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ON GENERALIZATION OF EXPLICIT EXPRESSIONS OF SOME HYBRID TRICOMI FUNCTIONS

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Abstract. In this paper, a general quadruple series identity is established. By using general quadruple series identity, a Laurent type hypergeometric generating relation is derived. Certain explicit expressions of some hybrid special functions related to the Tricomi functions introduced by Dattoli et al. and Khan et al. are established as applications.

Keywords: Generalized hypergeometric functions; Laurent type generating relations; Quadruple series identity; Tricomi functions.

AMS Subject Classification: 33B15, 33C10, 33C20.

1. Introduction and preliminaries

About five decades ago, Srivastava and Daoust [12] first considered the two-variable case of their multiple hypergeometric function [13, p.454]; see also [14]). For the sake of ready reference, we choose to recall here their definition only in the two-variable case as follows [12, p.199, Eq. (2.1)]:

$$F^{A:B;B'}_{C:D;D'}\left(\begin{array}{l}[(a_A):\theta,\phi]:[(b_B):\psi];[(b'_{B'}):\psi'];\\ \\ [(c_C):\xi,\eta]:[(d_D):\zeta];[(d'_{D'}):\zeta'];\end{array}\quad x,\,y\right)$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m\theta_j+n\phi_j}}{\prod_{j=1}^{C}(c_j)_{m\xi_j+n\eta_j}}\frac{\prod_{j=1}^{B}(b_j)_{m\psi_j}}{\prod_{j=1}^{D}(d_j)_{m\zeta_j}}\frac{\prod_{j=1}^{B'}(b'_j)_{n\psi'_j}}{\prod_{j=1}^{D'}(d'_j)_{n\zeta'_j}}\frac{x^m\,y^n}{m!\,n!},\tag{1}$$

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where, for convergence of the double hypergeometric series,

$$1+\sum_{j=1}^{C}\xi_j+\sum_{j=1}^{D}\zeta_j-\sum_{j=1}^{A}\theta_j-\sum_{j=1}^{B}\psi_j\geq 0 \tag{2}$$

and

$$1+\sum_{j=1}^{C}\eta_j+\sum_{j=1}^{D'}\zeta'_j-\sum_{j=1}^{A}\phi_j-\sum_{j=1}^{B'}\psi'_j\geq 0, \tag{3}$$

with equality only when $|x|$ and $|y|$ are constrained appropriately (see, for details, [14]). Here, for the sake of convenience, $[(a_A):\theta,\phi]$ represents the set of "A" number of parameters $[a_1:\theta_1,\phi_1]$, $[a_2:\theta_2,\phi_2]$, $\ldots$, $[a_A:\theta_A,\phi_A]$. The values of positive real coefficients θ_1, θ_2, $\ldots$, θ_A may be equal or different with similar interpretation for coefficients ϕ_1, ϕ_2, $\ldots$, ϕ_A and others.

Megumi Saigo's general quadruple hypergeometric function $F_M^{(4)}$

In 1988, M. Saigo defined a more general quadruple hypergeometric function [10, pp. 455-456 (16)], [11, pp. 15 (17)] $F_M^{(4)}$ (slightly modified notation) in the following form:

$$F_M^{(4)}\left[\begin{array}{l} (a_A) ::: (b_B) ; (d_D) ; (e_E) ; (g_G) :: (h_H) ; (m_M) ; (n_N) ; \\ (a'_{A'}) ::: (b'_{B'}) ; (d'_{D'}) ; (e'_{E'}) ; (g'_{G'}) :: (h'_{H'}) ; (m'_{M'}) ; (n'_{N'}) ; \end{array}\right.$$

$$\left.\begin{array}{l} (p_P) ; (q_Q) ; (r_R) : (s_S) ; (t_T) ; (u_U) ; (w_W) ; \\ (p'_{P'}) ; (q'_{Q'}) ; (r'_{R'}) : (s'_{S'}) ; (t'_{T'}) ; (u'_{U'}) ; (w'_{W'}) ; \end{array} x,y,z,c\right]$$

