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PRIME LABELING IN THE CONTEXT OF SUBDIVISION OF SOME CYCLE RELATED GRAPHS

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Abstract. A prime labeling on a graph G of order n is a bijection from the set of vertices of G into the set of first n positive integers such that any two adjacent vertices in G have relatively prime labels. The results about prime labeling of wheel, helm, flower, crown and union of crown graphs are very well-known. In this paper we obtain prime labeling of various graphs resulting from the subdivision of edges in these graphs.

Keywords: Prime labeling, Prime graphs, Subdivision of graphs, barycentric subdivision, Union of graphs.

AMS Subject Classification: 05C78

1. Introduction

We consider only finite, simple and undirected graphs. For a graph G, $V(G)$ and $E(G)$ denote its vertex set and edge set respectively whereas $|V(G)|$ and $|E(G)|$ denote the cardinalities of the respective sets. We refer to Gross and Yellen [5] for graph theoretic terminology and notations and Burton [2] for results related to number theory. We start with the definition of a prime labeling and a prime graph.

Definition 1.1. *Let G be a graph with n vertices. A bijection $f : V(G) \to \{1, 2, \ldots, n\}$ is said to be a prime labeling of G if $\gcd(f(u), f(v)) = 1$ whenever u and v are adjacent vertices of G. A graph that admits a prime labeling is called a prime graph.*

The simplest examples of prime graphs are path and cycle graphs whereas a complete graph with four or more vertices is not a prime graph due to following lemma which gives a necessary condition for a graph to be prime.

Lemma 1.1. [3] *Let $\beta_0(G)$ denote the independence number of G. If $\beta_0(G) < \left\lfloor \frac{|V(G)|}{2} \right\rfloor$, then G is not a prime graph.*

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Entringer originated prime labeling and Taut et al.[10] discussed it in a paper about thirty five years ago. Since then variety of graphs have been studied for prime labeling. Prime labeling is also extensively studied in the context of various graph operations. Some of the variants of prime labeling like prime cordial labeling [9] and neighborhood-prime labeling [7] are also interesting areas of research these days. For a complete survey of results related to prime labeling and its variants, we refer the reader to the dynamic survey of graph labeling by J. Gallian [4]. Here we shall study prime labeling in the context of graph operation of subdivision whose meaning is explained below.

Definition 1.2. *Let e be an edge with end vertices as u and v in a graph G. Then by subdivision of the edge $e = uv$ in G, we mean introduction of a new vertex w in G and where the edge $e = uv$ is replaced by two new edges $e' = uw$ and $e'' = wv$ in G. Thus, subdividing a single edge in G increases the cardinality of its vertex and edge set by one.*

Definition 1.3. *By subdivision of a graph G, we mean subdivision of all or some of the edges in G. Further, if all the edges of G are subdivided then the resultant subdivision is also known as the Barycentric subdivision of graph G.*

In the present paper, we derive prime labeling of graphs resulting from the subdivision of edges of some well-known graphs like wheel, helm, $C_n \odot K_1$ (i.e. crown graph), $(C_n \odot K_1) \cup (C_n \odot K_1)$ and flower graph. All the results are supported with appropriate examples and figures so that the theorems and their proofs are better understood. The following lemma which is based on Euclid's division algorithm has been stated over here since it is (directly or indirectly) used frequently throughout the paper.

Lemma 1.2. *If a and b are positive integers such that $a < b$, then*

$$\gcd(a, b) = \gcd(a, r),$$

where r is the remainder obtained on dividing the number b by the number a.

2. Main Results

It is proved in [10] that the wheel graph $W_n = C_n + K_1$ is prime if and only if n is even. Here we derive the results for graphs obtained by taking subdivision of a wheel graph.

Theorem 2.1. *The barycentric subdivision of the wheel graph W_n is a prime graph for all n.*

Proof. Consider the wheel graph W_n with vertex set $\{u_0, v_{2i-1} : i = 1, 2, \ldots, n\}$ and edge set $\{u_0 v_{2i-1}, v_{2i-1} v_{2i+1} : i = 1, 2, \ldots, n\}$ where suffixes are read as modulo $2n$. We call u_0 the apex vertex of W_n. Let G be the barycentric subdivision of W_n in which the edges $u_0 v_{2i-1}$ and $v_{2i-1} v_{2i+1}$ are subdivided by the newly added vertices u_i and v_{2i} respectively for $i = 1, 2, \ldots, n$. Thus $|V(G)| = 3n + 1$. For $i = 1, 2, ..., n$ and $j = 1, 2, ..., 2n$, define $f : V(G) \to \{1, 2, \ldots, 3n + 1\}$ as

$$\begin{aligned}
f(u_0) &= 1,\\
f(u_i) &= 3i, \quad i \text{ is even},\\
f(u_i) &= 3i - 1, \quad i \text{ is odd},\\
f(v_j) &= \frac{3j + 2}{2}, \quad j \text{ is even},\\
f(v_j) &= \frac{3j + 3}{2}, \quad j \equiv 1 \pmod 4,\\
f(v_j) &= \frac{3j + 1}{2}, \quad j \equiv 3 \pmod 4.
\end{aligned}$$

We claim that any two adjacent vertices have relatively prime labels.
If $j \equiv 0 \pmod 4$, then $\gcd(f(v_j), f(v_{j+1})) = \gcd\left(\frac{3j+2}{2}, \frac{3j+6}{2}\right) = \gcd\left(\frac{3j+2}{2}, 2\right) = 1$.
If $j \equiv 1 \pmod 4$, then $\gcd(f(v_j), f(v_{j+1})) = \gcd\left(\frac{3j+3}{2}, \frac{3j+5}{2}\right) = \gcd\left(\frac{3j+3}{2}, 1\right) = 1$.
If $j \equiv 2 \pmod 4$, then $\gcd(f(v_j), f(v_{j+1})) = \gcd\left(\frac{3j+2}{2}, \frac{3j+4}{2}\right) = \gcd\left(\frac{3j+2}{2}, 1\right) = 1$.
If $j \equiv 3 \pmod 4$, then $\gcd(f(v_j), f(v_{j+1})) = \gcd\left(\frac{3j+1}{2}, \frac{3j+5}{2}\right) = \gcd\left(\frac{3j+1}{2}, 2\right) = 1$ as $\frac{3j+1}{2}$ is an odd number.
Also $\gcd(f(v_1), f(v_{2n})) = \gcd\left(3, \frac{6n+2}{2}\right) = \gcd(3, 3n+1) = 1$. Except these cases, every pair of adjacent vertices in G have either consecutive labels or one of the labels is 1. Thus f defines a prime labeling on G. □

