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ON THE LOCAL CONVERGENCE OF WEERAKOON'S METHOD UNDER HÖLDER CONTINUITY CONDITION IN BANACH SPACES

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Abstract. In this manuscript, the study of local convergence analysis for the cubically convergent Weerakoon's method using Hölder continuity condition is presented to solve nonlinear equations in Banach spaces. Hölder continuity condition on the first derivative is assumed to extend the applicability of the method on such problems for which Lipschitz condition fails. This convergence analysis generalises the local convergence with Lipschitz continuity condition. A theorem showing existence and uniqueness of the solution with the error bounds is established. To verify our theoretical findings some numerical examples like Hammerstein integral equation and a system of nonlinear equations are solved.

Keywords: Banach space, Local convergence, Hölder continuity condition, Iterative methods.

AMS Subject Classification: 47H99, 65D10, 65D99, 65G49.

1. Introduction

The prime objective of the study presented in this manuscript is to find a locally unique solution x^* of the equation

$$F(x) = 0, \tag{1}$$

where $F : \Omega \subseteq X \to Y$ is a Fréchet differentiable function and Ω is a convex subset of X. X and Y are Banach spaces. In the field of applied science and engineering, a large number of problems can be solved by transforming them into nonlinear equations of the form (1). For instance, the boundary value problems occur in Kinetic theory of gases, the integral equations related to radiative transfer theory, problems in optimization and many others can be reduced to the problem of solving nonlinear equations. The most frequently used solution techniques are iterative in nature.

A widely known iterative procedure for solving (1) is Newton's algorithm, which can be expressed as:

$$x_{n+1} = x_n - [F'(x_n)]^{-1}F(x_n),\ n \geq 0. \tag{2}$$

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Evaluation of second and more order derivatives is a major drawback of higher-order iterative schemes and are not appropriate for practical use. Due to the calculation of F'' in each iteration the cubically convergent classical schemes are not suitable in terms of computational cost. Some classical third-order algorithms include Chebyshev's, the Halley's and Super-Halley's schemes are produced by putting ($\alpha = 0$), ($\alpha = \frac{1}{2}$) and ($\alpha = 1$) respectively in

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}(1 - \alpha H_F(x_n))^{-1} H_F(x_n)\right) [F'(x_n)]^{-1} F(x_n), \tag{3}$$

where $H_F(x_n) = F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n)$.
The local convergence analysis of many varieties of the methods defined in (3) has been studied by numerous authors in [5, 6, 7, 8]. In addition, the local convergence study of various iterative schemes is explored in Banach spaces in [10, 11, 12, 13, 14]. In this paper, we use the Hölder continuity condition only on the first derivative to generalize the local convergence analysis based on Lipschitz continuity condition and enhance the applicability of Weerakoon's method in Banach spaces. Basically, we have used the technique given by Argyros et al. [3] and the assumption that the first derivative is Hölder continuous.

In [18], the cubically convergent Weerakoon's method [17] is generalized to solve systems of nonlinear equations in $\mathbb{R}^n$. The method is given as:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= x_n - 2[F'(x_n) + F'(y_n)]^{-1}F(x_n) \end{aligned} \tag{4}$$

In this method, only the first-order derivative occurs in the iteration function but the convergence is proved with the assumption on third-order derivative for which the applicability of the method is restricted. In [3], the authors use Lipschitz condition on the first-order derivative. But there are numerous problems for which Lipschitz condition fails. For instance, consider the nonlinear integral equation [10] given by

$$F(x)(s) = x(s) - 3\int_0^1 G_1(s,t)x(t)^{\frac{5}{4}}dt,$$

where $x(s) \in C[0,1]$ and $G_1(s,t)$ is Green's function defined on $[0,1] \times [0,1]$ by

$$G_1(s,t) = \begin{cases} (1-s)t, & \text{if } t \le s \\ s(1-t), & \text{if } s \ge t \end{cases}.$$

Then,

$$||F'(x) - F'(y)|| \le \frac{15}{32}||x - y||^{\frac{1}{4}}$$

It is clear that Lipschitz condition does not hold for this problem. However, Hölder continuity condition holds on F' for $p = \frac{1}{4}$. In this paper, we provide the local convergence analysis of the method (4) using hypotheses only on F' to avoid the use of higher-order derivatives. Particularly, the Hölder continuity condition on the first derivative is assumed to extend the applicability of the method by generalizing the Lipschitz condition.

The rest part of this paper is arranged as follows: The local convergence analysis of the method (4) is placed in Section 2. Section 3 is devoted to demonstrating the applications of our theoretical outcomes on some numerical examples. Conclusions are discussed in the last section.