$$=\sum_{i,j,k,\ell=0}^{\infty}\frac{[(a_A)]_{i+j+k+\ell}\,[(b_B)]_{i+j+k}\,[(d_D)]_{j+k+\ell}\,[(e_E)]_{k+\ell+i}\,[(g_G)]_{\ell+i+j}\,[(h_H)]_{i+j}}{[(a'_{A'})]_{i+j+k+\ell}\,[(b'_{B'})]_{i+j+k}\,[(d'_{D'})]_{j+k+\ell}\,[(e'_{E'})]_{k+\ell+i}\,[(g'_{G'})]_{\ell+i+j}\,[(h'_{H'})]_{i+j}}$$

$$\times\frac{[(m_M)]_{i+k}\,[(n_N)]_{i+\ell}\,[(p_P)]_{j+k}\,[(q_Q)]_{j+\ell}\,[(r_R)]_{k+\ell}\,[(s_S)]_i\,[(t_T)]_j\,[(u_U)]_k\,[(w_W)]_\ell}{[(m'_{M'})]_{i+k}\,[(n'_{N'})]_{i+\ell}\,[(p'_{P'})]_{j+k}\,[(q'_{Q'})]_{j+\ell}\,[(r'_{R'})]_{k+\ell}\,[(s'_{S'})]_i\,[(t'_{T'})]_j\,[(u'_{U'})]_k\,[(w'_{W'})]_\ell}\,\frac{x^i\ y^j\ z^k\ c^\ell}{i!\ j!\ k!\ \ell!}, \tag{4}$$

where $[(a_A)]_{i+j+k+\ell}=\prod_{m=1}^{A}(a_m)_{i+j+k+\ell}$ with similar interpretations for others.

It is the generalization and unification of Exton's quadruple hypergeometric functions $K_1,\cdots K_{21}$ [4, 5] and Pathan's quadruple hypergeometric function $F_P^{(4)}$ [8] etc.

The n^{th} order Tricomi functions $C_n(x)$ are defined by the following generating function:

$$\exp\left(t-\frac{x}{t}\right)=\sum_{n=-\infty}^{\infty}C_n(x)\ t^n,\quad n\in\mathbb{Z} \tag{5}$$

and have the following series:

$$C_n(x)\ =\ \sum_{k=0}^{\infty}\frac{(-1)^k\ x^k}{k!\ (n+k)!},\quad n\in\mathbb{N}_0. \tag{6}$$

Tricomi functions $C_n(x)$ have the following hypergeometric representation:

$$C_n(x) = \frac{1}{n!}\ {}_0F_1\left[\begin{array}{ccc} — & ; & \\ & & -x \\ n+1 & ; & \end{array}\right], \quad n \in \mathbb{N}_0 \tag{7}$$

and

$$C_{-n}(x) = (-1)^n x^n C_n(x), \quad n \in \mathbb{N}. \tag{8}$$

The classical Hermite polynomials $H_j(z)$ are defined as [9, p. 187 and p. 191]:

$$H_j(z) = (2z)^j\ {}_2F_0\left[\begin{array}{ccc} \frac{-j}{2}, \frac{-j+1}{2} & ; & \\ & & -\frac{1}{z^2} \\ — & ; & \end{array}\right], \quad j \in \mathbb{N}_0 \tag{9}$$

$$= \sum_{k=0}^{[\frac{j}{2}]} \frac{(-1)^k\ j!\ (2z)^{j-2k}}{k!\ (j-2k)!}, \quad j \in \mathbb{N}_0. \tag{10}$$

The 2-variable Hermite-Kampé de Fériet polynomials (2VHKdFP) [1] $H_j(x,y)$ are defined by the following series:

$$H_j(x,y) = j! \sum_{r=0}^{[\frac{j}{2}]} \frac{y^r x^{j-2r}}{r!(j-2r)!}, \quad j \in \mathbb{N}_0 \tag{11}$$

and have the following relationship with the classical Hermite polynomials $H_j(x)$:

$$H_j(x,y) = (-i)^j y^{j/2} H_j\left(\frac{ix}{2\sqrt{y}}\right); \quad i = \sqrt{-1}, \quad j \in \mathbb{N}_0. \tag{12}$$

