TWMS J. App. and Eng. Math. V.11, N.3, 2021, pp. 884-892

NEW INTUITIONISTIC FUZZY 2- NORMED I-CONVERGENT DOUBLE SEQUENCE SPACES BY BOUNDED LINEAR OPERATOR

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ABSTRACT. Saadati and Park [Saadati. R, and Park J. H., Chaos, Solitons Fractals 2006] [14] introduced the notion of intuitionistic fuzzy normed space. Motivated by this, we introduce some intuitionistic fuzzy 2-normed I-convergent double sequence spaces defined by bounded linear operator. Furthermore, we study some basic topological and algebraic properties of these spaces.

Keywords: Ideal, filter, I-convergence, intuitionistic fuzzy normed spaces, intuitionistic fuzzy 2-normed spaces, bounded linear operator.

AMS Subject Classification: 46A70, 54A20, 54A40.

1. Introduction and Preliminaries

The first publication of fuzzy sets theory was given by Zadeh [19] and Goguen [5] shows the intention of the authors to generalize the classical notion of a set and a proposition to accommodate fuzziness. The concept of fuzzy norm on a linear space was introduced by Cheng and Mordeson [2] and some properties of the fuzzy norm has been studied in [1]. On the other hand, the work provided by George et al. [6] played a big role in the development of fuzzy theory. Fuzzy topology is one of the most important and useful tools for dealing with such situations where the classical theories breaks down. The concept of intuitionistic fuzzy normed space [14] and intuitionistic fuzzy 2-normed space [12] are the latest developments in fuzzy topology. Certainly there are some situations where the ordinary norm does not work and the concept of intuitionistic fuzzy norm seems to be more suitable in such cases, that is, we can deal with such situations by modelling the inexactness of the norm in some situations.

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§ Manuscript received: April 30, 2019; accepted: October 13, 2019.

The notion of statistical convergence is a very useful functional tool for studying the convergence problems through the concept of density. Initially, as a generalization of statistical convergence [4], the concept of ideal convergence was introduced and studied by Kostyrko et al.[11] by using the idea I of subsets of the set of natural numbers $\mathbb{N}$. Later on, it was studied by Šalát et al.[15, 16], Tripathy and Hazarika [17, 18], Khan et al. [7, 9, 10] and many others. Quite recently, Das et al.[3] studied the notion of I and I^*- convergence of double sequences in $\mathbb{R}$. Initially, as a generalization of statistical convergence by Mursaleen et al.[13], the notion of I-convergence for double sequence in intuitionistic fuzzy normed spaces was studied by Khan et al. [8] and Mursaleen and Lohani [12] generalized I-convergence and I-Cauchy for sequence in intuitionistic fuzzy 2-normed spaces.

Definition 1.1: (See [12]) Let $I \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a non trivial ideal and $(X, \mu, \nu, *, \diamond)$ be an IF-2-NS. Then a sequence $x = (x_{ij})$ is said to be I-convergent to $L \in X$ with respect to the intuitionistic fuzzy norm $(\mu, \nu)_2$, if for every $\epsilon > 0$ and $t > 0$, the set

$$\{(i,j) : \mu(x_{ij} - L, z; t) \leq 1 - \epsilon \text{ or } \nu(x_{ij} - L, t) \geq \epsilon\} \in I.$$

In this case L is called the I-limit of the sequence (x_{ij}) with respect to the intuitionistic fuzzy norm $(\mu, \nu)_2$ and we write $I_{(\mu,\nu)_2} - \lim x_{ij} = L$.

Definition 1.2: (See [12]) Let $(X, \mu, \nu, *, \diamond)$ be an IF-2-NS. Then a sequence $x = (x_{ij})$ is said to be a I-Cauchy sequence with respect to the intuitionistic fuzzy norm $(\mu, \nu)_2$, if for every $\epsilon > 0$ and $t > 0$, the set

$$\{(i,j) : \mu(x_{ij} - x_{mn}, z; t) \leq 1 - \epsilon \text{ or } \nu(x_{ij} - x_{mn}, z; t) \geq \epsilon\} \in I.$$

This paper explores the work of Zadeh. In this article, the most important generalization is the consideration of bounded linear operator. The significance of this work may lie more in its point of view than in any particular results. Our aim for this paper is to launch the concept of intuitionistic fuzzy 2-normed double sequence space defined by bounded linear operator which would provide a more suitable framework to deal with the inexactness of the norm or 2-norm in some situations. We prove here some topological results in this new set up.

