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COEFFICIENT ESTIMATES FOR BI-UNIVALENT MA-MINDA TYPE FUNCTIONS ASSOCIATED WITH q-DERIVATIVE

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Abstract. In this article, we consider a new subclasses of analytic and bi-univalent functions associated with q-derivative in the open unit disk. We obtain coefficient bounds for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of the functions from these new subclasses.

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1. Introduction

Let the collection of functions f that are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \ and \ |z| < 1\}$, and normalized by conditions $f(0) = f'(0) - 1 = 0$ be denoted by the symbol $\mathcal{A}$. Equivalently, if $f \in \mathcal{A}$, then the Taylor-Maclaurin series representation has the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in \mathcal{U}. \tag{1}$$

Furthermore, let us name by $\mathcal{S}$ the most basic sub-collection of the set $\mathcal{A}$ that are univalent in $\mathcal{U}$. The well-known Köebe one-quarter theorem [7] ensures that the image of $\mathcal{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Hence, every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z, \ z \in \mathcal{U}$ and

$$f^{-1}(f(\omega)) = \omega, \ \left(|\omega| < r_0(f), \ r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + ... \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if f and f^{-1} are univalent in $\mathcal{U}$. Let σ denote the class of bi-univalent functions defined in the unit disk $\mathcal{U}$. The familiar Köebe

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function is not an element of σ because it univalently maps the unit disk $\mathcal{U}$ onto the entire complex plane minus a slit along the line from $\frac{-1}{4}$ to $-\infty$. Hence, the image domain does not contain the unit disk $\mathcal{U}$. In 1985, Louis de Branges [6] proved the celebrated Bieberbach conjecture, which states that, for each $f \in \mathcal{S}$ given by the Taylor-Maclaurin series expansion (1), the following coefficient inequality is true

$$|a_n| \leq n \qquad (n \in \mathbb{N} - \{1\}),$$

where N is the set of positive integers. The class of analytic bi-univalent functions was first introduced and studied

by Lewin [9] who proved that $|a_2| < 1.51$. Later, Brannan and Clunie [4] improved Lewin's result to $|a_2| \leq \sqrt{2}$. Brannan and Taha [5] and Taha [15] considered certain subclasses of bi-univalent functions similar to the familiar subclasses of univalent functions formed by strongly starlike, starlike, and convex functions. They introduced bi-starlike functions and bi-convex functions and established non-sharp estimates for the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. For two analytic functions f and g in $\mathcal{U}$, the subordination between them is written as $f \prec g$. The function $f(z)$ is subordinate to $g(z)$ if there is a Schwarz function w with $w(0) = 0$, $|w(z)| < 1$, for all $z \in \mathcal{U}$, such that $f(z) = g(w(z))$ for all $z \in \mathcal{U}$. The q-difference operator which was introduced by Jackson [8] (see also [2, 3, 12, 14, 16, 17]) is defined as

$$D_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad z \in \mathcal{U} - \{0\}. \tag{3}$$

In addition, the q-derivative at zero defined for $|q| > 1$, $D_q f(0) = D_{q^{-1}} f(0)$. In some literature the q-derivative at zero is defined as $f'(0)$ if it exists.
Equivalently (3), may be written as

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0,$$

where

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1. \end{cases}$$

Making use of the q-derivative, we define the subclasses $S_q^*(\alpha)$ and $K_q(\alpha)$ of the class $\mathcal{A}$ for $0 \leq \alpha < 1$ by

Definition 1.1. *A function f of the form* (1) *is in the class $S_q^*(\alpha)$, if and only if*

$$\Re\left\{\frac{zD_q f(z)}{f(z)}\right\} > \alpha, \qquad for\ all\ z \in \mathcal{U}.$$

Definition 1.2. *A function f of the form* (1) *is in the class $K_q(\alpha)$, if and only if*

$$\Re\left\{1 + \frac{qzD_q^2 f(z)}{D_q f(z)}\right\} > \alpha, \qquad for\ all\ z \in \mathcal{U}.$$