The generating function of the 2-variable Hermite-Tricomi functions (2VHTF) ${}_HC_n(x,y)$ [3, p. 407, (44a)] are as follows:

$$\exp\left(t - \frac{x}{t} + \frac{y}{t^2}\right) = \sum_{n=-\infty}^{\infty} {}_HC_n(x,y)t^n, \quad n \in \mathbb{Z}. \tag{13}$$

We recall that the 2-variable Hermite-Tricomi functions (2VHTF) ${}_HC_n(x,y)$ are defined by the series [3, p. 407, (44b)]:

$${}_HC_n(x,y) = \sum_{j=0}^{\infty} \frac{(-1)^j H_j(x,y)}{j!\ (j+n)!}, \quad n \in \mathbb{N}_0. \tag{14}$$

In view of series definition (11), equation (14) becomes

$$
\begin{aligned}
{}_HC_n(x,y) &= \sum_{j=0}^{\infty}\sum_{r=0}^{[\frac{j}{2}]} \frac{(-1)^j\ y^r x^{j-2r}}{r!\ (j+n)!\ (j-2r)!} && (15)\\
&= \sum_{j=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-x)^j\ y^r}{r!\ j!\ (j+2r+n)!} && (16)\\
&= \sum_{j=0}^{\infty} \frac{(-x)^j}{j!\ (j+n)!}\ {}_0F_2\left[\begin{array}{ccc} \text{——} & ; & \\ & & \frac{y}{4} \\ \frac{1+j+n}{2},\ \ \frac{2+j+n}{2} & ; & \end{array}\right] && (17)\\
&= \frac{1}{n!}\ F^{0:0;0}_{1:0;0}\left(\begin{array}{cccccccc} \text{——} & : & \text{——} & ; & \text{——} & ; & & \\ & & & & & & -x,\ y & \\ [1+n:1,2] & ; & \text{——} & ; & \text{——} & ; & & \end{array}\right), \quad n\in\mathbb{N}_0. && (18)
\end{aligned}
$$

Also, we can have

$$
{}_HC_{-n}(x,y) = (-x)^n \sum_{j=0}^{\infty} \frac{(-x)^j}{j!\ (n+j)!}\ {}_2F_0\left[\begin{array}{ccc} \frac{-n-j}{2},\ \ \frac{-n-j+1}{2} & ; & \\ & & \frac{4y}{x^2} \\ \text{——} & ; & \end{array}\right], \quad n\in\mathbb{N}. \tag{19}
$$

Using relationship (12) in equation (14), we get

$$
{}_HC_n(x,y) = \sum_{j=0}^{\infty} \frac{i^j\ y^{j/2} H_j\left(\frac{ix}{2\sqrt{y}}\right)}{j!\ (j+n)!}, \quad n\in\mathbb{N}_0. \tag{20}
$$

We can also write

$$
\begin{aligned}
{}_HC_{-n}(x,y) &= \sum_{j=n}^{\infty} \frac{i^j\ y^{\frac{j}{2}} H_j\left(\frac{ix}{2\sqrt{y}}\right)}{j!\ (j-n)!} && (21)\\
&= i^n\ y^{\frac{n}{2}} \sum_{j=0}^{\infty} \frac{i^j\ y^{\frac{j}{2}} H_{j+n}\left(\frac{ix}{2\sqrt{y}}\right)}{j!\ (j+n)!}, \quad n\in\mathbb{N}. && (22)
\end{aligned}
$$

The Laguerre-Gould-Hopper-Tricomi functions ${}_{LH^{(m,r)}}C_n(x,y,z)$ are introduced and studied by Khan *et. al.* [7] by using the operational techniques. The Laguerre-Gould-Hopper-Tricomi functions ${}_{LH^{(m,r)}}C_n(x,y,z)$ are defined by the following generating function [7, p. 389, Eq. (14)]:

$$
C_0\left[(-1)^{m-1}\frac{x}{t^m}\right]\exp\left[\left(t-\frac{y}{t}\right)+(-1)^r\frac{z}{t^r}\right] = \sum_{n=-\infty}^{\infty} {}_{LH^{(m,r)}}C_n(x,y,z)t^n, \quad n\in\mathbb{Z}. \tag{23}
$$