2. Mean Results

In this section, we introduce the following double sequence spaces:

$${}_2S^I_{(\mu,\nu)_2}(B) = \{(x_{ij}) \in {}_2\ell_\infty : \{(i,j) : \mu(B(x_{ij}) - L, y; t) \leq 1 - \epsilon \text{ or } \nu(B(x_{ij}) - L, y; t) \geq \epsilon\} \in I\};$$

$${}_2S^I_{0(\mu,\nu)_2}(B) = \{(x_{ij}) \in {}_2\ell_\infty : \{(i,j) : \mu(B(x_{ij}), y; t) \leq 1 - \epsilon \text{ or } \nu(B(x_{ij}), y; t) \geq \epsilon\} \in I\}.$$

Let $x \in X$, $r \in (0,1)$ and for all $t > 0$, then the set

$${}_2\mathcal{B}_x(r,t)(B) = \{(y_{ij}) \in {}_2\ell_\infty : \{(i,j) : \mu(B(x_{ij}) - B(y_{ij}), z; t) > 1 - r \text{ or } \nu(B(x_{ij}) - B(y_{ij}), z; t) < r\} \in I\}$$

is called the open ball with center x and radius r with respect to t.

Theorem 3.1 If a sequence $x = (x_{ij}) \in {}_2S^I_{(\mu,\nu)_2}(B)$ and ${}_2S^I_{0(\mu,\nu)_2}(B)$ is I- convergent with respect to the intuitionistic fuzzy 2-norm $(\mu, \nu)_2$, then limit is unique.

Proof: Suppose that $I_{(\mu,\nu)_2} - \lim x = L_1$ and $I_{(\mu,\nu)_2} - \lim x = L_2$. Given $\epsilon > 0$, choose $r > 0$ such that $(1 - r) * (1 - r) > 1 - \epsilon$ and $r \diamond r < \epsilon$. Then, for any $t > 0$, define the following sets as:

$${}_2K_{(\mu,1)_2}(r,t)(B) = \{(i,j) : \mu(B(x_{ij} - L_1), z; t/2) \leq 1 - r\},$$

$$
\begin{aligned}
{}_2K_{(\mu,2)_2}(r,t)(B) &= \{(i,j) : \mu(B(x_{ij} - L_2), z; t/2) \le 1 - r\},\\
{}_2K_{(\nu,1)_2}(r,t)(B) &= \{(i,j) : \nu(B(x_{ij} - L_1), z; t/2) \ge r\},\\
{}_2K_{(\nu,2)_2}(r,t)(B) &= \{(i,j) : \nu(B(x_{ij} - L_2), z; t/2) \ge r\}.
\end{aligned}
$$

Since $I_{(\mu,\nu)_2} - \lim x = L_1$, we have

$$ {}_2K_{(\mu,1)_2}(r,t)(B) \ \text{ and } {}_2K_{(\nu,1)_2}(r,t)(B) \in I. $$

Furthermore, using $I_{(\mu,\nu)_2} - \lim x = L_2$, we get

$$ {}_2K_{(\mu,2)_2}(r,t)(B) \ \text{ and } {}_2K_{(\nu,2)_2}(r,t)(B) \in I. $$

Now, let

$$ {}_2K_{(\mu,\nu)_2}(r,t)(B) = ({}_2K_{(\mu,1)_2}(r,t)(B) \cup {}_2K_{(\mu,2)_2}(r,t)(B)) \cap ({}_2K_{(\nu,1)_2}(r,t)(B) \cup {}_2K_{(\nu,2)_2}(r,t)(B)) \in I, $$

then we see that ${}_2K_{(\mu,\nu)_2}(r,t)(B) \in I$. This implies that its complement ${}_2K^c_{(\mu,\nu)}(r,t)(B) \in \mathcal{F}(I)$. If $(i,j) \in {}_2K^c_{(\mu,\nu)}(r,t)(B)$, then we have two possible cases. That is, $(i,j) \in {}_2K^c_{(\mu,1)_2}(r,t)(B) \cap {}_2K^c_{(\mu,2)_2}(r,t)(B)$ or $(i,j) \in {}_2K^c_{(\nu,1)_2}(r,t)(B) \cap {}_2K^c_{(\nu,2)_2}(r,t)(B)$. We first consider that $(i,j) \in {}_2K^c_{(\mu,1)_2}(r,t)(B) \cap {}_2K^c_{(\mu,2)_2}(r,t)(B)$. Then, we have

$$ \mu(B(L_1 - L_2), z; t) \ge \mu(B(x_{ij} - L_1), z; t/2) * \mu(B(x_{ij} - L_2), z; t/2) > (1-r) * (1-r) > 1 - \epsilon. $$

