ f(+\infty)=0 \end{cases} \tag{8}$$

and R is the linear differential operator, N represents the general non-linear differential operator, and $g(\mu,\tau)$ is the source term.

Taking the ST on both sides of (6), we have

$$S\left[D_\tau^\delta\varphi(\mu,\tau)\right] + S\left[R\left[\varphi(\mu,\tau)\right]\right] + S\left[N\left[\varphi(\mu,\tau)\right]\right] = S\left[g(\mu,\tau)\right]. \tag{9}$$

Using the property of the ST, we obtain

$$S\left[\varphi(\mu,\tau)\right] = \varphi(\mu,0) + \omega^\delta S\left[g(\mu,\tau)\right] - \omega^\delta S\left[R\left[\varphi(\mu,\tau) + N\left[\varphi(\mu,\tau)\right]\right]\right]. \tag{10}$$

Operating with the ST on both sides of (10) gives

$$\varphi(\mu,\tau) = f(\mu) + S^{-1}\left(\omega^\delta S\left[g(\mu,\tau)\right]\right) - S^{-1}\left(\omega^\delta S\left[R\left[\varphi(\mu,\tau) + N\left[\varphi(\mu,\tau)\right]\right]\right]\right). \tag{11}$$

Now, we apply the HPM:

$$\varphi(\mu,\tau) = \sum_{n=0}^\infty p^n\varphi_n(\mu,\tau), \tag{12}$$

and the nonlinear term can be decomposed as

$$N\left[\varphi(\mu,\tau)\right] = \sum_{n=0}^\infty p^n H_n(\varphi_1,\varphi_2,\ldots,\varphi_n), \tag{13}$$

where

$$H_n(\varphi_1,\varphi_2,\ldots,\varphi_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^\infty p^i\varphi_i\right)\right]_{p=0}.$$

Substituting (12) and (13) in (11), we get

$$\begin{aligned}\sum_{n=0}^\infty p^n\varphi_n(\mu,\tau) &= f(\mu) + S^{-1}\left(\omega^\delta S\left[g(\mu,\tau)\right]\right) \\ &\quad -pS^{-1}\left(\omega^\delta S\left[R\left[\sum_{n=0}^\infty p^n\varphi_n(\mu,\tau)\right] + \sum_{n=0}^\infty p^n H_n\right]\right).\end{aligned} \tag{14}$$

Equating the terms with identical powers of p, we can obtain a series of equations as the follows:

$$\begin{aligned}
p^0 &: \varphi_0(\mu,\tau) = f(\mu) + S^{-1}\left(\omega^\delta S\left[g(\mu,\tau)\right]\right),\\
p^1 &: \varphi_1(\mu,\tau) = -S^{-1}\left(\omega^\delta S\left[R\left[\varphi_0(\mu,\tau)\right] + H_0\right]\right),\\
p^1 &: \varphi_2(\mu,\tau) = -S^{-1}\left(\omega^\delta S\left[R\left[\varphi_1(\mu,\tau)\right] + H_1\right]\right),\\
&\vdots\\
p^n &: \varphi_n(\mu,\tau) = -S^{-1}\left(\omega^\delta S\left[R\left[\varphi_{n-1}(\mu,\tau)\right] + H_{n-1}\right]\right), n \geq 1.
\end{aligned} \tag{15}$$

Finally, we approximate the analytical solution $\varphi(\mu,\tau)$ by truncated series:

$$\varphi(\mu,\tau) = \lim_{p\longrightarrow 1}\sum_{n=0}^{\infty} p^n \varphi_n(\mu,\tau). \tag{16}$$

4. Applications

In this section, we will implement the proposed method for solving Burger's and coupled Burger's equations.

Example 4.1. *First, we consider the fractional Burger's equation*

$$D_\tau^\delta \varphi(\mu,\tau) + \varphi(\mu,\tau)\varphi_\mu(\mu,\tau) = \varphi_{\mu\mu}(\mu,\tau), \tag{17}$$

subject to initial condition

$$\varphi(\mu,0) = \mu. \tag{18}$$

Taking the Sumudu transform (ST) on both sides of (17)*, we have*

$$S\left[D_\tau^\delta \varphi(\mu,\tau)\right] = S\left[\varphi_{\mu\mu}(\mu,\tau) - \varphi(\mu,\tau)\varphi_\mu(\mu,\tau)\right]. \tag{19}$$