For the Laguerre-Gould-Hopper-Tricomi functions ${}_{LH^{(m,r)}}C_n(x,y,z)$, the following explicit representation holds [7, p. 389, Eq.(20)]:

$$
{}_{LH^{(m,r)}}C_n(x,y,z) = \sum_{j=0}^{\infty}\sum_{\ell,s=0}^{m\ell+rs\le j} \frac{(-1)^j x^\ell z^s y^{j-m\ell-rs}}{s!(\ell!)^2(j-m\ell-rs)!(n+j)!}, \quad n\in\mathbb{N}_0, \tag{24}
$$

which can also be defined in terms of the Srivastava-Daoust double hypergeometric function as follows:

$$ {}_{L}H^{(m,r)}C_n(x,y,z) = \sum_{j=0}^{\infty} \frac{(-y)^j}{j!\,(n+j)!} F^{1:0;0}_{0:1;0}\left(\begin{array}{ccccc} [-j:m,r] & : & — & ; & — ; \\ — & : & [1:1] & ; & — ; \end{array} \frac{x}{(-y)^m}, \frac{z}{(-y)^r} \right), \tag{25} $$

where $n \in \mathbb{N}_0$.

For negative integers, we have

$$ {}_{L}H^{(m,r)}C_{-n}(x,y,z) = \sum_{j=n}^{\infty} \sum_{\ell,s=0}^{m\ell+rs\leq j} \frac{(-1)^j x^\ell z^s y^{j-m\ell-rs}}{s!(\ell!)^2(j-m\ell-rs)!(-n+j)!}, \quad n \in \mathbb{N}, \tag{26} $$

which can also be defined in terms of the Srivastava-Daoust double hypergeometric function as follows:

$$ {}_{L}H^{(m,r)}C_{-n}(x,y,z) = (-y)^n \sum_{j=0}^{\infty} \frac{(-y)^j}{j!\,(n+j)!} F^{1:0;0}_{0:1;0}\left(\begin{array}{ccccc} [-n-j:m,r] & : & — & ; & — ; \\ — & : & [1:1] & ; & — ; \end{array} \frac{x}{(-y)^m}, \frac{z}{(-y)^r} \right), \tag{27} $$

where $n \in \mathbb{N}$.

Lemma 1.1. [15, p.102, Eq.(16)] *For positive integers* $m_1, \cdots, m_r \quad (r \geq 1)$,

$$ \sum_{n=0}^{\infty} \sum_{k_1,\cdots,k_r=0}^{\infty} \Theta(k_1,\cdots,k_r;n) = \sum_{n=0}^{\infty} \sum_{k_1,\cdots,k_r=0}^{m_1k_1+\cdots+m_rk_r\leq n} \Theta(k_1,\cdots,k_r;n-m_1k_1-\cdots-m_rk_r), \tag{28} $$

provided that concerned multiple series are absolutely convergent.

Recently, a variety of polynomials, their extensions, and variants, have been extensively investigated, mainly due to their potential applications in diverse research areas. For instance, arbitrary complex order Hermite-Bernoulli polynomials and Hermite-Bernoulli numbers attached to a Dirichlet character χ have been introduced in [2]. The generalized Laguerre poly-Genocchi polynomials are considered in [6] and some of their properties and identities have been investigated. Further, the recent development in the field of hypergeometric functions have been studied by many authors. The basic properties of the extended τ-Gauss hypergeometric function, including integral and derivative formulas involving the Mellin transform and the operators of fractional calculus, are derived in [17]. Also, the introduction of extended Pochhammer symbol by using a known extension of the gamma function involving the modified Bessel (or Macdonald) function is a recent investigation, see for example, [16].