Since $\epsilon > 0$ was arbitrary, we get $\mu(B(L_1 - L_2), z; t) = 1$ for all $t > 0$, which yields $L_1 = L_2$, since $\mu(x, z; t) > 0$ for all $t > 0$ and B is a bounded linear operator. On the other hand, if $(i,j) \in {}_2K^c_{(\nu,1)_2}(r,t)(B) \cap {}_2K^c_{(\nu,2)_2}(r,t)(B)$, then we may write that

$$ \nu(B(L_1 - L_2), z; t) \le \nu(B(x_{ij} - L_1), z; t/2) \diamond \nu(B(x_{ij} - L_2), z; t/2) < r \diamond r < \epsilon. $$

Therefore, we have $\nu(B(L_1 - L_2), z; t) = 0$, for all $t > 0$, which implies that $L_1 = L_2$, since $\nu(x, z; t) > 0$ for all $t > 0$ and B is a bounded linear operator. Therefore, in all cases, we conclude that the limit is unique. This completes the proof of the theorem.

Theorem 3.2 Let $\chi(B)$ be an IF-2-NS defined by bounded linear operator B and I be an admissible ideal. Then
(i) if $I_{(\mu,\nu)_2} - \lim x_{ij} = L_1$ and $I_{(\mu,\nu)_2} - \lim y_{ij} = L_2$, then $I_{(\mu,\nu)_2} - \lim(x_{ij} + y_{ij}) = L_1 + L_2$
(ii) if $I_{(\mu,\nu)_2} - \lim x_{ij} = L$ then $I_{(\mu,\nu)_2} - \lim \alpha x_{ij} = \alpha L$
where α is a scalar and $\chi = {}_2S^I_{(\mu,\nu)_2}$ and ${}_2S^I_{0(\mu,\nu)_2}$.

Proof: Let $I_{(\mu,\nu)_2} - \lim x_{ij} = L_1$ and $I_{(\mu,\nu)_2} - \lim y_{ij} = L_2$. For a given $\epsilon > 0$, choose $r > 0$ such that $(1-r) * (1-r) > 1 - \epsilon$ and $r \diamond r < \epsilon$. Then for any $t > 0$, define the following sets as:

$$
\begin{aligned}
{}_2K_{(\mu,1)_2}(r,t)(B) &= \{(i,j) : \mu(B(x_{ij} - L_1), z; t/2) \le 1 - r\},\\
{}_2K_{(\mu,2)_2}(r,t)(B) &= \{(i,j) : \mu(B(y_{ij} - L_2), z; t/2) \le 1 - r\},\\
{}_2K_{(\nu,1)_2}(r,t)(B) &= \{(i,j) : \nu(B(x_{ij} - L_1), z; t/2) \ge r\},\\
{}_2K_{(\nu,2)_2}(r,t)(B) &= \{(i,j) : \nu(B(y_{ij} - L_2), z; t/2) \ge r\}.
\end{aligned}
$$

Since, $I_{(\mu,\nu)_2} - \lim x_{ij} = L_1$, we have

$$ {}_2K_{(\mu,1)_2}(r,t)(B) \text{ and } {}_2K_{(\nu,1)_2}(r,t)(B) \in I. $$

Furthermore, using $I_{(\mu,\nu)_2} - \lim y_{ij} = L_2$, we get

$$ {}_2K_{(\mu,2)_2}(r,t)(B) \text{ and } {}_2K_{(\nu,2)_2}(r,t)(B) \in I. $$

Now, let

$${}_2K_{(\mu,\nu)_2}(r,t)(B) = \Big({}_2K_{(\mu,1)_2}(r,t)(B) \cup {}_2K_{(\mu,2)_2}(r,t)(B)\Big) \bigcap \Big({}_2K_{(\nu,1)_2}(r,t)(B) \cup {}_2K_{(\nu,2)_2}(r,t)(B)\Big) \in I$$

which implies that ${}_2K^c_{(\mu,\nu)_2}(r,t)(B)$ is a non empty set in $F(I)$. Now we have to show that

$${}_2K^c_{(\mu,\nu)_2}(r,t)(B) \subset \Big\{(i,j) : \mu(B(x_{ij}+y_{ij}) - B(L_1+L_2), z; t) > 1-\epsilon \text{ and } \nu(B(x_{ij}+y_{ij}) - B(L_1+L_2), z; t) < \epsilon\Big\}.$$