Applying the properties of the ST in (19)*, we get*

$$S\left[\varphi(\mu,\tau)\right] = \mu + \omega^\delta S\left[\varphi_{\mu\mu}(\mu,\tau) - \varphi(\mu,\tau)\varphi_\mu(\mu,\tau)\right]. \tag{20}$$

Operating with the ST inverse on both sides of (20)*, we obtain*

$$\varphi(\mu,\tau) = \mu + S^{-1}\left(\omega^\delta S\left[\varphi_{\mu\mu}(\mu,\tau) - \varphi(\mu,\tau)\varphi_\mu(\mu,\tau)\right]\right). \tag{21}$$

By applying HPM, and substituting

$$\varphi(\mu,\tau) = \sum_{n=0}^{\infty} p^n \varphi_n(\mu,\tau),$$

and

$$\varphi(\mu,\tau)\varphi_\mu(\mu,\tau) = \sum_{n=0}^{\infty} p^n H_n,$$

where

$$\begin{aligned}
H_0 &= \varphi_0\varphi_{0\mu},\\
H_1 &= \varphi_0\varphi_{1\mu} + \varphi_1\varphi_{0\mu},\\
H_2 &= \varphi_0\varphi_{2\mu} + \varphi_1\varphi_{1\mu} + \varphi_2\varphi_{0\mu},\\
&\vdots
\end{aligned}$$

in (21)*, we have*

$$\sum_{n=0}^{\infty} p^n \varphi_n(\mu,\tau) = \mu + S^{-1}\left(\omega^\delta S\left[\frac{\partial^2}{\partial\mu^2}\left(\sum_{n=0}^{\infty} p^n \varphi_n(\mu,\tau)\right) - \sum_{n=0}^{\infty} p^n H_n\right]\right). \tag{22}$$

Equating the terms with identical powers of p, we obtain

$$\begin{aligned}
p^0 : \varphi_0(\mu,\tau) &= \mu,\\
p^1 : \varphi_1(\mu,\tau) &= S^{-1}\left(\omega^\delta S\left[\frac{\partial^2}{\partial\mu^2}\varphi_0(\mu,\tau) - H_0\right]\right),\\
p^2 : \varphi_2(\mu,\tau) &= S^{-1}\left(\omega^\delta S\left[\frac{\partial^2}{\partial\mu^2}\varphi_1(\mu,\tau) - H_1\right]\right),\\
p^3 : \varphi_3(\mu,\tau) &= S^{-1}\left(\omega^\delta S\left[\frac{\partial^2}{\partial\mu^2}\varphi_2(\mu,\tau) - H_2\right]\right),\\
&\vdots
\end{aligned}$$

Hence, we have:

$$\begin{aligned}
p^0 : \varphi_0(\mu,\tau) &= \mu,\\
p^1 : \varphi_1(\mu,\tau) &= -\mu\frac{\tau^\delta}{\Gamma(\delta+1)},\\
p^2 : \varphi_2(\mu,\tau) &= 2\mu\frac{\tau^{2\delta}}{\Gamma(2\delta+1)},\\
p^3 : \varphi_3(\mu,\tau) &= -\mu\frac{\Gamma(2\delta+1)}{\Gamma^2(\delta+1)}\frac{\tau^{3\delta}}{\Gamma(3\delta+1)} - 4\mu\frac{\tau^{3\delta}}{\Gamma(3\delta+1)},\\
&\vdots
\end{aligned}$$

Thererfore, the solution of (17) *is given by*

$$\begin{aligned}
\varphi(\mu,\tau) &= \lim_{p\longrightarrow 1}\sum_{n=0}^{\infty} p^n\varphi_n(\mu,\tau)\\
&= \mu\left[1 - \frac{\tau^\delta}{\Gamma(\delta+1)} + 2\frac{\tau^{2\delta}}{\Gamma(2\delta+1)} - \frac{\Gamma(2\delta+1)}{\Gamma^2(\delta+1)}\frac{\tau^{3\delta}}{\Gamma(3\delta+1)} - 4\frac{\tau^{3\delta}}{\Gamma(3\delta+1)} + \cdots\right]
\end{aligned} \tag{23}$$

The Eq. (23) *is approximate to the form* $\varphi(\mu,\tau) = \dfrac{\mu}{1-\tau}$ *for* $\delta = 1$*, which is the exact solution of Eq.* (17) *for* $\delta = 1$*. The result is same as VIM* [19].