2. General quadruple series identity

Theorem 2.1. *Let* $\{\Delta(\ell)\}$, $\{\Theta_1(\ell)\}$, $\{\Theta_2(\ell)\}$, $\{\Theta_3(\ell)\}$, $\{\Theta_4(\ell)\}$, $\{\Phi_1(\ell)\}$, $\{\Phi_2(\ell)\}$, $\{\Phi_3(\ell)\}$, $\{\Phi_4(\ell)\}$, $\{\Phi_5(\ell)\}$, $\{\Phi_6(\ell)\}$, $\{\Psi_1(\ell)\}$, $\{\Psi_2(\ell)\}$, $\{\Psi_3(\ell)\}$ *and* $\{\Psi_4(\ell)\}$ *are fifteen bounded sequences of arbitrary complex numbers such that* $\Delta(0) \neq 0$, $\Theta_i(0) \neq 0 (i = 1,2,3,4)$,

$\Phi_i(0) \neq 0 (i = 1, 2, 3, 4, 5, 6),\ \Psi_i(0) \neq 0 (i = 1, 2, 3, 4)$. *Then*

$$\begin{aligned}
&\sum_{\ell,j,k,s=0}^{\infty} \Delta(\ell+j+k+s)\ \Theta_1(\ell+j+k)\ \Theta_2(j+k+s)\ \Theta_3(k+s+\ell)\ \Theta_4(s+\ell+j) \\
&\times\Phi_1(\ell+j)\ \Phi_2(\ell+k)\ \Phi_3(\ell+s)\ \Phi_4(j+k)\ \Phi_5(j+s)\ \Phi_6(k+s)\ \Psi_1(\ell)\ \Psi_2(j)\ \Psi_3(k) \\
&\times\Psi_4(s)\ \frac{\left(\frac{\alpha x}{t^m}\right)^\ell \left(\frac{\beta y}{t}\right)^j (\mu u t)^k \left(\frac{\lambda z}{t^r}\right)^s}{\ell!\ j!\ k!\ s!} \\
= &\sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}\sum_{\ell,s=0}^{m\ell+rs\leq j+n^\star} \Delta(2j+2n^\star+n-(m-1)\ell-(r-1)s) \\
&\times\Theta_1(2j+2n^\star+n-(m-1)\ell-rs) \\
&\times\Theta_2(2j+2n^\star+n-m\ell-(r-1)s)\ \Theta_3(j+n^\star+n+s+\ell) \\
&\times\Theta_4(j+n^\star-(m-1)\ell-(r-1)s)\ \Phi_1(j+n^\star-(m-1)\ell-rs)\ \Phi_2(\ell+j+n^\star+n) \\
&\times\Phi_3(\ell+s)\ \Phi_4(2j+2n^\star+n-m\ell-rs)\ \Phi_5(j+n^\star-m\ell-(r-1)s) \\
&\times\Phi_6(j+n^\star+n+s)\ \Psi_1(\ell)\ \Psi_2(j+n^\star-m\ell-rs)\ \Psi_3(j+n^\star+n)\ \Psi_4(s) \\
&\times\frac{(\alpha x)^\ell (\beta y)^{j+n^\star-m\ell-rs} (\mu u)^{j+n^\star+n} (\lambda z)^s}{\ell!\ (j+n^\star-m\ell-rs)!\ (j+n^\star+n)!\ s!}\ t^n, \qquad (29)
\end{aligned}$$

where m and r are positive integers and $n^\star$ is defined as:

$$n^\star = \max\{0, -n\} = \begin{cases} -n, & \textit{when}\ \ n = \cdots, -3, -2, -1 \\ 0, & \textit{when}\ n = 0, 1, 2, \cdots, \end{cases} \qquad (30)$$

provided that each of the multiple series involved is absolutely convergent.

Proof. Suppose the l.h.s. of equation (29) is denoted by Λ. Then, we have

$$\begin{aligned}
\Lambda = &\sum_{\ell,j,k,s=0}^{\infty} \Delta(\ell+j+k+s)\ \Theta_1(\ell+j+k)\ \Theta_2(j+k+s)\ \Theta_3(k+s+\ell)\ \Theta_4(s+\ell+j) \\
&\times\Phi_1(\ell+j)\ \Phi_2(\ell+k)\ \Phi_3(\ell+s)\ \Phi_4(j+k)\ \Phi_5(j+s)\ \Phi_6(k+s)\ \Psi_1(\ell)\ \Psi_2(j)\ \Psi_3(k) \\
&\times\Psi_4(s)\ \frac{(\alpha x)^\ell (\beta y)^j (\mu u)^k (\lambda z)^s}{\ell!\ j!\ k!\ s!} t^{k-(j+m\ell+rs)}. \qquad (31)
\end{aligned}$$