If $(i,j) \in {}_2K^c_{(\mu,\nu)_2}(r,t)(B)$, then we have $\mu(B(x_{ij}-L_1), z; t/2) > 1-r$, $\mu(B(y_{ij}-L_2), z; t/2) > 1-r, \nu(B(x_{ij}-L_1), z; t/2) < r$ and $\nu(B(y_{ij}-L_2), z; t/2) < r$. Therefore

$$\begin{aligned}\mu(B(x_{ij}+y_{ij}) - B(L_1+L_2), z; t) &\geq \mu(B(x_{ij}-L_1), z; t/2) * \mu(B(y_{ij}-L_2), z; t/2)\\ &> (1-r)*(1-r) > 1-\epsilon\end{aligned}$$

and
$$\begin{aligned}\nu(B(x_{ij}+y_{ij}) - B(L_1+L_2), z; t) &\leq \nu(B(x_{ij}-L_1), z; t/2) \diamond \nu(B(y_{ij}-L_2), z; t/2)\\ &< r \diamond r < \epsilon.\end{aligned}$$

This shows that

$${}_2K^c_{(\mu,\nu)_2}(r,t)(B) \subset \Big\{(i,j) : \mu(B(x_{ij}+y_{ij}) - B(L_1+L_2), z; t) > 1-\epsilon \text{ and } \nu(B(x_{ij}+y_{ij}) - B(L_1+L_2), z; t) < \epsilon\Big\}.$$

Since ${}_2K^c_{(\mu,\nu)_2}(r,t)(B) \in \mathcal{F}(I)$. So we have $I_{(\mu,\nu)_2} - \lim(x_{ij}+y_{ij}) = L_1 + L_2$. Since B is a bounded linear operator.

(ii) This is obvious for $\alpha = 0$. Now let $\alpha \neq 0$. Then for a given $\epsilon > 0$ and $t > 0$,

$$B(\epsilon) = \Big\{(i,j) : \mu(B(x_{ij}-L), z; t) > 1-\epsilon \text{ and } \nu(B(x_{ij}-L), z; t) < \epsilon\Big\} \in \mathcal{F}(I).$$

It is sufficient to prove that, for each $\epsilon > 0$ and $t > 0$,

$$B(\epsilon) \subset \Big\{(i,j) : \mu(B(\alpha x_{ij} - \alpha L), z; t) > 1-\epsilon \text{ and } \nu(B(\alpha x_{ij} - \alpha L), z; t) < \epsilon\Big\}. \quad (1)$$

Let $(i,j) \in B(\epsilon)$. Then we have

$$\mu(B(x_{ij}-L), z; t) > 1-\epsilon \text{ and } \nu(B(x_{ij}-L), z; t) < \epsilon.$$

So, we have

$$\begin{aligned}\mu(B(\alpha x_{ij} - \alpha L), z; t) &= \mu(B(x_{ij}-L), z; \tfrac{t}{|\alpha|}) \geq \mu(B(x_{ij}-L), z; t) * \mu(0, z; \tfrac{t}{|\alpha|} - t)\\ &= \mu(B(x_{ij}-L), z; t) * 1 = \mu(B(x_{ij}-L), z; t) > 1-\epsilon.\end{aligned}$$

Furthermore,

$$\begin{aligned}\nu(B(\alpha x_{ij} - \alpha L), z; t) &= \nu(B(x_{ij}-L), z; \tfrac{t}{|\alpha|}) \leq \mu(B(x_{ij}-L), z; t) \diamond \nu(0, z; \tfrac{t}{|\alpha|} - t)\\ &= \nu(B(x_{ij}-L), z; t) \diamond 0 = \nu(B(x_{ij}-L), z; t) < \epsilon.\end{aligned}$$

Hence, we obtain

$$B(\epsilon) \subset \Big\{(i,j) : \mu(B(\alpha x_{ij} - \alpha L), z; t) > 1-\epsilon \text{ and } \nu(B(\alpha x_{ij} - \alpha L), z; t) < \epsilon\Big\},$$

and from (3.1) we conclude that $I_{(\mu,\nu)_2} - \lim \alpha x_{ij} = \alpha L$.
This complete the proof of the theorem.

Theorem 3.3: ${}_2S^I_{(\mu,\nu)_2}(B)$ and ${}_2S^I_{0(\mu,\nu)_2}(B)$ are linear spaces.