Observe that $f \in K_q(\alpha)$ if and only if $zD_q f \in S_q^*(\alpha)$
and

$$\lim_{q \to 1^-} S_q^*(\alpha) = S^*(\alpha),$$
$$\lim_{q \to 1^-} K_q(\alpha) = K(\alpha),$$

where $S^*(\alpha)$, $K(\alpha)$ are the classes of starlike and convex functions of order α respectively. These classes is introduced and studied by Seoudy and Aouf [13]. In the present work, we deduce estimates for the initial coefficients $|a_2|$ and $|a_3|$ of two new subclass of the class of bi-univalent functions σ. Let φ be an analytic function with positive real part in $\mathcal{U}$ such that $\varphi(0) = 1,\ \varphi(0) > 0$ and $\varphi(\mathcal{U})$ is symmetric with respect to real axis. Such a function has a series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ...,\ (B_1 > 0). \tag{4}$$

With this brief introduction, we define the following class of bi-univalent functions and finding the coefficient estimates with the help of q-derivative.

In order to derive our main results, we have to recall here the following lemma.

Lemma 1.1. [11] *If the function $p \in \mathcal{P}$ is given by the series*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ..., \tag{5}$$

where $\mathcal{P}$ is the family of all functions $p(z)$ analytic in $\mathcal{U}$ and satisfy $\Re\{p(z)\} > 0$. Then the following sharp estimate holds:

$$|c_n| \le 2 \quad (n = 1, 2, ...)$$

2. Main results

Definition 2.1. *A function $f \in \sigma$ is said to be in the class $\mathcal{H}_{\sigma,q}(\varphi)$ if the following subordinations hold*

$D_q f(z) \prec \varphi(z)$ and $D_q g(\omega) \prec \varphi(\omega)$, where $g(\omega) = f^{-1}(\omega)$.

Theorem 2.1. *Let $f \in \mathcal{H}_{\sigma,q}(\varphi)$ and given by (1). Then*

$$|a_2| \le \frac{B_1^{\frac{3}{2}}}{\sqrt{|[3]_q B_1^2 - [2]_q^2 B_2 + [2]_q^2 B_1|}} \text{ and } |a_3| \le \frac{B_1}{[3]_q} + \frac{B_1^2}{[2]_q^2}. \tag{6}$$

Proof. Let $f \in \mathcal{H}_{\sigma,q}(\varphi)$ and $g = f^{-1}$. Then there are holomorphic functions $r, s : \mathcal{U} \to \mathcal{U}$, with $r(0) = s(0) = 0$, satisfying

$$D_q f(z) = \varphi(r(z)) \quad \text{and} \quad D_q g(\omega) = \varphi(s(\omega)). \tag{7}$$

Define the functions p_1 and p_2 by

$p_1(z) = \frac{1+r(z)}{1-r(z)} = 1 + c_1 z + c_2 z^2 + ...$ and $p_2(z) = \frac{1+s(z)}{1-s(z)} = 1 + b_1 z + b_2 z^2 + ...,$

or, equivalently,

$$r(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left(c_1 z + (c_2 - \frac{c_1^2}{2})z^2 + ...\right) \tag{8}$$

and

$$s(z) = \frac{p_2(z) - 1}{p_2(z) + 1} = \frac{1}{2}\left(b_1 z + (b_2 - \frac{b_1^2}{2})z^2 + ...\right). \tag{9}$$

It is clear that p_1 and p_2 are analytic in $\mathcal{U}$ and $p_1(0) = p_2(0) = 1$. Also p_1 and p_2 have positive real part in $\mathcal{U}$ and hence $|b_i| \le 2$ and $|c_i| \le 2,\ (i \in \mathbb{N})$.