2. Local convergence analysis

This section deals with the local convergence analysis of Weerakoon's method defined in (4). Denote by $B(c, \rho)$ and $\bar{B}(c, \rho)$ the open and closed balls respectively with center

c and radius $\rho > 0$. Suppose $p \in (0,1]$, $k_0 > 0$ and $k > 0$ be given parameters with $k_0 \leq k$. Some functions and parameters are defined for the local convergence analysis of the method (4). Define the functions J_1 and K_1 on the interval $[0, (\frac{1}{k_0})^{\frac{1}{p}})$ by

$$J_1(s) = \frac{ks^p}{(p+1)(1-k_0s^p)}, \quad K_1(s) = J_1(s) - 1. \tag{5}$$

Now, $K_1(0) = -1 < 0$ and $\lim_{s \to ((\frac{1}{k_0})^{\frac{1}{p}})^-} K_1(t) = +\infty$. The intermediate value theorem confirms that the interval $(0, (\frac{1}{k_0})^{\frac{1}{p}})$ contains the zeros of the function $K_1(s)$. Let the smallest zero of $K_1(s)$ in $(0, (\frac{1}{k_0})^{\frac{1}{p}})$ is R_1. So, $J_1(R_1) = 1$ and $0 \leq J_1(s) < 1$ for $s \in [0, R_1)$. One can explicitly obtain $R_1 = \left(\frac{p+1}{(p+1)k_0+k}\right)^{\frac{1}{p}}$ using $J_1(R_1) = 1$. Again, define functions J_2 and K_2 on $[0, (\frac{1}{k_0})^{\frac{1}{p}})$ by

$$J_2(s) = \frac{k_0}{2}(1 + J_1(s)^p)s^p \tag{6}$$

and

$$K_2(s) = J_2(s) - 1.$$

Now, $K_2(0) = -1 < 0$ and $\lim_{s \to ((\frac{1}{k_0})^{\frac{1}{p}})^-} K_2(s) = +\infty$. According to the intermediate value theorem, the interval $(0, (\frac{1}{k_0})^{\frac{1}{p}})$ contains the zeros of the function $K_2(s)$. Let the smallest zero of $K_2(s)$ in $(0, (\frac{1}{k_0})^{\frac{1}{p}})$ is R_2. Again, define functions J_3 and K_3 on $[0, R_2)$ by

$$J_3(s) = \frac{k[\frac{2}{p+1} + J_1(s)^p]s^p}{2(1 - J_2(s))} \tag{7}$$

and

$$K_3(s) = J_3(s) - 1.$$

Now, $K_3(0) = -1 < 0$ and $\lim_{s \to R_2^-} K_3(s) = +\infty$. So, the interval $(0, R_2)$ contains the zeros of the function $K_3(s)$. Let the smallest zero of $K_3(s)$ in $(0, R_2)$ is R_3. Consider

$$R = min\{R_1, R_3\} \tag{8}$$

Now, we have

$$0 \leq J_1(s) < 1, \tag{9}$$

$$0 \leq J_2(s) < 1 \tag{10}$$

and

$$0 \leq J_3(s) < 1 \tag{11}$$

for each $s \in [0, R)$.

Next, the local convergence analysis of the method (4) is presented in Theorem 1.

Theorem 2.1. *Let $F : \Omega \subseteq X \to Y$ be a Fréchet differentiable function and $x^* \in \Omega$. Suppose there exist parameters $p \in (0,1]$, $k_0 > 0$ and $k > 0$ such that*

$$F(x^*) = 0, \ F'(x^*)^{-1} \in BL(Y, X), \tag{12}$$

$$||F'(x^*)^{-1}(F'(x) - F'(x^*))|| \leq k_0||x - x^*||^p, \ \forall x \in \Omega, \tag{13}$$

$$||F'(x^*)^{-1}(F'(x) - F'(y))|| \leq k||x - y||^p, \ \forall x, y \in \Omega \tag{14}$$

and

$$\bar{B}(x^*, R) \subseteq \Omega, \tag{15}$$

where R is defined in (8). Starting from $x_0 \in B(x^*, R)$ the method (4) generates the sequence of iterates $\{x_n\}$ which is well defined, $\{x_n\}_{n\geq 0} \in B(x^*, R)$ and converges to the solution x^* of (1). Moreover, the following estimations hold $\forall n \geq 0$

$$||y_n - x^*|| \leq J_1(||x_n - x^*||)||x_n - x^*|| < ||x_n - x^*|| < R, \tag{16}$$

$$||[F'(x_n) + F'(y_n)]^{-1}F'(x^*)|| \leq \frac{1}{2(1 - J_2(||x_n - x^*||))}, \tag{17}$$

and

$$||x_{n+1} - x^*|| \leq J_3(||x_n - x^*||)||x_n - x^*|| < ||x_n - x^*|| < R, \tag{18}$$

where the functions J_1, J_2 and J_3 are given in (5), (6) and (7) respectively. Furthermore, the solution x^* of the equation $F(x) = 0$ is unique in $\bar{B}(x^*, \Delta) \cap \Omega$, where $\Delta \in [R, (\frac{p+1}{k_0})^{\frac{1}{p}})$.