Proof: We shall prove for space ${}_2S^I_{(\mu,\nu)_2}(B)$. On a similar way, we can prove the other space. Let $x = (x_{ij}), y = (y_{ij}) \in {}_2S^I_{(\mu,\nu)_2}(B)$ and α, β be scalars. Then for a given $\epsilon > 0$, we have

$$A_1 = \Big\{(i,j) : \mu\Big(B(x_{ij}) - L_1, z; \tfrac{t}{2|\alpha|}\Big) \leq 1-\epsilon \text{ or } \nu\Big(B(x_{ij}) - L_1, z; \tfrac{t}{2|\alpha|}\Big) \geq \epsilon\Big\} \in I;$$

$$A_2 = \Big\{(i,j) : \mu\Big(B(y_{ij}) - L_2, z; \tfrac{t}{2|\beta|}\Big) \leq 1-\epsilon \text{ or } \nu\Big(B(y_{ij}) - L_2, z; \tfrac{t}{2|\beta|}\Big) \geq \epsilon\Big\} \in I.$$

$$A_1^c = \Big\{(i,j) : \mu\Big(B(x_{ij}) - L_1, z; \tfrac{t}{2|\alpha|}\Big) > 1-\epsilon \text{ or } \nu\Big(B(x_{ij}) - L_1, z; \tfrac{t}{2|\alpha|}\Big) < \epsilon\Big\} \in \mathcal{F}(I);$$

$$A_2^c = \Big\{(i,j) : \mu\Big(B(y_{ij}) - L_2, z; \tfrac{t}{2|\beta|}\Big) > 1-\epsilon \text{ or } \nu\Big(B(y_{ij}) - L_2, z; \tfrac{t}{2|\beta|}\Big) < \epsilon\Big\} \in \mathcal{F}(I).$$

Define the set $A_3 = A_1 \cup A_2$, so that $A_3 \in I$. It follows that A_3^c is a non-empty set in $\mathcal{F}(I)$. We shall show that for each $(x_{ij}), (y_{ij}) \in {}_2S^I_{(\mu,\nu)_2}(B)$.

$$A_3^c \subset \{(i,j) : \mu((\alpha B(x_{ij}) + \beta B(y_{ij})) - (\alpha L_1 + \beta L_2), z; t) > 1-\epsilon$$
$$\text{or } \nu((\alpha B(x_{ij}) + \beta B(y_{ij})) - (\alpha L_1 + \beta L_2), z; t) < \epsilon\}.$$

Let $(m,n) \in A_3^c$. In this case

$$\mu\Big(B(x_{mn}) - L_1, z; \frac{t}{2\mid\alpha\mid}\Big) > 1-\epsilon \text{ or } \nu\Big(B(x_{mn}) - L_1, z; \frac{t}{2\mid\alpha\mid}\Big) < \epsilon$$

and

$$\mu\Big(B(y_{mn}) - L_2, z; \frac{t}{2\mid\beta\mid}\Big) > 1-\epsilon \text{ or } \nu\Big(B(y_{mn}) - L_2, z; \frac{t}{2\mid\beta\mid}\Big) < \epsilon.$$

We have

$$\mu\Big((\alpha B(x_{mn}) + \beta B(y_{mn})) - (\alpha L_1 + \beta L_2), z; t\Big) \geq \mu\Big(\alpha B(x_{mn}) - \alpha L_1, z; \tfrac{t}{2}\Big) * \mu\Big(\beta B(x_{mn}) - \beta L_2, z; \tfrac{t}{2}\Big)$$
$$= \mu\Big(B(x_{mn}) - L_1, z; \frac{t}{2\mid\alpha\mid}\Big) * \mu\Big(B(x_{mn}) - L_2, z; \frac{t}{2\mid\beta\mid}\Big)$$
$$> (1-\epsilon) * (1-\epsilon) = 1-\epsilon.$$

and

$$\nu\Big((\alpha B(x_{mn}) + \beta B(y_{mn})) - (\alpha L_1 + \beta L_2), z; t\Big) \leq \nu\Big(\alpha B(x_{mn}) - \alpha L_1, z; \tfrac{t}{2}\Big) \diamond \nu\Big(\beta B(x_{mn}) - \beta L_2, z; \tfrac{t}{2}\Big)$$
$$= \mu\Big(B(x_{mn}) - L_1, z; \frac{t}{2\mid\alpha\mid}\Big) \diamond \mu\Big(B(x_{mn}) - L_2, z; \frac{t}{2\mid\beta\mid}\Big)$$
$$< \epsilon \diamond \epsilon = \epsilon.$$

This implies that

$$A_3^c \subset \Big\{(i,j) : \mu((\alpha B(x_{ij}) + \beta B(y_{ij})) - (\alpha L_1 + \beta L_2), z; t) > 1-\epsilon$$
$$\text{or } \nu((\alpha B(x_{ij}) + \beta B(y_{ij})) - (\alpha L_1 + \beta L_2), z; t) < \epsilon\Big\}.$$

Hence ${}_2S^I_{(\mu,\nu)_2}(B)$ is a linear space.