Example 2.1. *Prime labeling of the barycentric subdivision of W_9 is shown in Figure 1.*

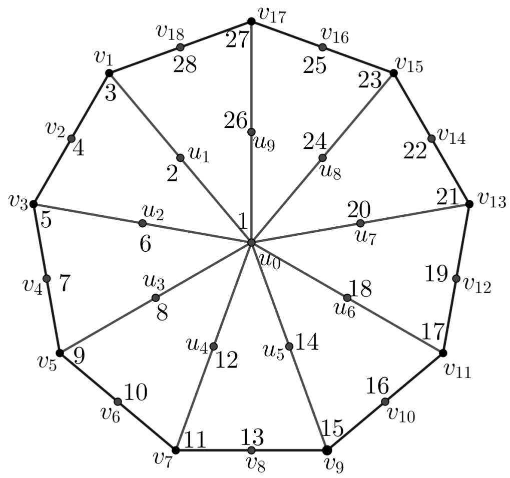

FIGURE 1

Theorem 2.2. *The graph obtained by taking subdivision of the edges adjacent to the apex vertex of W_n is a prime graph for all n.*

Proof. Consider the wheel graph W_n with vertex set $\{u_0, v_i : i = 1, 2, \dots, n\}$ and edge set $\{u_0v_i, v_iv_{i+1} : i = 1, 2, \dots, n\}$ where suffixes are read as modulo n. Let G be a graph obtained from the wheel graph W_n by subdividing edges u_0v_i with the newly added vertices u_i. Then $|V(G)| = 2n + 1$. For $i = 1, 2, ..., n$, define $f : V(G) \to \{1, 2, \dots, 2n + 1\}$ as

$$\begin{aligned} f(u_0) &= 1, \\ f(u_1) &= 3, \\ f(v_1) &= 2 \\ f(u_i) &= 2i, \quad i \neq 1, \\ f(v_i) &= 2i + 1, \quad i \neq 1. \end{aligned}$$

It is easy to verify that f is a prime labeling of G. □

Example 2.2. *Prime labeling of the graph obtained from W_7 by taking subdivision of every edge adjacent to its apex vertex is shown in Figure 2.*

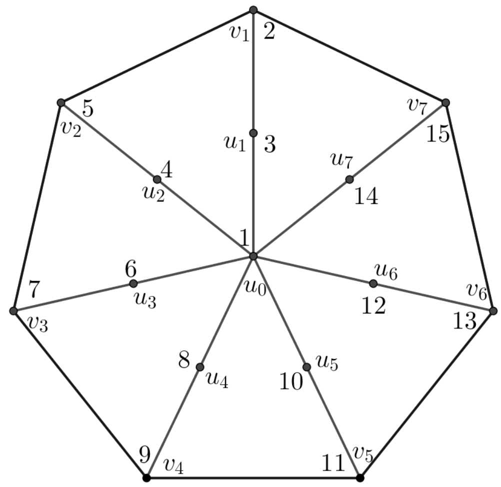

FIGURE 2

Note that the graph obtained by taking the subdivision of cycle edges in W_n is a gear graph which is known to be prime.
The helm graph H_n is the graph obtained from wheel graph $W_n = C_n + K_1$ by attaching a pendant edge to every vertex of the cycle C_n in W_n. It is proved in [8] that H_n is a prime graph for all n. Our next few results are about subdivision of helm graphs.

Theorem 2.3. *Let G be a graph obtained by taking subdivision of every pendant edge in the helm graph H_n. Then G is prime if and only if n is even.*

Proof. Consider the helm graph H_n with vertex set $\{u_0, u_i, w_i : i = 1, 2, \ldots, n\}$ and edge set $\{u_0u_i, u_iw_i, u_iu_{i+1} : i = 1, 2, \ldots, n\}$ where suffixes are read as modulo n. Let G be the graph obtained from H_n by subdividing the edges u_iw_i with the newly added vertices v_i for $i = 1, 2, \ldots n$. Then $|V(G)| = 3n + 1$.

First we prove that G is not prime when n is odd. Suppose $n = 2k + 1$ for some k. It may be verified that the independence number $\beta_0(G)$ of the graph G is equal to $3k + 1$. But $|V(G)| = 6k + 4$ and therefore

$$\beta_0(G) < \left\lfloor \frac{|V(G)|}{2} \right\rfloor.$$

So in view of Lemma 1.1, we conclude that G is not a prime graph.

Now assume that $n = 2k$ for some k. For $i = 1, 2, \ldots, 2k$, define $f : V(G) \to \{1, 2, \ldots, 6k + 1\}$ as

$$\begin{aligned} f(u_0) &= 1, \\ f(u_i) &= 3i - 1, \\ f(v_i) &= 3i, \\ f(w_i) &= 3i + 1. \end{aligned}$$

Observe that
$\gcd(f(u_i), f(u_{i+1})) = \gcd(3i - 1, 3i + 2) = \gcd(3i - 1, 3) = 1$.
$\gcd(f(u_1), f(u_{2k+1})) = \gcd(2, 3(2k) - 1) = \gcd(2, 6k - 1) = 1$. Except these cases, any two adjacent vertices in G have either consecutive labels or one of them has 1 as a label. Thus f defines a prime labeling on G. □

Example 2.3. *Prime labeling of the graph obtained from H_8 by taking subdivision of every pendant edge, is shown in Figure 3.*

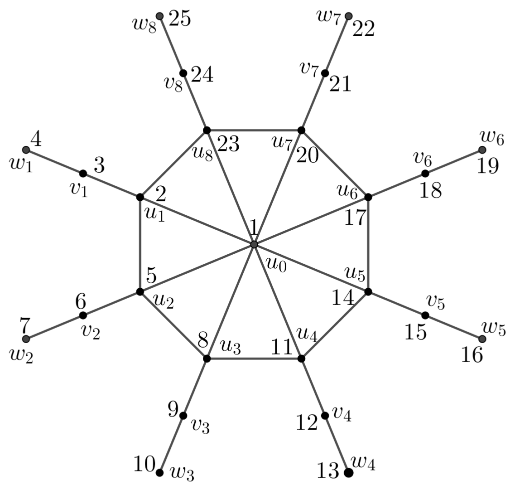

FIGURE 3

Theorem 2.4. *The graph obtained from the helm graph H_n by subdivision of every edge of the cycle C_n in H_n is a prime graph.*