Now, replacing j by $j - m\ell - rs$ and using Lemma 1.1, we get

$$\begin{aligned}
\Lambda = &\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\ell,s=0}^{m\ell+rs\leq j} \Delta(k+j-(m-1)\ell-(r-1)s)\ \Theta_1(k+j-(m-1)\ell-rs) \\
&\times\Theta_2(k+j-m\ell-(r-1)s)\ \Theta_3(k+s+\ell)\ \Theta_4(j-(m-1)\ell-(r-1)s) \\
&\times\Phi_1(j-(m-1)\ell-rs)\ \Phi_2(\ell+k)\ \Phi_3(\ell+s)\ \Phi_4(k+j-m\ell-rs) \\
&\times\Phi_5(j-m\ell-(r-1)s)\ \Phi_6(k+s)\ \Psi_1(\ell)\ \Psi_2(j-m\ell-rs)\ \Psi_3(k)\ \Psi_4(s) \\
&\times\frac{(\alpha x)^\ell (\beta y)^{j-m\ell-rs} (\mu u)^k (\lambda z)^s}{\ell!\ (j-m\ell-rs)!\ k!\ s!}\ t^{k-j}. \qquad (32)
\end{aligned}$$

Putting $k = j + n$, we obtain

$$\begin{aligned}\Lambda &= \sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}\sum_{\ell,s=0}^{m\ell+rs\leq j} \Delta(2j+n-(m-1)\ell-(r-1)s)\ \Theta_1(2j+n-(m-1)\ell-rs)\\ &\times\Theta_2(2j+n-m\ell-(r-1)s)\ \Theta_3(j+n+s+\ell)\ \Theta_4(j-(m-1)\ell-(r-1)s)\\ &\times\Phi_1(j-(m-1)\ell-rs)\ \Phi_2(\ell+j+n)\ \Phi_3(\ell+s)\ \Phi_4(2j+n-m\ell-rs)\\ &\times\Phi_5(j-m\ell-(r-1)s)\ \Phi_6(j+n+s)\ \Psi_1(\ell)\ \Psi_2(j-m\ell-rs)\ \Psi_3(j+n)\ \Psi_4(s)\\ &\times\frac{(\alpha x)^{\ell}\,(\beta y)^{j-m\ell-rs}\,(\mu u)^{j+n}\,(\lambda z)^{s}}{\ell!\,(j-m\ell-rs)!\,(j+n)!\,s!}\ t^{n}. \end{aligned} \tag{33}$$

Since n varies from $-\infty$ to ∞ and j varies from 0 to ∞, therefore due to the presence of $(j+n)!$ in denominator of above equation, equation (33) can be modified in the following form:

$$\begin{aligned}\Lambda &= \sum_{n=-\infty}^{\infty}\sum_{j=n^{\star}}^{\infty}\sum_{\ell,s=0}^{m\ell+rs\leq j} \Delta(2j+n-(m-1)\ell-(r-1)s)\ \Theta_1(2j+n-(m-1)\ell-rs)\\ &\times\Theta_2(2j+n-m\ell-(r-1)s)\ \Theta_3(j+n+s+\ell)\ \Theta_4(j-(m-1)\ell-(r-1)s)\\ &\times\Phi_1(j-(m-1)\ell-rs)\ \Phi_2(\ell+j+n)\ \Phi_3(\ell+s)\ \Phi_4(2j+n-m\ell-rs)\\ &\times\Phi_5(j-m\ell-(r-1)s)\ \Phi_6(j+n+s)\ \Psi_1(\ell)\ \Psi_2(j-m\ell-rs)\ \Psi_3(j+n)\ \Psi_4(s)\\ &\times\frac{(\alpha x)^{\ell}\,(\beta y)^{j-m\ell-rs}\,(\mu u)^{j+n}\,(\lambda z)^{s}}{\ell!\,(j-m\ell-rs)!\,(j+n)!\,s!}\ t^{n}, \end{aligned} \tag{34}$$

where $n^{\star}$ is defined by equation (30).

On replacing j by $j + n^{\star}$ in equation (34), we get equation (29).