Clearly, upon substituting from (8) and (9) into (7), if we make use of (4), we obtain

$$D_q f(z) = \varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right) z^2 + ... \tag{10}$$

and

$$D_q g(\omega) = \varphi\left(\frac{p_2(\omega) - 1}{p_2(\omega) + 1}\right) = 1 + \frac{1}{2}B_1 b_1 \omega + \left(\frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2 b_1^2\right) \omega^2 + ... \tag{11}$$

Since $f \in \sigma$ has the Maclaurin series given by (1), a computation shows that its inverse $g = f^{-1}$ has the expansion $g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 + ...$ Since

$D_q f(z) = 1 + [2]_q a_2 z + [3]_q a_3 z^2 + ...$ and $D_q g(\omega) = 1 - [2]_q a_2 \omega + [3]_q (2a_2^2 - a_3)\omega^2 + ...$, it follows from (10) and (11) that

$$a_2 = \frac{B_1 c_1}{2[2]_q}. \tag{12}$$

$$[3]_q a_3 = \frac{1}{2} B_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2. \tag{13}$$

$$a_2 = \frac{B_1 b_1}{-2[2]_q}. \tag{14}$$

$$[3]_q \left(2a_2^2 - a_3\right) = \frac{1}{2} B_1 \left(b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4} B_2 b_1^2. \tag{15}$$

From (12) and (14), we obtain

$$c_1 = -b_1, \tag{16}$$

and

$$2a_2^2 = \frac{B_1^2 (c_1^2 + b_1^2)}{4[2]_q^2}. \tag{17}$$

Now, by adding equation (13) and equation (15) and using (17), we get

$$a_2^2 = \frac{B_1^3 \left(b_2 + c_2\right)}{4\left[[3]_q B_1^2 - [2]_q^2 B_2 + [2]_q^2 B_1\right]}.$$

Applying Lemma 5 for the coefficients b_2 and c_2, we immediately have

$$|a_2| \leq \frac{B_1^{\frac{3}{2}}}{\sqrt{|[3]_q B_1^2 - [2]_q^2 B_2 + [2]_q^2 B_1|}}.$$

This gives us the bound on $|a_2|$ as asserted in (18). Next, in order to find the bound on $|a_3|$, by subtracting (15) from (13) and also from (16), we get $c_1^2 = b_1^2$, hence

$$a_3 = \frac{1}{4[3]_q} B_1 (c_2 - b_2) + \frac{1}{4[2]_q^2} (B_1^2 c_1^2).$$

Using (17) and applying Lemma 5 once again for the coefficients b_2 and c_2, we have

$$|a_3| \leq \frac{B_1}{[3]_q} + \frac{B_1^2}{[2]_q^2}.$$

This completes the proof of Theorem 2.1. □

As $q \to 1^-$, we get the following result, introduced by Rosihan et al. [1].

Corollary 2.1. *Let $f \in \mathcal{H}_\sigma(\varphi)$ and given by* (1). *Then*

$$|a_2| \leq \frac{B_1^{\frac{3}{2}}}{\sqrt{|3B_1^2 - 4B_2 + 4B_1|}} \text{ and } |a_3| \leq \frac{B_1}{3} + \frac{B_1^2}{4}. \tag{18}$$

Definition 2.2. *A function $f \in \sigma$ is said to be in the class $\mathcal{ST}_{\sigma,q}(\alpha, \varphi)$, $\alpha \geq 0$, if the following subordinations hold*

$$\frac{zD_q f(z)}{f(z)} + \frac{\alpha z^2 D_q^2 f(z)}{f(z)} \prec \varphi(z), \qquad (z \in \mathcal{U}),$$

and

$$\frac{\omega D_q g(\omega)}{g(\omega)} + \frac{\alpha\omega^2 D_q^2 g(\omega)}{g(\omega)} \prec \varphi(\omega), \qquad (\omega \in \mathcal{U}),$$

where $g(\omega) = f^{-1}(\omega)$.