Proof. Consider the helm graph H_n with vertex set $\{u_0, u_{2i-1}, v_i : i = 1, 2, \ldots, n\}$ and edge set $\{u_0u_{2i-1}, u_{2i-1}v_i, u_{2i-1}u_{2i+1} : i = 1, 2, \ldots, n\}$ where suffixes are read as modulo $2n$. Let G be the graph obtained from H_n by subdividing the edges $u_{2i-1}u_{2i+1}$ with the newly added vertices u_{2i} for $i = 1, 2, \ldots n$.
For $i = 1, 2, 3, ..., 2n$ and $j = 1, 2, 3, ..., n$, define $f : V(G) \to \{1, 2, \ldots, 3n + 1\}$ as

$$\begin{aligned}
f(u_0) &= 1, \\
f(u_i) &= 3\left(\frac{i+1}{2}\right), \quad i \equiv 1 \pmod 4 \\
f(u_i) &= 3\left(\frac{i+1}{2}\right) - 1, \quad i \equiv 3 \pmod 4 \\
f(u_i) &= 3\left(\frac{i}{2}\right) + 1, \quad i \text{ is even} \\
f(v_j) &= 3j - 1, \quad j \text{ is odd} \\
f(v_j) &= 3j, \quad j \text{ is even.}
\end{aligned}$$

Since $f(u_0) = 1$, we see that $\gcd(f(u), f(u_0)) = 1$ for every vertex u which is adjacent to u_0. In other cases, it may be verified that the labels of any two adjacent vertices in G are either consecutive integers or consecutive odd integers and so f defines a prime labeling on G. $\square$

Example 2.4. *Prime labeling of the graph obtained from H_7 by taking subdivision of every edge of the cycle C_7 is shown in Figure 4.*

Theorem 2.5. *The graph obtained from the helm graph H_n by subdivision of every edge incident to its apex vertex is a prime graph.*

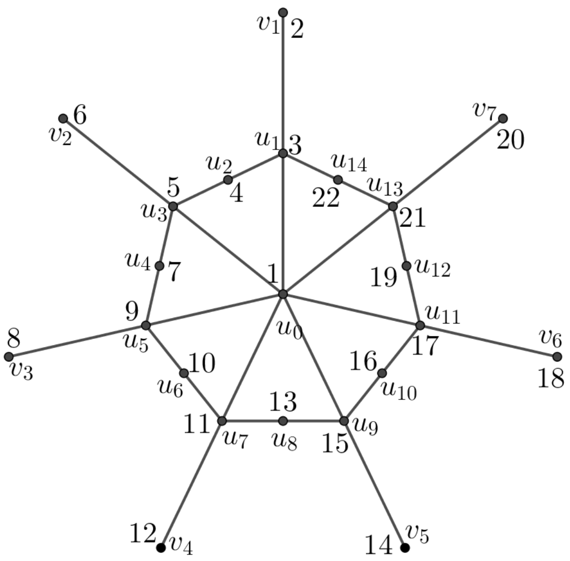

FIGURE 4

Proof. Consider the helm graph H_n with vertex set $\{u_0, v_i, w_i : i = 1, 2, \ldots, n\}$ and edge set $\{u_0v_i, v_iw_i, v_iv_{i+1} : i = 1, 2, \ldots, n\}$ where suffixes are read as modulo n. Let G be the graph obtained from H_n by subdividing the edges u_0v_i with the newly added vertices u_i for $i = 1, 2, \ldots n$.
For $i = 1, 2, 3, ..., n$, define $f : V(G) \to \{1, 2, \ldots, 3n + 1\}$ as

$$
\begin{aligned}
f(u_0) &= 1,\\
f(u_i) &= i + 2, \quad i = 1, 2,\\
f(v_i) &= 3i - 1, \quad i = 1, 2,\\
f(w_i) &= 8 - i, \quad i = 1, 2,\\
f(u_i) &= 3i - 1, \quad \text{all } i > 2,\\
f(v_i) &= 3i, \quad i > 1 \text{ is odd}\\
f(w_i) &= 3i + 1, \quad i > 1 \text{ is odd},\\
f(v_i) &= 3i + 1, \quad i > 2 \text{ is even},\\
f(w_i) &= 3i, \quad i > 2 \text{ is even}.
\end{aligned}
$$

We claim that $\gcd(f(u), f(v)) = 1$ for any two adjacent vertices u and v.
If n is odd then $\gcd(f(v_1), f(v_n)) = \gcd(2, 3n) = 1$.
If n is even then $\gcd(f(v_1), f(v_n)) = \gcd(2, 3n + 1) = 1$.
If $i > 1$ is odd then $\gcd(f(v_i), f(v_{i+1})) = \gcd(3i, 3(i + 1) + 1) = \gcd(3i, 4) = 1$.
If $i > 2$ is even then $\gcd(f(v_i), f(v_{i+1})) = \gcd(3i + 1, 3(i + 1)) = \gcd(3i + 1, 2) = 1$.
The remaining cases are easy to verify and thus we conclude that f is a prime labeling on G. □

Example 2.5. *Prime labeling of the graph obtained from H_7 by taking subdivision of every edge incident to its apex vertex, is shown in Figure 5.*

Theorem 2.6. *The graph obtained from the helm graph H_n by subdivision of every pendant edge and every edge of the cycle C_n is a prime graph.*

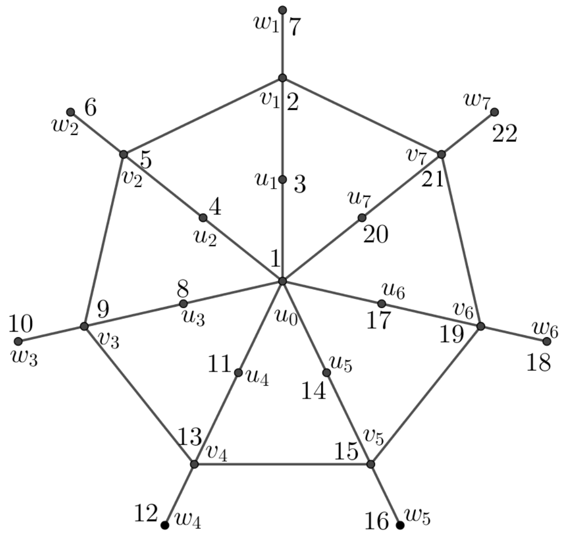

FIGURE 5

Proof. Consider the helm graph H_n with vertex set $\{u_0, u_{2i-1}, w_i : i = 1, 2, \ldots, n\}$ and edge set $\{u_0u_{2i-1}, u_{2i-1}w_i, u_{2i-1}u_{2i+1} : i = 1, 2, \ldots, n\}$ where suffixes are read as modulo $2n$. Let G be the graph obtained from H_n by subdividing the edges $u_{2i-1}u_{2i+1}$ and $u_{2i-1}w_i$ with the newly added vertices u_{2i} and v_i respectively for $i = 1, 2, \ldots n$. Thus $|V(G)| = 4n + 1$.
For $i = 1, 2, 3, ..., 2n$ and $j = 1, 2, 3, ..., n$, define $f : V(G) \to \{1, 2, \ldots, 4n + 1\}$ as