□

3. Laurent type hypergeometric generating relations

Theorem 3.1. *The following generating function (in terms of general quadruple hypergeometric series $F_M^{(4)}$) for the Srivastava-Daoust double hypergeometric function F holds true:*

$$F_M^{(4)}\left[\begin{array}{l} (a_A)\ :::\ (b_B)\ ;\ (d_D)\ ;\ (e_E)\ ;\ (g_G)\ ::\ (h_H)\ ;\ (m_M)\ ;\ (n_N)\ ;\\ (a'_{A'})\ :::\ (b'_{B'})\ ;\ (d'_{D'})\ ;\ (e'_{E'})\ ;\ (g'_{G'})\ ::\ (h'_{H'})\ ;\ (m'_{M'})\ ;\ (n'_{N'})\ ; \end{array}\right.$$

$$\left.\begin{array}{l} (p_P)\ ;\ (q_Q)\ ;\ (r_R)\ :\ (s_S)\ ;\ (t_T)\ ;\ (u_U)\ ;\ (w_W)\ ;\\ (p'_{P'})\ ;\ (q'_{Q'})\ ;\ (r'_{R'})\ :\ (s'_{S'})\ ;\ (t'_{T'})\ ;\ (u'_{U'})\ ;\ (w'_{W'})\ ; \end{array}\ \frac{\alpha x}{t^m}, \frac{\beta y}{t}, \mu u t, \frac{\lambda z}{t^r}\right]$$

$$= \sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} \Omega(j,n^\star,n)$$

$$\times F^{\rho:\sigma;\tau}_{\gamma:\eta;\delta}\left(\begin{array}{l} [1-(a'_{A'})-2j-2n^\star-n:m-1,r-1],\ [1-(b'_{B'})-2j-2n^\star-n:m-1,r], \\ [1-(a_A)-2j-2n^\star-n:m-1,r-1],\ [1-(b_B)-2j-2n^\star-n:m-1,r], \end{array}\right.$$

$$\begin{array}{l} [1-(d'_{D'})-2j-2n^\star-n:m,r-1],\ [(e_E)+j+n^\star+n:1,1], \\ [1-(d_D)-2j-2n^\star-n:m,r-1],\ [(e'_{E'})+j+n^\star+n:1,1], \end{array}$$

$$\begin{array}{l} [1-(g'_{G'})-j-n^\star:m-1,r-1],\ [1-(h'_{H'})-j-n^\star:m-1,r],\ [(n_N):1,1], \\ [1-(g_G)-j-n^\star:m-1,r-1],\ [1-(h_H)-j-n^\star:m-1,r],\ [(n'_{N'}):1,1], \end{array}$$

$$\begin{array}{l} [1-(p'_{P'})-2j-2n^\star-n:m,r],\ [1-(q'_{Q'})-j-n^\star:m,r-1], \\ [1-(p_P)-2j-2n^\star-n:m,r],\ [1-(q_Q)-j-n^\star:m,r-1], \end{array}$$

$$\begin{array}{lcl} [1-(t'_{T'})-j-n^\star:m,r],\ [-j-n^\star:m,r] & : & [(m_M)+j+n^\star+n:1], \\ [1-(t_T)-j-n^\star:m,r] & : & [(m'_{M'})+j+n^\star+n:1], \end{array}$$

$$\left.\begin{array}{lclcl} [(s_S):1] & ; & [(r_R)+j+n^\star+n:1],\ [(w_W):1] & ; & \\ & & & & (-1)^\theta \frac{\alpha x}{(\beta y)^m},\ (-1)^\phi \frac{\lambda z}{(\beta y)^r} \\ [(s'_{S'}):1] & ; & [(r'_{R'})+j+n^\star+n:1],\ [(w'_{W'}):1] & ; & \end{array}\right), \qquad (35)$$

where

$$\Omega(j,n^\star,n) = \frac{\prod_{i=1}^{A}(a_i)_{2j+2n^\star+n}}{\prod_{i=1}^{A'}(a'_i)_{2j+2n^\star+n}} \frac{\prod_{i=1}^{B}(b_i)_{2j+2n^\star+n}}{\prod_{i=1}^{B'}(b'_i)_{2j+2n^\star+n}} \frac{\prod_{i=1}^{D}(d_i)_{2j+2n^\star+n}}{\prod_{i=1}^{D'}(d'_i)_{2j+2n^\star+n}} \frac{\prod_{i=1}^{E}(e_i)_{j+n^\star+n}}{\prod_{i=1}^{E'}(e'_i)_{j+n^\star+n}} \frac{\prod_{i=1}^{G}(g_i)_{j+n^\star}}{\prod_{i=1}^{G'}(g'_i)_{j+n^\star}}$$