Theorem 2.2. *Let f given by (1) be in the class $\mathcal{ST}_{\sigma,q}(\alpha,\varphi)$. Then*

$$|a_2| \leq \frac{B_1^{\frac{3}{2}}}{\sqrt{\left|\left[([3]_q-[2]_q)+[2]_q([3]_q-1)\alpha\right]B_1^2+(B_1-B_2)\left[([2]_q-1)+[2]_q\alpha\right]^2\right|}}. \tag{19}$$

and

$$|a_3| \leq \frac{B_1+|B_2-B_1|}{\left[([3]_q-[2]_q)+[2]_q([3]_q-1)\alpha\right]}. \tag{20}$$

Proof. Let $f \in \mathcal{ST}_{\sigma,q}(\alpha,\varphi)$. Then there are holomorphic functions $r, s : \mathcal{U} \to \mathcal{U}$, with $r(0) = s(0) = 0$, satisfying

$$\frac{zD_q f(z)}{f(z)} + \frac{\alpha z^2 D_q^2 f(z)}{f(z)} = \varphi(r(z)), \qquad (z \in \mathcal{U}), \tag{21}$$

and

$$\frac{\omega D_q g(\omega)}{g(\omega)} + \frac{\alpha\omega^2 D_q^2 g(\omega)}{g(\omega)} = \varphi(s(\omega)), \qquad (\omega \in \mathcal{U}), \tag{22}$$

where $g(\omega) = f^{-1}(\omega)$. By (21), we have

$$z + [2]_q a_2(1+\alpha)z^2 + [3]_q a_3(1+[2]_q\alpha)z^3 + ... =$$
$$\left\{1+\frac{1}{2}B_1c_1z+\left(\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right)z^2+...\right\}\left\{z+a_2z^2+a_3z^3+...\right\}.$$

Equating the coefficients on both sides we have

$$\left[([2]_q-1)+[2]_q\alpha\right]a_2 = \frac{B_1c_1}{2}. \tag{23}$$

$$\left[([3]_q-1)+[2]_q[3]_q\alpha\right]a_3 - \left[([2]_q-1)+[2]_q\alpha\right]a_2^2 = \frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2. \tag{24}$$

Also, from (22), we have

$$\omega - [2]_q a_2(1+\alpha)\omega^2 + [3]_q(2a_2^2-a_3)(1+[2]_q\alpha)\omega^3 + ... =$$
$$\left\{1+\frac{1}{2}B_1b_1\omega+\left(\frac{1}{2}B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}B_2b_1^2\right)\omega^2+...\right\}\left\{\omega-a_2\omega^2+(2a_2^2-a_3)\omega^3+...\right\}.$$

Equating the coefficients on both sides we have

$$-\left[([2]_q-1)+[2]_q\alpha\right]a_2 = \frac{B_1b_1}{2}. \tag{25}$$

$$\left[(2[3]_q-[2]_q1-)+(2[2]_q[3]_q-[2]_q)\right]a_2^2-\left[([3]_q-1)+[2]_q[3]_q\alpha\right]a_3 = \frac{1}{2}B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}B_2b_1^2. \tag{26}$$

From (23) and (25), we obtain

$$c_1 = -b_1, \tag{27}$$

and

$$2a_2^2 = \frac{B_1^2(c_1^2 + b_1^2)}{4\left[([2]_q - 1) + [2]_q\alpha\right]^2}. \tag{28}$$

Now, by adding equation (24) and equation (26) and using (28), we get

$$a_2^2 = \frac{B_1^3\,(b_2 + c_2)}{4\left[\left(([3]_q - [2]_q) + [2]_q\,([3]_q - 1)\,\alpha\right)B_1^2 + (B_1 - B_2)\left(([2]_q - 1) + [2]_q\alpha\right)^2\right]}.$$

Applying Lemma 5 for the coefficients b_2 and c_2, we immediately get

$$|a_2^2| \leq \frac{B_1^3}{\left|\left(([3]_q - [2]_q) + [2]_q\,([3]_q - 1)\,\alpha\right)B_1^2 + (B_1 - B_2)\left(([2]_q - 1) + [2]_q\alpha\right)^2\right|}.$$