$$
\begin{aligned}
f(u_0) &= 1,\\
f(u_i) &= 2i + 2, \text{ if } i \equiv 1 \pmod 6 \text{ and } i \equiv 3 \pmod 6,\\
f(u_i) &= 2i, \quad i \equiv 5 \pmod 6,\\
f(u_i) &= 2i + 1, \quad i \text{ is even},\\
f(v_j) &= 4j - 1, \quad \text{all } j,\\
f(w_j) &= 4j - 2, \quad j \not\equiv 0 \pmod 3,\\
f(w_j) &= 4j, \quad j \equiv 0 \pmod 3.
\end{aligned}
$$

Observe that $\gcd(f(u_1), f(u_{2n})) = \gcd(4, 2(2n) + 1) = 1$.
Further, if i is even with $i \not\equiv 4 \pmod 6$ then

$$\gcd(f(u_i), f(u_{i+1})) = \gcd(2i + 1, 2(i + 1) + 2) = \gcd(2i + 1, 3) = 1,$$

and if i is even with $i \equiv 4 \pmod 6$ then $\gcd(f(u_i), f(u_{i+1})) = \gcd(2i + 1, 2(i + 1)) = 1$. Moreover, if i is odd with $i \not\equiv 5 \pmod 6$ then $\gcd(f(u_i), f(u_{i+1})) = \gcd(2i + 2, 2(i + 1) + 1) = \gcd(2i + 2, 1) = 1$ and if i is odd with $i \equiv 5 \pmod 6$, then $\gcd(f(u_i), f(u_{i+1})) = \gcd(2i, 2(i + 1) + 1) = \gcd(2i, 3) = 1$ because $i \equiv 5 \pmod 6 \Rightarrow i \equiv 2 \pmod 3$. Almost similar reasons show that if i is odd $\gcd(f(u_i), f(v_{\frac{i+1}{2}})) = 1$.
Except these cases, any two adjacent vertices in G have either consecutive labels or one of them has 1 as a label. Thus f is a prime labeling on G. □

Example 2.6. *Prime labeling of the graph obtained from H_7 by taking subdivision of every pendant edge and cycle edge, is shown in Figure 6.*

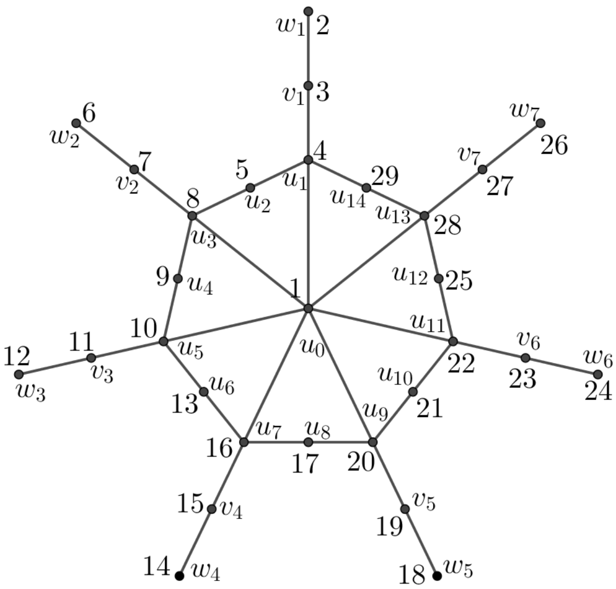

FIGURE 6

Theorem 2.7. *The graph obtained from the helm graph H_n by subdivision of every pendant edge and every edge incident to its apex vertex is a prime graph.*

Proof. Consider the helm graph H_n with vertex set $\{u_0, v_i, x_i : i = 1, 2, \ldots, n\}$ and edge set $\{u_0v_i, v_ix_i, v_iv_{i+1} : i = 1, 2, \ldots, n\}$ where suffixes are read as modulo n. Let G be the graph obtained from H_n by subdividing the edges u_0v_i and v_ix_i with the newly added vertices u_i and w_i respectively for $i = 1, 2, \ldots n$.
For $i = 1, 2, 3, ..., n$, define $f : V(G) \to \{1, 2, \ldots, 4n + 1\}$ as

$$\begin{aligned} f(u_0) &= 1, \\ f(u_1) &= 3, \\ f(v_1) &= 2, \\ f(w_1) &= 5, \\ f(x_1) &= 4, \\ f(u_i) &= 4i - 2, \quad i \neq 1 \\ f(v_i) &= 4i - 1, \quad i \neq 1 \\ f(w_i) &= 4i, \quad i \neq 1 \\ f(x_i) &= 4i + 1, \quad i \neq 1. \end{aligned}$$

The reader may easily verify that f defines a prime labeling on G. □

Example 2.7. *Prime labeling of the graph obtained from H_7 by taking subdivision of every pendant edge and every edge incident to its apex vertex, is shown in Figure 7.*

Theorem 2.8. *The graph obtained from the helm graph H_n by subdivision of every edge of the cycle C_n and every edge incident to its apex vertex is a prime graph.*

Proof. Consider the helm graph H_n with vertex set $\{u_0, v_{2i-1}, w_i : i = 1, 2, \ldots, n\}$ and edge set $\{u_0v_{2i-1}, v_{2i-1}w_i, v_{2i-1}v_{2i+1} : i = 1, 2, \ldots, n\}$ where suffixes are read as modulo $2n$. Let G be the graph obtained from H_n by subdividing the edges $v_{2i-1}v_{2i+1}$ and u_0v_{2i-1}