Example 4.2. *Consider the following coupled fractional Burger's equations*

$$\begin{aligned}
D_\tau^\delta\varphi(\mu,\tau) - \varphi_{\mu\mu}(\mu,\tau) - 2\varphi(\mu,\tau)\varphi_\mu(\mu,\tau) + (\varphi\psi)_\mu = 0,\\
D_\tau^\gamma\psi(\mu,\tau) - \psi_{\mu\mu}(\mu,\tau) - 2\psi(\mu,\tau)\psi_\mu(\mu,\tau) + (\varphi\psi)_\mu = 0,
\end{aligned} \tag{24}$$

subject to initial conditions

$$\begin{aligned}
\varphi(\mu,0) = \sin(\mu),\\
\psi(\mu,0) = \sin(\mu),
\end{aligned} \tag{25}$$

Taking the Sumudu transform (ST) on both sides of (24)*, we have*

$$\begin{aligned}
S\left[D_\tau^\delta\varphi(\mu,\tau)\right] = S\left[\varphi_{\mu\mu}(\mu,\tau) + 2\varphi\varphi_\mu - (\varphi\psi)_\mu\right],\\
S\left[D_\tau^\gamma\psi(\mu,\tau)\right] = S\left[\psi_{\mu\mu}(\mu,\tau) + 2\psi\psi_\mu - (\varphi\psi)_\mu\right].
\end{aligned} \tag{26}$$

Using the property of the Sumudu transform and the initial condition in (25)*, we obtain*

$$\begin{aligned} S\left[\varphi(\mu,\tau)\right] &= \sin(\mu) + \omega^{\delta} S\left[\varphi_{\mu\mu}(\mu,\tau) + 2\varphi\varphi_{\mu} - (\varphi\psi)_{\mu}\right], \\ S\left[\psi(\mu,\tau)\right] &= \sin(\mu) + \omega^{\gamma} S\left[\psi_{\mu\mu}(\mu,\tau) + 2\psi\psi_{\mu} - (\varphi\psi)_{\mu}\right]. \end{aligned} \tag{27}$$

Operating with the Sumudu inverse on both sides of (27)*, we have*

$$\begin{aligned} \varphi(\mu,\tau) &= \sin(\mu) + S^{-1}\left(\omega^{\delta} S\left[\varphi_{\mu\mu}(\mu,\tau) + 2\varphi\varphi_{\mu} - (\varphi\psi)_{\mu}\right]\right), \\ \psi(\mu,\tau) &= \sin(\mu) + S^{-1}\left(\omega^{\gamma} S\left[\psi_{\mu\mu}(\mu,\tau) + 2\psi\psi_{\mu} - (\varphi\psi)_{\mu}\right]\right). \end{aligned} \tag{28}$$

Suppose that

$$\varphi(\mu,\tau) = \sum_{n=0}^{\infty} p^n \varphi_n(\mu,\tau), \tag{29}$$

$$\psi(\mu,\tau) = \sum_{n=0}^{\infty} p^n \psi_n(\mu,\tau), \tag{30}$$

$$\varphi\varphi_{\mu} = \sum_{n=0}^{\infty} p^n H_n, \tag{31}$$

$$\psi\psi_{\mu} = \sum_{n=0}^{\infty} p^n K_n, \tag{32}$$

$$(\varphi\psi)_{\mu} = \sum_{n=0}^{\infty} p^n G_n. \tag{33}$$

By applying the HPM, and substituting (29)-(33) *in* (28)*, we get*

$$\begin{aligned} \sum_{n=0}^{\infty} p^n \varphi_n &= \sin(\mu) + S^{-1}\left(\omega^{\delta} S\left[\frac{\partial^2}{\partial\mu^2}\left(\sum_{n=0}^{\infty} p^n \varphi_n\right) + 2\sum_{n=0}^{\infty} p^n H_n - \sum_{n=0}^{\infty} p^n G_n\right]\right), \\ \sum_{n=0}^{\infty} p^n \psi_n &= \sin(\mu) + S^{-1}\left(\omega^{\delta} S\left[\frac{\partial^2}{\partial\mu^2}\left(\sum_{n=0}^{\infty} p^n \psi_n\right) + 2\sum_{n=0}^{\infty} p^n K_n - \sum_{n=0}^{\infty} p^n G_n\right]\right). \end{aligned} \tag{34}$$