Proof. Using the definition of R, the equation (13) and the assumption $x_0 \in B(x^*, R)$, we obtain

$$||F'(x^*)^{-1}(F'(x_0) - F'(x^*))|| \leq k_0||x_0 - x^*||^p < k_0R^p < 1. \tag{19}$$

Now, we get $F'(x_0)^{-1} \in BL(Y, X)$ applying Banach Lemma on invertible functions [1, 2, 9, 15, 16]. Also,

$$||F'(x_0)^{-1}F'(x^*)|| \leq \frac{1}{1 - k_0||x_0 - x^*||^p} < \frac{1}{1 - k_0R^p}. \tag{20}$$

Hence, it follows from the first step of the method (4) for $n = 0$ that y_0 is well defined. Again,

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &= -\left[F'(x_0)^{-1}F'(x^*)\right]\left[\int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)d\theta\right]. \end{aligned} \tag{21}$$

The equations (5), (8), (9) and (14) are employed to produce

$$\begin{aligned} ||y_0 - x^*|| &\leq \left[||F'(x_0)^{-1}F'(x^*)||\right]\left[||\int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)d\theta||\right] \\ &\leq \frac{k||x_0 - x^*||^p}{(p+1)(1 - k_0||x_0 - x^*||^p)}||x_0 - x^*|| \\ &= J_1(||x_0 - x^*||)||x_0 - x^*|| < ||x_0 - x^*|| < R. \end{aligned} \tag{22}$$

and this shows (16) for $n = 0$. Then we show $[F'(x_0) + F'(y_0)]^{-1} \in BL(Y, X)$. We use (6), (10), (13) and (22) to obtain

$$\begin{aligned} &||(2F'(x^*))^{-1}(F'(x_0) + F'(y_0) - 2F'(x^*))|| \\ &\leq \frac{1}{2}[||F'(x^*)^{-1}(F'(x_0) - F'(x^*))|| + ||F'(x^*)^{-1}(F'(y_0) - F'(x^*))||] \\ &\leq \frac{k_0}{2}[||x_0 - x^*||^p + ||y_0 - x^*||^p] \\ &\leq \frac{k_0}{2}[||x_0 - x^*||^p + (J_1(||x_0 - x^*||)^p||x_0 - x^*||^p] \\ &= \frac{k_0}{2}[1 + J_1(||x_0 - x^*||)^p]||x_0 - x^*||^p \\ &= J_2(||x_0 - x^*||) < 1. \end{aligned}$$

Using Banach Lemma on invertible functions, we have $[F'(x_0) + F'(y_0)]^{-1} \in BL(Y, X)$ with

$$||[F'(x_0) + F'(y_0)]^{-1}F'(x^*)|| \leq \frac{1}{2(1 - J_2(||x_0 - x^*||))}. \tag{23}$$

Thus, we establish (17) for $n = 0$. Now, it follows from the last step of the method (4) for $n = 0$ that x_1 is well defined. Using the definition of R, (4), (7), (11), (14), (22) and (23), we get