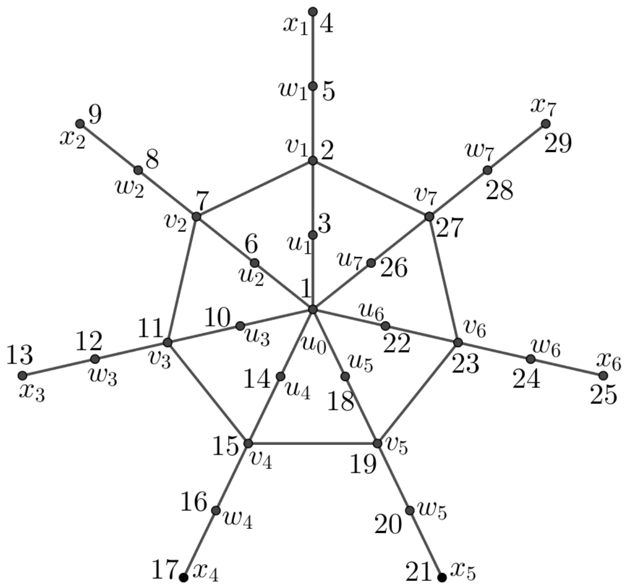

FIGURE 7

with the newly added vertices v_{2i} and u_i respectively for $i = 1, 2, \ldots n$. We consider two cases over here.

Case 1: $n \not\equiv 2 \pmod 3$.

For $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, 2n$, define $f : V(G) \to \{1, 2, \ldots, 4n + 1\}$ as

$$\begin{aligned} f(u_0) &= 1, \\ f(u_i) &= 4i - 2, \\ f(w_i) &= 4i, \\ f(v_j) &= 2j + 1. \end{aligned}$$

Since $n \not\equiv 2 \pmod 3$, f clearly defines a prime labeling. However, if $n \equiv 2 \pmod 3$, then $\gcd(f(v_1), f(v_{2n})) = 3$ and so we need to modify f.

Case 2: $n \equiv 2 \pmod 3$.

Define $g : V(G) \to \{1, 2, \ldots, 4n + 1\}$ as follows:

$$\begin{aligned} g(x) &= f(x), \quad x \neq v_{2n}, w_n, \\ g(v_{2n}) &= f(w_n), \\ g(w_n) &= f(v_{2n}). \end{aligned}$$

Observe that g defines a prime labeling on G when $n \equiv 2 \pmod 3$. □

Example 2.8. *Prime labeling of the graph obtained from H_7 by taking subdivision of every cycle edge and edges incident to its apex vertex, is shown in Figure 8.*

The flower graph Fl_n is a graph obtained from the helm graph H_n by joining its each and every pendant vertex to its apex vertex. Seoud et al. [8] proved that Fl_n is a prime graph for all n. Since flower graph is obtained by adding specific edges to a helm graph, the reader may verify that the prime labelings derived in Theorem 2.3 upto Theorem 2.8 are the prime labelings of the corresponding subdivision in flower graphs also. We now obtain a prime labeling for the barycentric subdivision of flower graph.

Theorem 2.9. *The barycentric subdivision of flower graph is a prime graph for all n.*

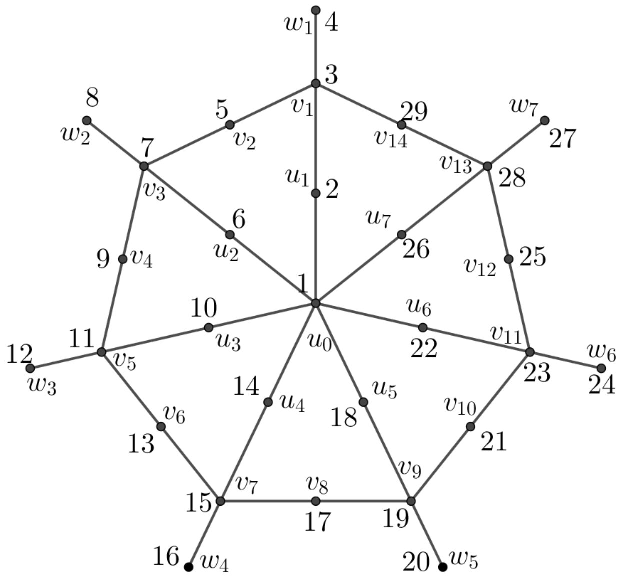

FIGURE 8

Proof. Consider the flower graph Fl_n with vertex set $\{v_0, v_i, u_i : i = 1, 2, \ldots, n\}$ and edge set $\{v_0v_i, v_0u_i, v_iu_i, v_iv_{i+1} : i = 1, 2, \ldots, n\}$, where the suffixes are read as modulo n. Let G be the barycentric subdivision of flower graph in which the edges v_0u_i, v_iu_i, v_0v_i and v_iv_{i+1} are subdivided by the newly added vertices x_i, y_{3i-1}, y_{3i-2} and y_{3i} respectively. Thus $|V(G)| = 6n + 1$. We shall obtain prime labeling of graph G by considering the following two cases.

Case 1 $n \not\equiv 4 \pmod 5$.

In this case define $f : V(G) \to \{1, 2, \ldots 6n + 1\}$ by

$$f(v_0) = 1,$$
$$f(v_i) = 6i - 1, \text{ for } i = 1, 2, \ldots, n,$$
$$f(u_i) = 6i - 3, \text{ for } i = 1, 2, \ldots, n,$$
$$f(x_i) = \begin{cases} 6i - 5, & \text{for } i = 2, \ldots, n; \\ 6n + 1, & \text{for } i = 1, \end{cases}$$
$$f(y_j) = \begin{cases} 2j - 4, & \text{for } j \equiv 0 \pmod 3; \\ 2j + 4, & \text{for } j \equiv 1 \pmod 3; \\ 2j, & \text{for } j \equiv 2 \pmod 3), \text{where } j = 1, 2, \ldots, 3n. \end{cases}$$

Then for $i = 1, 2, \ldots, n$

$$\begin{aligned}
&\gcd(f(v_0), f(y_{3i-2})) = 1 = \gcd(f(v_0), f(x_i)) \text{ as } f(v_0) = 1,\\
&\gcd(f(v_i), f(y_{3i})) = \gcd(6i-1, 6i-4) = \gcd(3, 6i-4) = 1,\\
&\gcd(f(v_i), f(y_{3i-1})) = \gcd(6i-1, 6i-2) = 1,\\
&\gcd(f(v_i), f(y_{3i-2})) = \gcd(6i-1, 6i) = 1,\\
&\gcd(f(v_i), f(y_{3i-3})) = \gcd(6i-1, 6i-10) = \gcd(9, 6i-10) = 1, \text{ for } i \geq 2,\\
&\gcd(f(v_1), f(y_{3n})) = \gcd(5, 6n-4) = 1 \text{ since } n \not\equiv 4 \pmod 5,\\
&\gcd(f(x_i), f(u_i)) = \begin{cases} \gcd(6i-5, 6i-3) = 1 \text{ for } i \neq 1;\\ \gcd(6n+1, 3) = 1 \text{ for } i = 1,\end{cases}\\
&\gcd(f(u_i), f(y_{3i-1})) = \gcd(6i-3, 6i-2) = 1.
\end{aligned}$$

Case 2 $n \equiv 4 \pmod 5$.