Theorem 3.4: Every open ball ${}_2\mathcal{B}_x(r,t)(B)$ is an open set in ${}_2S^I_{(\mu,\nu)_2}(B)$.
Proof: Let ${}_2\mathcal{B}_x(r,t)(B)$ be an open ball with centre x and radius r with respect to t. That is

$${}_2\mathcal{B}_x(r,t)(B) = \Big\{y = (y_{ij}) \in {}_2\ell_\infty : \{(i,j) : \mu(B(x_{ij}) - B(y_{ij}), z; t) > 1-r$$
$$\text{or } \nu(B(x_{ij}) - B(y_{ij}), z; t) < r\} \in I\Big\}.$$

Let $y \in {}_2\mathcal{B}_x(r,t)(B)$, then $\mu(B(x_{ij}) - B(y_{ij}), z; t) > 1 - r$ and $\nu(B(x_{ij}) - B(y_{ij}), z; t) < r$. Since $\mu(B(x_{ij}) - B(y_{ij}), z; t) > 1 - r$, there exists $t_0 \in (0,t)$ such that $\mu(B(x_{ij}) - B(y_{ij}), z; t_0) > 1 - r$ and $\nu(B(x_{ij}) - B(y_{ij}), z; t_0) < r$. Putting $r_0 = \mu(B(x_{ij}) - B(y_{ij}), t_0)$, we have $r_0 > 1 - r$, there exists $s \in (0,1)$ such that $r_0 > 1 - s > 1 - r$.
For $r_0 > 1 - s$, we have $r_1, r_2 \in (0,1)$ such that $r_0 * r_1 > 1 - s$ and $(1 - r_0) \diamond (1 - r_0) \leq s$. Putting $r_3 = \max\{r_1, r_2\}$. Consider the ball ${}_2\mathcal{B}_y(1 - r_3, t - t_0)(B)$. We prove that

$$ {}_2\mathcal{B}_y(1 - r_3, t - t_0)(B) \supset {}_2\mathcal{B}_x(r,t)(B). $$

Let $w = (w_{ij}) \in {}_2\mathcal{B}_y(1 - r_3, t - t_0)(B)$, then $\mu(B(y_{ij}) - B(w_{ij}), z; t - t_0) > r_3$ and $\nu(B(y_{ij}) - B(w_{ij}), z; t - t_0) < 1 - r_3$.
Therefore

$$ \begin{aligned} \mu(B(x_{ij}) - B(w_{ij}), z; t) &\geq \mu(B(x_{ij}) - B(y_{ij}), z; t_0) * \mu(B(y_{ij}) - B(w_{ij}), z; t - t_0) \\ &\geq (r_0 * r_3) \geq (r_0 * r_1) \geq (1 - s) \geq (1 - r) \end{aligned} $$

and

$$ \begin{aligned} \nu(B(x_{ij}) - B(w_{ij}), z; t) &\leq \nu(B(x_{ij}) - B(y_{ij}), z; t_0) \diamond \nu(B(y_{ij}) - B(w_{ij}), z; t - t_0) \\ &\leq (1 - r_0) \diamond (1 - r_3) \leq (1 - r_0) \diamond (1 - r_2) \leq s \leq r. \end{aligned} $$

Thus $w = (w_{ij}) \in {}_2\mathcal{B}_x(r,t)(B)$ and hence

$$ {}_2\mathcal{B}_y(1 - r_3, t - t_0)(B) \subset {}_2\mathcal{B}_x(r,t)(B). $$

Remark 3.5 ${}_2S^I_{(\mu,\nu)_2}(B)$ is an IF-2-NS.
Define

$$ {}_2\tau^I_{(\mu,\nu)_2}(B) = \Big\{ A \subset {}_2S^I_{(\mu,\nu)_2}(B) : \text{ for each } x \in A \text{ there exists } t > 0 \\ \text{and } r \in (0,1) \text{ such that } {}_2\mathcal{B}_x(r,t)(B) \subset A \Big\}. $$

Then, ${}_2\tau^I_{(\mu,\nu)_2}(B)$ is a topology on ${}_2S^I_{(\mu,\nu)_2}(B)$.