Since $B_1 > 0$, the last inequality gives the desired estimate on $|a_2|$ given in (19). Next, in order to find the bound on $|a_3|$, by subtracting (26) from (24) and also from (27), we get $c_1^2 = b_1^2$, hence

$$a_3 = \frac{B_1\left[\left((2[3]_q - [2]_q - 1) + [2]_q(2[3]_q - 1)\alpha\right)c_2 + \left(([2]_q - 1) + [2]_q\alpha\right)b_2\right]}{4\left[([3]_q - 1) + [2]_q[3]_q\alpha)\right]\left[([3]_q - [2]_q) + [2]_q([3]_q - 1)\alpha\right]}$$

$$+\frac{b_1^2(B_2 - B_1)\left[([3]_q - 1) + [2]_q[3]_q\alpha\right]}{8\left[([3]_q - 1) + [2]_q[3]_q\alpha)\right]\left[([3]_q - [2]_q) + [2]_q([3]_q - 1)\alpha\right]}.$$

Using (28) and applying Lemma 5 once again for the coefficients b_2 and c_2, we obtain

$$|a_3| \leq \frac{B_1 + |B_2 - B_1|}{\left[([3]_q - [2]_q) + [2]_q([3]_q - 1)\alpha\right]}. \quad .$$

This is precisely the estimate in (20). □

As $q \to 1^-$, we get the following result, introduced by Rosihan et al. [1].

Corollary 2.2. *Let f given by* (1) *be in the class $\mathcal{ST}_\sigma(\alpha, \varphi)$. Then*

$$|a_2| \leq \frac{B_1^{\frac{3}{2}}}{\sqrt{\left|\left[(1 + 4\alpha)B_1^2 + (B_1 - B_2)(1 + 2\alpha)^2\right|\right.}},$$

and

$$|a_3| \leq \frac{B_1 + |B_2 - B_1|}{(1 + 4\alpha)}.$$

As $q \to 1^-$ and for $\alpha = 0$, we get the following coefficient estimates for Ma-Minda bi-starlike functions [10].

Corollary 2.3. *Let f given by* (1) *be in the class $\mathcal{ST}_\sigma(\varphi)$. Then*

$$|a_2| \leq \frac{B_1^{\frac{3}{2}}}{\sqrt{\left|B_1^2 + (B_1 - B_2\right|}},$$

and

$$|a_3| \leq B_1 + |B_2 - B_1|.$$

Definition 2.3. *A function $f \in \sigma$ is said to be in the class $\mathcal{M}_{\sigma,q}(\alpha,\varphi)$, $\alpha \geq 0$, if the following subordinations hold*

$$(1-\alpha)\frac{zD_qf(z)}{f(z)} + \alpha\left(1 + \frac{qzD_q^2f(z)}{D_qf(z)}\right) \prec \varphi(z), \qquad (z \in \mathcal{U}),$$

and

$$(1-\alpha)\frac{\omega D_qg(\omega)}{g(\omega)} + \alpha\left(1 + \frac{q\omega D_q^2g(\omega)}{D_qg(\omega)}\right) \prec \varphi(\omega), \qquad (\omega \in \mathcal{U}),$$

where $g(\omega) = f^{-1}(\omega)$.