In this case define $g : V(G) \to \{1, 2, \ldots 6n+1\}$ by

$$\begin{aligned}
g(u) &= f(u), \text{ for } u \neq y_{3n}, y_{3n-3},\\
g(y_{3n}) &= f(y_{3n-3})\\
g(y_{3n-3}) &= f(y_{3n}).
\end{aligned}$$

Observe that g is a prime labeling on G in this case.

□

Example 2.9. *Prime labeling of barycentric subdivision of flower graph Fl_6 is shown in Figure 9.*

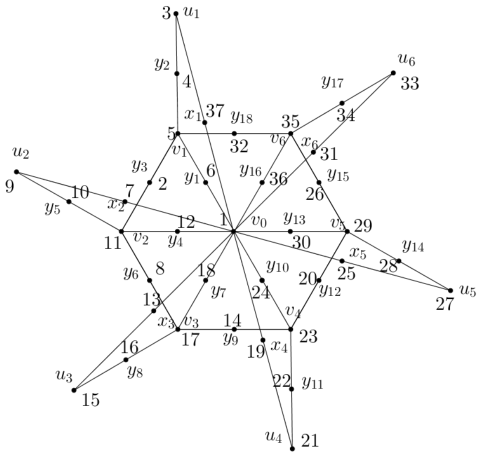

Figure 9

The crown graph with $2n$ vertices and $2n$ edges is a graph obtained by attaching a pendant edge to each vertex of the cycle C_n. It is also viewed as a corona product of the cycle C_n and K_1 and it is denoted by $C_n \odot K_1$. It is known that crown graph $C_n \odot K_1$ and also union of its two copies are prime graphs for all n. See for instance [10] and [6]. Here we consider the subdivision of crown graph as well as the subdivision of union of crown graphs and derive the results regarding prime labeling of the resultant graphs.

Theorem 2.10. *The graph obtained by taking subdivision of pendant edges in crown graph $C_n \odot K_1$ is prime for all n.*

Proof. Consider the crown graph $C_n \odot K_1$ with vertex set $\{v_i, u_i : i = 1, 2, \dots, n\}$ and edge set $\{v_i v_{i+1}, v_i u_i : i = 1, 2, \dots, n\}$, where suffixes are read as modulo n. Let G be the graph obtained from $C_n \odot K_1$ by subdividing its pendant edges $u_i v_i$ with the newly added vertices y_i. Thus $V(G) = \{u_i, v_i, y_i : i = 1, 2, \dots, n\}$.

For $i = 1, 2, \dots, n$, define $f : V(G) \to \{1, 2, \dots, 3n\}$ as

$$f(v_i) = 3i - 2,\ f(u_i) = 3i,\ f(y_i) = 3i - 1.$$

Then for $i = 1, 2, \dots, n$, $\gcd(f(v_i), f(y_i)) = \gcd(3i-2, 3i-1) = 1$ and $\gcd(f(u_i), f(y_i)) = \gcd(3i, 3i-1) = 1$. Also for $i < n$, $\gcd(f(v_i), f(v_{i+1})) = \gcd(3i-2, 3i+1) = \gcd(3, 3i+1) = 1$ and $\gcd(f(v_n), f(v_1)) = \gcd(3n-2, 1) = 1$. Thus f is a prime labeling on G. □

Example 2.10. *Prime labeling of the graph obtained from $C_4 \odot K_1$ by taking subdivision of its pendant edges is shown the Figure 10.*

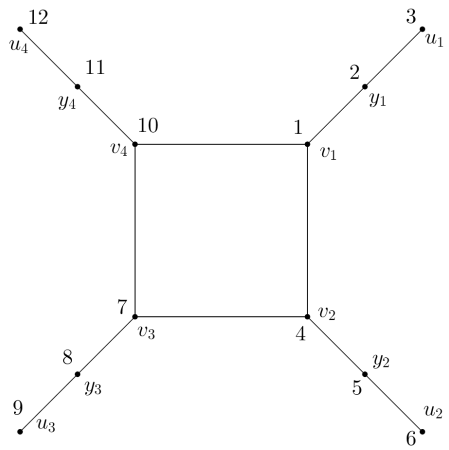

FIGURE 10

Theorem 2.11. *The graph obtained by taking subdivision of every edge of cycle in crown graph $C_n \odot K_1$ is prime for all n.*

Proof. Consider the crown graph $C_n \odot K_1$ with vertex set $\{v_i, u_i : i = 1, 2, \dots, n\}$ and edge set $\{v_i v_{i+1}, v_i u_i : i = 1, 2, \dots, n\}$, where suffixes are read as modulo n. Let G be the graph obtained from $C_n \odot K_1$ by subdividing edges $v_i v_{i+1}$ with the newly added vertices x_i for $i = 1, 2, \dots, n$. Thus $V(G) = \{u_i, v_i, x_i : i = 1, 2, \dots, n\}$.
Now define $f : V(G) \to \{1, 2, \dots, 3n\}$ as

$$f(x_i) = 3i \text{ for all } i,$$

$$f(v_i) = \begin{cases} 3i - 1; & i \text{ is even}, \\ 3i - 2; & i \text{ is odd}, \end{cases}$$

$$f(u_i) = \begin{cases} 3i - 1; & i \text{ is odd}, \\ 3i - 2; & i \text{ is even}. \end{cases}$$

It is not difficult to verify that f is a prime labeling on G and so we skip the details. Then one can check that $\gcd(f(u), f(v)) = 1$ whenever u and v are adjacent. Thus G is a prime graph. □

Example 2.11. *Prime labeling of the graph obtained by taking subdivision of every edge of cycle in crown graph $C_6 \odot K_1$ is shown in Figure 11.*