$$\begin{aligned}
||x_1 - x^*|| &\leq \left(||[F'(x_0) + F'(y_0)]^{-1}F'(x^*)||\right)\Bigg(||\int_0^1 F'(x^*)^{-1}\left(F'(x_0) - F'(x^* + \theta(x_0 - x^*))\right)(x_0 - x^*)d\theta|| \\
&\quad + ||\int_0^1 F'(x^*)^{-1}\left(F'(y_0) - F'(x^* + \theta(x_0 - x^*))\right)(x_0 - x^*)d\theta||\Bigg) \\
&\leq \frac{\frac{k}{p+1}||x_0 - x^*||^{p+1} + k\int_0^1(||y_0 - x^* - \theta(x_0 - x^*)||^p)d\theta||x_0 - x^*||}{2(1 - J_2(||x_0 - x^*||))} \\
&\leq \frac{\frac{k}{p+1}||x_0 - x^*||^{p+1} + k(||y_0 - x^*||^p + \frac{||x_0 - x^*||^p}{p+1})||x_0 - x^*||}{2(1 - J_2(||x_0 - x^*||))} \\
&\leq \frac{\frac{k}{p+1}||x_0 - x^*||^{p+1} + k[J_1(||x_0 - x^*||)^p||x_0 - x^*||^p + \frac{||x_0 - x^*||^p}{p+1}]||x_0 - x^*||}{2(1 - J_2(||x_0 - x^*||))} \\
&\leq \frac{(\frac{2k}{p+1}||x_0 - x^*||^p + kJ_1(||x_0 - x^*||)^p||x_0 - x^*||^p)||x_0 - x^*||}{2(1 - J_2(||x_0 - x^*||))} \\
&= \frac{[(\frac{2k}{p+1} + kJ_1(||x_0 - x^*||)^p)||x_0 - x^*||^p]||x_0 - x^*||}{2(1 - J_2(||x_0 - x^*||))} \\
&= \frac{k[(\frac{2}{p+1} + J_1(||x_0 - x^*||)^p)||x_0 - x^*||^p]||x_0 - x^*||}{2(1 - J_2(||x_0 - x^*||))} \\
&= J_3(||x_0 - x^*||)||x_0 - x^*|| < ||x_0 - x^*|| < R.
\end{aligned}$$

Hence, we show the estimate (18) for $n = 0$. We get the estimates (16)-(18) by substituting x_n, y_n and x_{n+1} in place of x_0, y_0 and x_1 in the previous estimations. Using the fact $||x_{n+1} - x^*|| \leq J_3(R)||x_n - x^*|| < R$, we confirm that $x_{n+1} \in B(x^*, R)$ and $\lim_{n\to\infty} x_n = x^*$. Now, we want to show the uniqueness of the solution x^*. Suppose there exist another solution $y^*(\neq x^*)$ of $F(x) = 0$ in $B(x^*, \Delta)$. Consider $Q = \int_0^1 F'(y^* + \theta(x^* - y^*))d\theta$. From equation (13), we get

$$\begin{aligned}
||F'(x^*)^{-1}(Q - F'(x^*))|| &\leq \int_0^1 k_0||y^* + \theta(x^* - y^*) - x^*||^p d\theta \\
&\leq \frac{k_0}{p+1}||x^* - y^*||^p \\
&\leq \frac{k_0\Delta^p}{p+1} < 1.
\end{aligned}$$

Applying Banach Lemma, we find $Q^{-1} \in BL(Y, X)$. Now, Using the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, it is concluded that $x^* = y^*$. This ends the proof. □

3. Numerical Examples

In this section, numerical examples are provided to validate the theoretical results. We consider the Examples (1, 2 and 3) from the research paper of Argyros and George [6].

The examples 4 and 5 are selected from [10].
Example 1 Consider $S = \mathbb{R}$ and define F on $\Omega = [-\frac{1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

We have $x^* = 1$. Also, $p = 1$ and $k_0 = k = 146.6629$. The value of R is determined using the definitions of "J" functions.

Table 1: Radius of convergence for example 1

WM
$R_1 = 0.004545$
$R_2 = 0.005209$
$R_3 = 0.003994$
$R = 0.003994$

Example 2 Let us define F on $\Omega = [1, 3]$ by

$$F(x) = \frac{2}{3}x^{\frac{3}{2}} - x$$

We have $x^* = \frac{9}{4}$. Also, we have $p = 0.5$, $k_0 = 1$ and $k = 2$. R is determined using "J" functions.

Table 2: Radius of convergence for example 2

WM
$R_1 = 0.183673$
$R_2 = 0.505824$
$R_3 = 0.107877$
$R = 0.107877$

Example 3 Let F is defined on $\bar{B}(0, 1)$ for $(x_1, x_2, x_3)^t$ by

$$F(x) = (e^{x_1} - 1, \frac{e-1}{2}x_2^2 + x_2, x_3)^t$$

We have $x^* = (0, 0, 0)^t$. Also, we have $p = 1$, $k_0 = e - 1$ and $k = e$. We determine the value of R using "J" functions.

Table 3: Radius of convergence for example 3

WM
$R_1 = 0.324947$
$R_2 = 0.407903$
$R_3 = 0.268633$
$R = 0.268633$

Example 4 Consider the nonlinear Hammerstein type integral equation given by

$$F(x)(s) = x(s) - 5\int_0^1 stx(t)^{\frac{3}{2}} dt,$$

where $x(s) \in C[0, 1]$. We have $x^* = 0$. Also, $p = 0.5$ and $k_0 = k = \frac{15}{4}$. Using the definitions of "J" functions the value of R is determined.