Theorem 3.6: ${}_2S^I_{(\mu,\nu)_2}(B)$ and ${}_2S^I_{0(\mu,\nu)_2}(B)$ are Hausdorff spaces.
Proof: We prove the result for ${}_2S^I_{(\mu,\nu)_2}(B)$. Similarly the proof follows for ${}_2S^I_{0(\mu,\nu)_2}(B)$. Let $x, y \in {}_2S^I_{(\mu,\nu)_2}(B)$ such that $x \neq y$. Then $0 < \mu(B(x) - B(y), z; t) < 1$ and $0 < \nu(B(x) - B(y), z; t) < 1$. Putting $r_1 = \mu(B(x) - B(y), z; t)$, $r_2 = \nu(B(x) - B(y), z; t)$ and $r = \max\{r_1, 1 - r_2\}$.
For each $r_0 \in (r, 1)$ there exists r_3 and r_4 such that $r_3 * r_4 \geq r_0$ and $(1 - r_3) \diamond (1 - r_4) \leq (1 - r_0)$. Putting $r_5 = \max\{r_3, r_4\}$ and consider the open balls ${}_2\mathcal{B}_x(1 - r_5, \frac{t}{2})$ and ${}_2\mathcal{B}_y(1 - r_5, \frac{t}{2})$. Then clearly ${}_2\mathcal{B}^c_x(1 - r_5, \frac{t}{2}) \cap {}_2\mathcal{B}^c_y(1 - r_5, \frac{t}{2}) = \emptyset$. For if there exists $w \in {}_2\mathcal{B}^c_x(1 - r_5, \frac{t}{2}) \cap {}_2\mathcal{B}^c_y(1 - r_5, \frac{t}{2})$, then

$$ r_1 = \mu(B(x) - B(y), z; t) \geq \mu(B(x) - B(z), z; \frac{t}{2}) * \mu(B(z) - B(y), z; \frac{t}{2}) \geq r_5 * r_5 \geq r_3 * r_3 \geq r_0 > r_1 $$

and

$$ \begin{aligned} r_2 = \nu(B(x) - B(y), z; t) &\leq \nu(B(x) - B(z), z; \frac{t}{2}) \diamond \nu(B(z) - B(y), z; \frac{t}{2}) \leq (1 - r_5) \diamond (1 - r_5) \\ &\leq (1 - r_4) \diamond (1 - r_4) \leq (1 - r_0) < r_2 \end{aligned} $$

which is a contradiction. Hence ${}_2S^I_{(\mu,\nu)_2}(B)$ is Haussdorff.

Theorem 3.7: ${}_2S^I_{(\mu,\nu)_2}(B)$ is an IFNS and ${}_2\tau^I_{(\mu,\nu)_2}(B)$ is a topology on ${}_2S^I_{(\mu,\nu)_2}(B)$. Then a sequence $(x_{ij}) \in {}_2S^I_{(\mu,\nu)_2}(B), x_{ij} \to x$ if and only if $\mu(B(x_{ij}) - B(x), z; t) \to 1$ and $\nu(B(x_{ij}) - B(x), z; t) \to 0$ as $i, j \to \infty$.

Proof: Fix $t_0 > 0$. Suppose $x_{ij} \to x$. Then for $r \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $(x_{ij}) \in {}_2\mathcal{B}_x(r, t)(B)$ for all $i, j \geq n_0$,

$${}_2\mathcal{B}_x(r, t)(B) = \{(i, j) : \mu(B(x_{ij}) - B(x), z; t) \leq 1 - r \text{ or } \nu(B(x_{ij}) - B(x), z; t) \geq r\} \in I,$$

such that ${}_2\mathcal{B}^c_x(r, t)(B) \in \mathcal{F}(I)$. Then $1 - \mu(B(x_{ij}) - B(x), z; t) < r$ and $\nu(B(x_{ij}) - B(x), z; t) < r$.
Hence $\mu(B(x_{ij}) - B(x), z; t) \to 1$ and $\nu(B(x_{ij}) - B(x), z; t) \to 0$ as $i, j \to \infty$.
Conversely, if for each $t > 0, \mu(B(x_{ij}) - B(x), z; t) \to 1$ and $\nu(B(x_{ij}) - B(x), z; t) \to 0$ as $i, j \to \infty$, then for $r \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $1 - \mu(B(x_{ij}) - B(x), z; t) < r$ and $\nu(B(x_{ij}) - B(x), z; t) < r$, for all $i, j \geq n_0$. It follows that $\mu(B(x_{ij}) - B(x), z; t) > 1 - r$ and $\nu(B(x_{ij}) - B(x), z; t) < r$ for all $i, j \geq n_0$. Thus $(x_{ij}) \in {}_2\mathcal{B}^c_x(r, t)(B)$ for all $i, j \geq n_0$ and hence $x_{ij} \to x$.