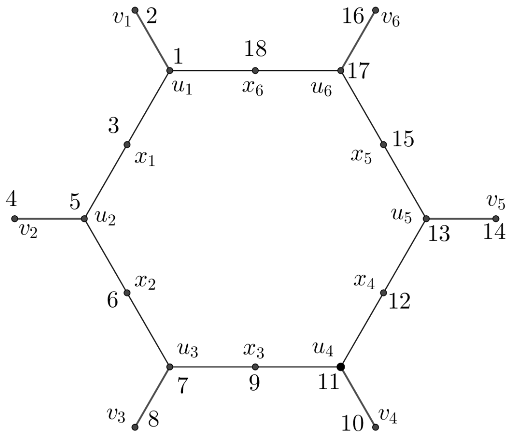

FIGURE 11

Theorem 2.12. *Let G be the graph obtained by subdividing the pendant edges in $(C_n \odot K_1) \cup (C_n \odot K_1)$. Then G is a prime graph if and only if n is even.*

Proof. Let $H = (C_n \odot K_1) \cup (C_n \odot K_1)$, where the vertex set and edge set of H are as follows: $V(H) = \{v_i, u_i : i = 1, 2, \dots, n\} \cup \{v_i, u_i : i = n+1, n+2, \dots, 2n\}$ and $E(H) = \{v_iv_{i+1}, v_iu_i : i = 1, 2, \dots n\} \cup \{v_iv_{i+1}, v_iu_i : i = n+1, n+2, \dots 2n\}$, where suffixes in the first set of $E(H)$ are read as modulo n and in second set as modulo $2n$. Let G be the graph obtained from H by subdividing all its pendant edges u_iv_i with the help of the newly added vertices y_i for $i = 1, 2, \dots, 2n$. Thus $V(G) = \{u_i, v_i, y_i : i = 1, 2, \dots, 2n\}$ and $|V(G)| = 6n$.

If n is odd, then it may be verified that $\beta_0(G) = 3n - 1$ and so

$$\beta_0(G) < \left\lfloor \frac{V(G)}{2} \right\rfloor = 3n.$$

Hence by Lemma 1.1, G is not a prime graph.

Now assume that n is even. We show that $f : V(G) \to \{1, 2, \dots, 6n\}$ defined by

$$f(v_i) = 3i - 2,\ f(u_i) = 3i,\ f(y_i) = 3i - 1$$

is a prime labeling on G. For all $i \neq n, 2n$;

$$\gcd(f(v_i), f(v_{i+1})) = \gcd(3i - 2, 3i + 1) = \gcd(3i - 2, 3) = 1,$$

whereas $\gcd(f(v_n), f(v_1)) = \gcd(3i-2, 1) = 1$ and $\gcd(f(v_{2n}), f(v_{n+1})) = \gcd(6n-2, 3n+1) = \gcd(3n-3, 3n+1) = \gcd(3n-3, 4) = 1$ because $3n-3$ is odd (since n is even). The remaining pairs of adjacent vertices have consecutive labels and so we are through. $\square$

Example 2.12. *Prime labeling of the graph obtained by taking subdivision of pendant edges of $(C_8 \odot K_1) \cup (C_8 \odot K_1)$ is shown in Figure 12*

Theorem 2.13. *Let G be the graph obtained by subdividing cycle edges in $(C_n \odot K_1) \cup (C_n \odot K_1)$. Then G is prime graph.*

Proof. Let $H = (C_n \odot K_1) \cup (C_n \odot K_1)$, whose vertex set and edge set are defined as in Theorem 2.12. Let G be the graph obtained from H by subdividing all its (cycle) edges v_iv_{i+1} with the help of the newly added vertices x_i for $i = 1, 2, \dots, 2n$. Thus $V(G) = \{u_i, v_i, x_i : i = 1, 2, \dots, 2n\}$ and $|V(G)| = 6n$.

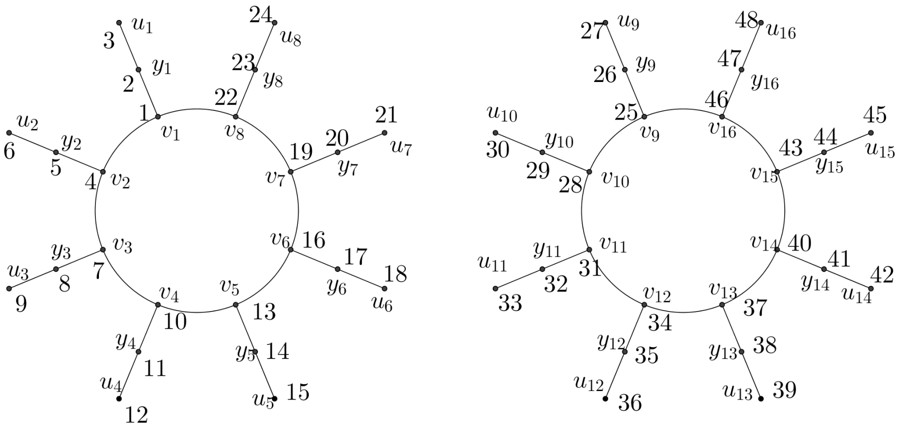

FIGURE 12

For $i = 1, 2, \ldots, 2n$, define $f : V(G) \to \{1, 2, \ldots, 6n\}$ as

$$f(x_i) = 3i,$$

$$f(v_i) = \begin{cases} 3i-1; & i \text{ is even}, \ i \neq n+1; \\ 3i-2; & i \text{ is odd}, \ i \neq 1, n+1, \end{cases}$$

$$f(u_i) = \begin{cases} 3i-1; & i \text{ is odd}, \ i \neq 1, n+1; \\ 3i-2; & i \text{ is even}. \ i \neq n+1, \end{cases}$$

and further define

$$f(v_{n+1}) = 1,$$
$$f(u_{n+1}) = 2,$$
$$f(v_1) = 3n+1,$$
$$f(u_1) = 3n+2.$$

Then it is not difficult to check that f is prime labeling on G. Thus G is a prime graph. □

Example 2.13. *Prime labeling of the graph obtained by taking subdivision of cycle edges of* $(C_8 \odot K_1) \cup (C_8 \odot K_1)$ *is shown in Figure 13.*

Theorem 2.14. *The barycentric subdivision of the union of two copies of crown graph* $C_n \odot K_1$ *is a prime graph for all* n.

Proof. Let $H = (C_n \odot K_1) \cup (C_n \odot K_1)$, whose vertex set and edge set are defined as in Theorem 2.12. Let G be the barycentric subdivision of H in which the edges $v_i v_{i+1}$ and $v_i u_i$ are subdivided with the help of the newly added vertices x_i and y_i respectively. Thus $V(G) = \{u_i, v_i, y_i, x_i : i = 1, 2, \ldots, 2n\}$ and $|V(G)| = 8n$.