Theorem 2.3. *Let f given by (1) be in the class $\mathcal{M}_{\sigma,q}(\alpha,\varphi)$. Then*

$$|a_2| \leq \frac{\sqrt{2B_1^3}}{\sqrt{\left|MB_1^2 + 2(B_1 - B_2)\Big(([2]_q - 1) + ((q-1)[2]_q + 1)\alpha\Big)^2\right|}}. \tag{29}$$

and

$$|a_3| \leq \frac{2\big(B_1 + |B_2 - B_1|\big)}{\Big(2([3]_q - [2]_q) + \big[(2[3]_q - [2]_q)(q[2]_q - 1) - [2]_q(2 - [2]_q(q+1))\big]\alpha\Big)}. \tag{30}$$

where $M = \Big(2([3]_q - [2]_q) + \big[(2[3]_q - [2]_q)(q[2]_q - 1) - [2]_q(2 - [2]_q(q+1))\big]\alpha\Big)$

Proof. Let $f \in \mathcal{M}_{\sigma,q}(\alpha,\varphi)$. Then there are holomorphic functions $r, s : \mathcal{U} \to \mathcal{U}$, with $r(0) = s(0) = 0$, satisfying

$$(1-\alpha)\frac{zD_qf(z)}{f(z)} + \alpha\left(1 + \frac{qzD_q^2f(z)}{D_qf(z)}\right) = \varphi(r(z)), \qquad (z \in \mathcal{U}), \tag{31}$$

and

$$(1-\alpha)\frac{\omega D_qg(\omega)}{g(\omega)} + \alpha\left(1 + \frac{q\omega D_q^2g(\omega)}{D_qg(\omega)}\right) = \varphi(s(\omega)), \qquad (\omega \in \mathcal{U}), \tag{32}$$

where $g(\omega) = f^{-1}(\omega)$. By (31), we have

$$z + a_2\Big(2[2]_q + ([2]_q(q-1)+1)\alpha\Big)z^2 + \Big\{[2]_q\Big([2]_q + 2\alpha - [2]_q\alpha\Big)a_2^2 + [3]_q\Big((2 - \alpha + [2]_q\alpha) + q\alpha\Big)a_3\Big\}z^3 + \ldots$$

$$= \left\{1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^2 + \ldots\right\}\left\{z + ([2]_q+1)a_2z^2 + \big[[2]_qa_2^2 + ([3]_q+1)a_3\big]z^3 + \ldots\right\}.$$

Equating the coefficients on both sides we have

$$\Big[([2]_q - 1) + ([2]_q(q-1)+1)\alpha\Big]a_2 = \frac{B_1c_1}{2}. \tag{33}$$

$$\Big[([3]_q-1) + ([2]_q[3]_q - [3]_q + q)\alpha\Big]a_3 - \Big[([2]_q-1) + ([2]_q^2 - [2]_q + 1)\alpha\Big]a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2. \tag{34}$$

Also, from (32), we have

$$\omega-\Big(2[2]_q+([2]_q(q-1)+1)\alpha\Big)a_2\omega^2+\Big\{\Big[(4[3]_q+[2]_q^2)+((q+1)[2]_q-2[3]_q+2[2]_q^2+2q[2]_q[3]_q)\alpha\Big]a_2^2+\Big[-2[3]_q+([3]_q-q[2]_q[3]_q-1)\alpha\Big]a_3\Big\}\omega^3+...$$

$$=\Big\{1+\frac{1}{2}B_1b_1\omega+\Big(\frac{1}{2}B_1\Big(b_2-\frac{b_1^2}{2}\Big)+\frac{1}{4}B_2b_1^2\Big)\omega^2+...\Big\}\Big\{\omega-([2]_q+1)a_2\omega^2+\big[[3]_q(2a_2^2-a_3)+[2]_qa_2^2+2a_2^2-a_3\big]\omega^3+...\Big\}.$$

Equating the coefficients on both sides we have

$$-\Big[([2]_q-1)+([2]_q(q-1)+1)\alpha\Big]a_2=\frac{B_1b_1}{2}. \tag{35}$$

$$\Big[(2[3]_q-[2]_q-1)+((2[3]_q-[2]_q)(q[2]_q-1)+1)\alpha\Big]a_2^2-\Big[([3]_q-1)-([3]_q(q[2]_q-1)+1)\alpha\Big]a_3=\frac{1}{2}B_1\Big(b_2-\frac{b_1^2}{2}\Big)+\frac{1}{4}B_2b_1^2. \tag{36}$$