Equating the terms with identical powers of p, we obtain

$$p^0 : \begin{cases} \varphi_0(\mu,\tau) = \sin(\mu), \\ \psi_0(\mu,\tau) = \sin(\mu). \end{cases} \tag{35}$$

$$p^1 : \begin{cases} \varphi_1(\mu,\tau) = S^{-1}\left(\omega^{\delta} S\left[\dfrac{\partial^2 \varphi_0}{\partial\mu^2} + 2H_0 - G_0\right]\right), \\ \psi_1(\mu,\tau) = S^{-1}\left(\omega^{\gamma} S\left[\dfrac{\partial^2 \psi_0}{\partial\mu^2} + 2K_0 - G_0\right]\right), \end{cases} \tag{36}$$

$$p^2 : \begin{cases} \varphi_2(\mu,\tau) = S^{-1}\left(\omega^{\delta} S\left[\dfrac{\partial^2 \varphi_1}{\partial\mu^2} + 2H_1 - G_1\right]\right), \\ \psi_2(\mu,\tau) = S^{-1}\left(\omega^{\gamma} S\left[\dfrac{\partial^2 \psi_1}{\partial\mu^2} + 2K_1 - G_1\right]\right), \end{cases} \tag{37}$$

$$p^3: \begin{cases} \varphi_3(\mu,\tau) = S^{-1}\left(\omega^\delta S\left[\dfrac{\partial^2 \varphi_2}{\partial \mu^2} + 2H_2 - G_2\right]\right), \\ \psi_3(\mu,\tau) = S^{-1}\left(\omega^\gamma S\left[\dfrac{\partial^2 \psi_2}{\partial \mu^2} + 2K_2 - G_2\right]\right), \end{cases} \tag{38}$$

$$\vdots$$

Then, we have

$$p^0: \begin{cases} \varphi_0(\mu,\tau) = \sin(\mu), \\ \psi_0(\mu,\tau) = \sin(\mu). \end{cases} \tag{39}$$

$$p^1: \begin{cases} \varphi_1(\mu,\tau) = -\sin(\mu)\dfrac{\tau^\delta}{\Gamma(\delta+1)}, \\ \psi_1(\mu,\tau) = -\sin(\mu)\dfrac{\tau^\gamma}{\Gamma(\gamma+1)}, \end{cases} \tag{40}$$

$$p^2: \begin{cases} \varphi_2(\mu,\tau) = \dfrac{\tau^{2\delta}}{\Gamma(2\delta+1)}\sin(\mu) - \dfrac{\tau^{2\delta}}{\Gamma(2\delta+1)}\sin(2\mu) + \dfrac{\tau^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}\sin(2\mu), \\ \psi_2(\mu,\tau) = \dfrac{\tau^{2\gamma}}{\Gamma(2\gamma+1)}\sin(\mu) - \dfrac{\tau^{2\gamma}}{\Gamma(2\gamma+1)}\sin(2\mu) + \dfrac{\tau^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}\sin(2\mu), \end{cases} \tag{41}$$

$$\vdots$$

and so on.

Therefore, the solution of (24) *is given by*

$$\begin{aligned} \varphi(\mu,\tau) &= \sin(\mu)\left[1 - \frac{\tau^\delta}{\Gamma(\delta+1)} + \frac{\tau^{2\delta}}{\Gamma(2\delta+1)}\cdots\right] \\ &\quad + \sin(\mu)\cos(\mu)\left[-2\frac{\tau^{2\delta}}{\Gamma(2\delta+1)} + 2\frac{\tau^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}\cdots\right], \\ \psi(\mu,\tau) &= \sin(\mu)\left[1 - \frac{\tau^\gamma}{\Gamma(\gamma+1)} + \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)}\cdots\right] \\ &\quad + \sin(\mu)\cos(\mu)\left[-2\frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)} + 2\frac{\tau^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}\cdots\right]. \end{aligned} \tag{42}$$

Setting $\delta = \gamma$ *in* (42), *we obtain*

$$\begin{aligned} \varphi(\mu,\tau) &= \sin(\mu)\left[1 - \frac{\tau^\delta}{\Gamma(\delta+1)} + \frac{\tau^{2\delta}}{\Gamma(2\delta+1)}\cdots\right] \\ \psi(\mu,\tau) &= \sin(\mu)\left[1 - \frac{\tau^\gamma}{\Gamma(\gamma+1)} + \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)}\cdots\right] \\ &= E_\delta(\tau^\delta)\sin(\mu), \\ &= E_\gamma(\tau^\gamma)\sin(\mu). \end{aligned} \tag{43}$$

The Eq. (43) *is approximate to the form* $\varphi(\mu,\tau) = \psi(\mu,\tau) = e^{-\tau}\sin(\mu)$ *for* $\delta = \gamma = 1$, *which is the exact solution of Eq.* (24) *for* $\delta = \gamma = 1$. *The result is same as q-HATM* [20] *and HPM* [21].

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