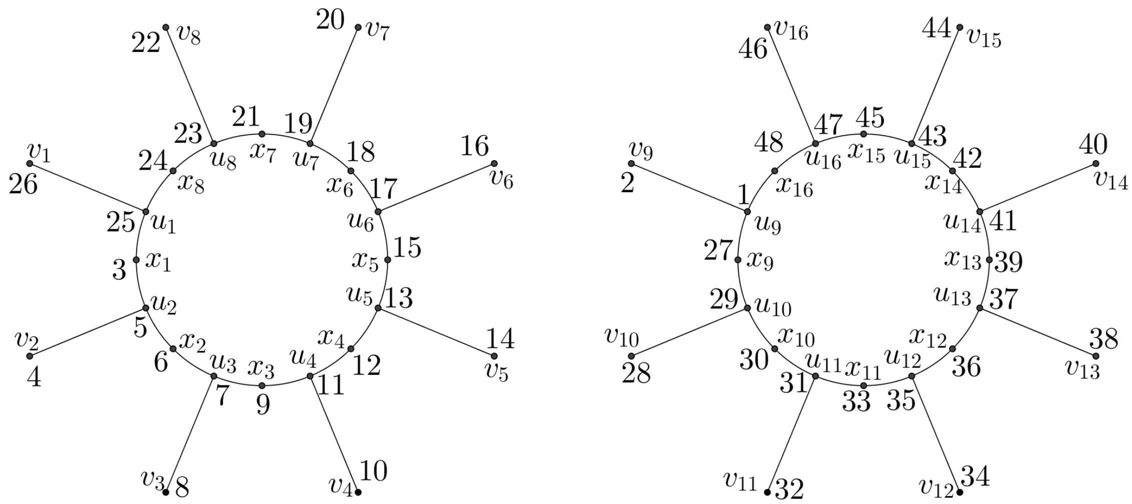

FIGURE 13

For $i = 1, 2, \ldots, 2n$, define $f : V(G) \to \{1, 2, , \ldots, 8n\}$ as

$$f(u_i) = \begin{cases} 4i, & \text{if } i \not\equiv 2 \pmod 3 \text{ and } i < 2n; \\ 4i-2, & \text{if } i \equiv 2 \pmod 3 \text{ and } i < 2n; \\ 8n, & \text{if } i = 2n, \end{cases}$$

$$f(v_i) = \begin{cases} 4i-2, & \text{if } i \not\equiv 2 \pmod 3 \text{ and } i < 2n; \\ 4i, & \text{if } i \equiv 2 \pmod 3 \text{ and } i < 2n \\ 8n-2, & \text{if } i = 2n, \end{cases}$$

$$f(x_i) = \begin{cases} 4i+1, & \text{if } i \neq n, 2n; \\ 1, & \text{if } i = n; \\ 4n+1, & \text{if } i = 2n, \end{cases}$$

$$f(y_i) = 4i - 1.$$

Then for $i = 1, 2, \ldots, n-1, n+1, \ldots, 2n-1$;

$$\gcd(f(x_i), f(v_i)) = \begin{cases} \gcd(4i+1, 4i-2) = 1, & \text{if } i \not\equiv 2 \pmod 3; \\ \gcd(4i+1, 4i) = 1, & \text{if } i \equiv 2 \pmod 3, \end{cases}$$

$$\gcd(f(x_i), f(v_{i+1})) = \begin{cases} \gcd(4i+1, 4i+2) = 1, & \text{if } i+1 \not\equiv 2 \pmod 3; \\ \gcd(4i+1, 4i+4) = 1, & \text{if } i+1 \equiv 2 \pmod 3. \end{cases}$$

Also, $\gcd(f(x_n), f(v_n)) = 1 = \gcd(f(x_n), f(v_1))$ as $f(x_n) = 1$ and

$$\gcd(f(x_{2n}), f(v_{2n})) = \begin{cases} \gcd(4n+1, 8n-2) = 1, & \text{if } n \not\equiv 1 \pmod 3; \\ \gcd(4n+1, 8n) = 1, & \text{if } n \equiv 1 \pmod 3, \end{cases}$$

$$\gcd(f(x_{2n}), f(v_{n+1})) = \begin{cases} \gcd(4n+1, 4n+2) = 1, & \text{if } n+1 \not\equiv 2 \pmod 3; \\ \gcd(4n+1, 4n+4) = 1, & \text{if } n+1 \equiv 2 \pmod 3. \end{cases}$$

Further, for $i = 1, 2, \ldots, 2n$; $\gcd(f(u_i), f(y_i)) = \gcd(f(v_i), f(y_i)) = 1$ as they are consecutive integers. Hence f is a prime labeling on G. □

Example 2.14. *Prime labeling of barycentric subdivision of* $(C_8 \odot K_1) \cup (C_8 \odot K_1)$ *is shown in Figure 14.*

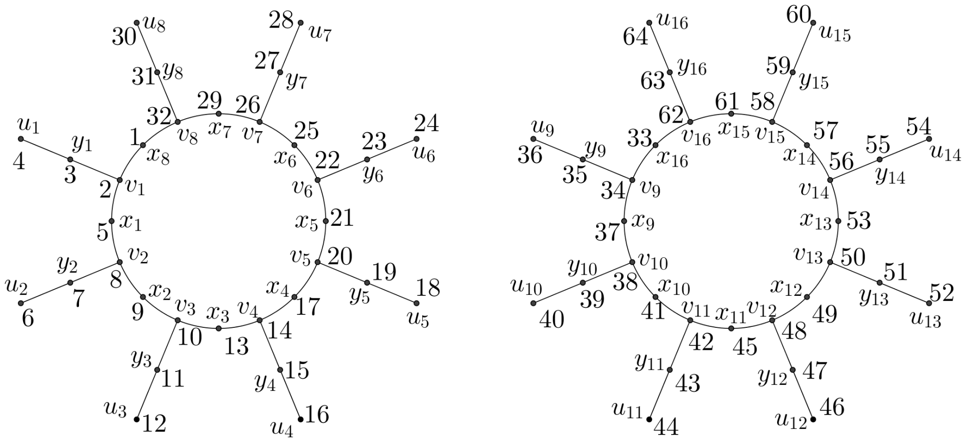

FIGURE 14

Note that the restriction of the function f to the first copy of crown graph in Theorem 2.14, gives a prime labeling of barycentric subdivision of crown graph.

3. Conclusions:

We have studied prime labeling in the context of various subdivision of cycle related graphs. There are many more graph families in which this type of investigation can be carried out.

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