Table 4: Radius of convergence for example 4

WM
$R_1 = 0.025599$
$R_2 = 0.043644$
$R_3 = 0.017992$
$R = 0.017992$

Example 5 Consider the nonlinear integral equation given by

$$F(x)(s) = x(s) - 3\int_0^1 G_1(s,t)x(t)^{\frac{5}{4}}dt,$$

where $x(s) \in C[0,1]$ and $G_1(s,t)$ is Green's function. We have $x^* = 0$. Also, $p = 0.25$ and $k_0 = k = \frac{15}{32}$. Using the definitions of "J" functions the value of R is determined.

Table 5: Radius of convergence for example 5

WM
$R_1 = 1.973080$
$R_2 = 9.863415$
$R_3 = 0.879329$
$R = 0.879329$

4. Conclusions

We studied the local convergence analysis of the method (4) to find a locally unique solution of a nonlinear equation in Banach spaces. The Hölder continuity condition on the first derivative is used to enhance the applicability of these methods. The theoretical outcomes are validated by solving numerical examples like Hammerstein equation and a system of nonlinear equations.

References

[1] Argyros, I. K., (2008), Convergence and Application of Newton-type Iterations, Springer.

[2] Argyros, I. K., Hilout, S., (2013), Computational methods in nonlinear Analysis, World Scientific Publ. House, New Jersey, USA.

[3] Argyros, I. K., Cho, Y. J., George, S., (2016), Local convergence for some third order iterative methods under weak conditions, J. Korean Math. Soc., 53 (4), pp. 781-793.

[4] Argyros, I. K., González, D., (2015), Local convergence for an improved Jarratt-type method in Banach space, Int. J. Interact. Multimed. Artif. Intell., 3(Special Issue on Teaching Mathematics Using New and Classic Tools), pp. 20-25.

[5] Argyros, I. K., Magreñán, Á. A., (2015), A study on the local convergence and the dynamics of Chebyshev–Halley–type methods free from second derivative, Numer. Algor., 71(1), pp. 1-23.

[6] Argyros, I. K., George, S., (2015), Local convergence of deformed Halley method in Banach space under Hölder continuity conditions, J. Nonlinear Sc. Appl., 8, pp/ 246-254.

[7] Argyros, I. K., George, S., Magreñán, Á. A, (2015), Local convergence for multi-point-parametric Chebyshev–Halley–type methods of higher convergence order, J. Comput. Appl. Math., 282, pp. 215-224.

[8] Argyros, I. K., George, S., (2015), Local convergence of modified Halley-like methods with less computation of inversion, Novi. Sad. J. Math., 45, pp. 47-58.

[9] Petković, M. S., Neta, B., Petković, L., Džunić, D., (2013), Multipoint methods for solving nonlinear equations, Elsevier.

[10] Singh, S., Gupta, D. K., Badoni, R. P., Martínez, E., Hueso, J. L., (2016), Local convergence of a parameter based iteration with Hölder continuous derivative in Banach spaces, Calcolo, DOI: 10.1007/s10092-016-0197-9.

[11] Martínez, E., Singh, S., Hueso, J. L., Gupta, D. K., (2016), Enlarging the convergence domain in local convergence studies for iterative methods in Banach spaces, Appl. Math. Comput., 281, pp. 252-265.

[12] Argyros, I. K., Hilout, S., (2013), On the local convergence of fast two-step Newton-like methods for solving nonlinear equations, J. Comput. Appl. Math., 245, pp. 1-9.

[13] Argyros, I. K., Khattri, S. K., (2015), Local convergence for a family of third order methods in Banach spaces, Appl. Math. Comput., 251, pp. 396-403.

[14] Cordero, A., Ezquerro, J. A., Hernández-Verón, M. A., (2014), On the local convergence of a fifth-order iterative method in Banach spaces, J. Math., 46, pp. 53-62.

[15] Rall, L. B., (1979), Computational solution of nonlinear operator equations, Robert E. Krieger, New York.

[16] Traub, J.F., (1964), Iterative Methods for Solution of Equations, Prentice-Hall, Englewood Cliffs.
[17] Weerakoon, S., Fernando, T. G. I., (2000), A variant of Newton's method with accelerated third-order convergence, Appl. Math. Lett., 13(8), pp.87-93.
[18] Nishani, H. P. S., Weerakoon, S., Fernando, T. G. I., Liyanag, M., (2018), Weerakoon-Fernando Method with accelerated third-order convergence for systems of nonlinear equations, International Journal of Mathematical Modelling and Numerical Optimisation, 8(3), pp. 287-304.

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