Theorem 3.8: A sequence $x = (x_{ij}) \in {}_2S^I_{(\mu,\nu)_2}(B)$ is I-convergent if and only if for every $\epsilon > 0$ and $t > 0$ there exists a number $M = M(x, \epsilon, t), N = N(x, \epsilon, t)$ such that

$$\{(M, N) : \mu(B(x_{MN}) - L, z; \frac{t}{2}) > 1 - \epsilon \text{ or } \nu(B(x_{MN}) - L, z; \frac{t}{2}) < \epsilon\} \in \mathcal{F}(I).$$

Proof: Suppose that $I_{(\mu,\nu)_2} - \lim x = L$ and let $\epsilon > 0$ and $t > 0$. For a given $\epsilon > 0$, choose $s > 0$ such that $(1-\epsilon) * (1-\epsilon) > 1-s$ and $\epsilon \diamond \epsilon < s$. Then for each $x = (x_{ij}) \in {}_2S^I_{(\mu,\nu)_2}(B)$,

$$P = \{(i, j) : \mu(B(x_{ij}) - L, z; \frac{t}{2}) \leq 1 - \epsilon \text{ or } \nu(B(x_{ij}) - L, z; \frac{t}{2}) \geq \epsilon\} \in I,$$

which implies that

$$P^c = \{(i, j) : \mu(B(x_{ij}) - L, z; \frac{t}{2}) > 1 - \epsilon \text{ or } \nu(B(x_{ij}) - L, z; \frac{t}{2}) < \epsilon\} \in \mathcal{F}(I).$$

Conversely let us choose $(M, N) \in P$. Then

$$\mu(B(x_{MN}) - L, z; \frac{t}{2}) > 1 - \epsilon \text{ or } \nu(B(x_{MN}) - L, z; \frac{t}{2}) < \epsilon.$$

Now we want to show that there exists a number $M = (x, \epsilon, t), N = N(x, \epsilon, t)$ such that

$$\{(i, j) : \mu(B(x_{ij}) - B(x_{MN}), z; t) \leq 1 - s \text{ or } \nu(B(x_{ij}) - B(x_{MN}), z; t) \geq s\} \in I.$$

For this, define for each $x \in {}_2S^I_{(\mu,\nu)_2}(B)$

$$Q = \{(i, j) : \mu(B(x_{ij}) - B(x_{MN}), z; t) \leq 1 - s \text{ or } \nu(B(x_{ij}) - B(x_{MN}), z; t) \geq s\} \in I.$$

Now we have to show that $Q \subset P$. Suppose that Q is not a subset of P. Then there exists $(m, n) \in Q$ and $(m, n) \notin P$.
Therefore we have

$$\mu(B(x_{mn}) - B(x_{MN}), z; t) \leq 1 - s \text{ or } \mu(B(x_{mn}) - L, z; \frac{t}{2}) > 1 - \epsilon.$$

In particular

$$\mu(B(x_{MN}) - L, z; \frac{t}{2}) > 1 - \epsilon.$$

Therefore we have

$$1-s \geq \mu(B(x_{mn})-B(x_{MN}), z; t) \geq \mu(B(x_{mn})-L, z; \frac{t}{2})*\mu(B(x_{MN})-L, z; \frac{t}{2}) \geq (1-\epsilon)*(1-\epsilon) > 1-s,$$

which is not possible. On the other hand

$$\nu(B(x_{mn}) - B(x_{MN}), z; t) \geq s \text{ or } \nu(B(x_{mn}) - L, z; \frac{t}{2}) < \epsilon$$

In particular

$$\nu(B(x_{MN}) - L, z; \frac{t}{2}) < \epsilon.$$

Therefore we have

$$s \leq \nu(B(x_{mn}) - B(x_{MN}), z; t) \leq \nu(B(x_{mn}) - L, z; \frac{t}{2}) \diamond \nu(B(x_{MN}) - L, z; \frac{t}{2}) \leq \epsilon \diamond \epsilon < s,$$

which is not possible.
Hence $Q \subset P$. $P \in I$ implies $Q \in I$.

3. Conclusions

In this paper, we have introduced and studied some new double sequence spaces defined by bounded linear operator with respect to intuitionistic fuzzy 2- normed spaces via ideal convergence that is ${}_2S^I_{(\mu,\nu)_2}(B)$ and ${}_2S^I_{0(\mu,\nu)_2}(B)$ in order to prove that a bounded linear operator with respect to intuitionistic fuzzy 2- normed spaces preserves some basic topological and algebraic properties of these spaces. These definitions and results which we define in this paper provide a bigger setting to deal with the uncertainity, vagueness and convergence problems of double sequences arising in many branches of engineering and science.

Acknowledgement. The authors would like to thank the referees and the editor for their careful reading and their valuable comments.

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