From (33) and (35), we obtain

$$c_1=-b_1, \tag{37}$$

and

$$2a_2^2=\frac{B_1^2(c_1^2+b_1^2)}{4\left[([2]_q-1)+([2]_q(q-1)+1)\alpha\right]^2}. \tag{38}$$

Now, by adding equation (34) and equation (36) and using (38), we get

$$a_2^2=\frac{B_1^3\,(b_2+c_2)}{2\Big[MB_1^2+2(B_1-B_2)\Big(([2]_q-1)+((q-1)[2]_q+1)\alpha\Big)^2\Big]},$$

where $M=\Big(2([3]_q-[2]_q)+\big[(2[3]_q-[2]_q)(q[2]_q-1)-[2]_q(2-[2]_q(q+1))\big]\alpha\Big)$. Applying Lemma 5 for the coefficients b_2 and c_2, we immediately get

$$|a_2^2|\le\frac{2B_1^3}{\Big|MB_1^2+2(B_1-B_2)\Big(([2]_q-1)+((q-1)[2]_q+1)\alpha\Big)^2\Big|}.$$

which yields the desired estimate on $|a_2|$ as described in (29). Next, in order to find the bound on $|a_3|$, by subtracting (36) from (34) and also from (37), we get $c_1^2=b_1^2$, hence

$$a_3=\frac{B_1\Big[\Big((2[3]_q-[2]_q-1)+((2[3]_q-[2]_q)(q[2]_q-1)+1)\alpha\Big)c_2+\Big(([2]_q-1)+([2]_q^2-[2]_q+1)\alpha\Big)b_2\Big]}{2\Big[2([3]_q-[2]_q)+\big[2[3]_q(q[2]_q-1)+[2]_q(2-[2]_q(q+1))\big]\alpha\Big]\Big[([3]_q-1)+([2]_q[3]_q-[3]_q+1)\alpha\Big]}$$

$$+\frac{b_1^2(B_2-B_1)\Big[(2[3]_q-1)+(2[3]_q(q[2]_q-1)-[2]_q^2(q+1)+2)\alpha\Big]}{2\Big[2([3]_q-[2]_q)+\big[2[3]_q(q[2]_q-1)+[2]_q(2-[2]_q(q+1))\big]\alpha\Big]\Big[([3]_q-1)+([2]_q[3]_q-[3]_q+1)\alpha\Big]}.$$

Using (38) and applying Lemma 5 once again for the coefficients b_2 and c_2, we obtain

$$|a_3|\le\frac{2\big(B_1+|B_2-B_1|\big)}{\Big(2([3]_q-[2]_q)+\big[(2[3]_q-[2]_q)(q[2]_q-1)-[2]_q(2-[2]_q(q+1))\big]\alpha\Big)}.$$

This is precisely the estimate in (30). $\square$

As $q \to 1^-$, we get the following result, introduced by Rosihan et al. [1].

Corollary 2.4. *Let f given by* (1) *be in the class $\mathcal{M}_\sigma(\alpha, \varphi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|(1+\alpha)B_1^2 + (B_1 - B_2)(1+\alpha)^2\right|}},$$

and

$$|a_3| \leq \frac{B_1 + |B_2 - B_1|}{(1+\alpha)}.$$

As $q \to 1^-$ and for $\alpha = 0$, we get the coefficient estimates for Ma-Minda bi-starlike functions, while for $\alpha = 1$, we get the following estimates for Ma-Minda bi-convex functions [10].

Corollary 2.5. *Let f given by* (1) *be in the class $\mathcal{CV}_\sigma(\varphi)$. Then*

$$|a_2| \leq \frac{B_1^{\frac{3}{2}}}{\sqrt{2\left|B_1^2 + 2B_1 - 2B_2\right|}},$$

and

$$|a_3| \leq \frac{1}{2}(B_1 + |B_2 - B_1|).$$

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