$$\times \frac{\prod_{i=1}^{H}(h_i)_{j+n^\star}}{\prod_{i=1}^{H'}(h'_i)_{j+n^\star}} \frac{\prod_{i=1}^{M}(m_i)_{j+n^\star+n}}{\prod_{i=1}^{M'}(m'_i)_{j+n^\star+n}} \frac{\prod_{i=1}^{P}(p_i)_{2j+2n^\star+n}}{\prod_{i=1}^{P'}(p'_i)_{2j+2n^\star+n}} \frac{\prod_{i=1}^{Q}(q_i)_{j+n^\star}}{\prod_{i=1}^{Q'}(q'_i)_{j+n^\star}} \frac{\prod_{i=1}^{R}(r_i)_{j+n^\star+n}}{\prod_{i=1}^{R'}(r'_i)_{j+n^\star+n}} \frac{\prod_{i=1}^{T}(t_i)_{j+n^\star}}{\prod_{i=1}^{T'}(t'_i)_{j+n^\star}}$$

$$\times \frac{\prod_{i=1}^{U}(u_i)_{j+n^\star+n}}{\prod_{i=1}^{U'}(u'_i)_{j+n^\star+n}} \frac{(\beta y)^{j+n^\star}\,(\mu u)^{j+n^\star+n}}{(j+n^\star)!\,(j+n^\star+n)!},$$

$$\rho = A'+B'+D'+E+G'+H'+N+P'+Q'+T'+1,$$

$$\sigma = M+S,$$

$$\tau = R+W,$$

$$\gamma = A+B+D+E'+G+H+N'+P+Q+T,$$

$$\eta = M' + S',$$

$$\delta = R' + W',$$

$$\theta = (A-A'+B-B'+G-G'+H-H')(m-1)+(D-D'+P-P'+Q-Q'+T-T'+1)m,$$

$$\phi = (A-A'+D-D'+G-G'+Q-Q')(r-1)+(B-B'+H-H'+P-P'+T-T'+1)r$$

and $n^\star$ *is defined by equation* (30).

Proof. Taking

$$\Delta(\ell) = \frac{\prod_{i=1}^{A}(a_i)_\ell}{\prod_{i=1}^{A'}(a'_i)_\ell}, \quad \Theta_1(\ell) = \frac{\prod_{i=1}^{B}(b_i)_\ell}{\prod_{i=1}^{B'}(b'_i)_\ell}, \quad \Theta_2(\ell) = \frac{\prod_{i=1}^{D}(d_i)_\ell}{\prod_{i=1}^{D'}(d'_i)_\ell},$$

$$\Theta_3(\ell) = \frac{\prod_{i=1}^{E}(e_i)_\ell}{\prod_{i=1}^{E'}(e'_i)_\ell}, \quad \Theta_4(\ell) = \frac{\prod_{i=1}^{G}(g_i)_\ell}{\prod_{i=1}^{G'}(g'_i)_\ell},$$

$$\Phi_1(\ell) = \frac{\prod_{i=1}^{H}(h_i)_\ell}{\prod_{i=1}^{H'}(h'_i)_\ell}, \quad \Phi_2(\ell) = \frac{\prod_{i=1}^{M}(m_i)_\ell}{\prod_{i=1}^{M'}(m'_i)_\ell}, \quad \Phi_3(\ell) = \frac{\prod_{i=1}^{N}(n_i)_\ell}{\prod_{i=1}^{N'}(n'_i)_\ell},$$

$$\Phi_4(\ell) = \frac{\prod_{i=1}^{P}(p_i)_\ell}{\prod_{i=1}^{P'}(p'_i)_\ell}, \quad \Phi_5(\ell) = \frac{\prod_{i=1}^{Q}(q_i)_\ell}{\prod_{i=1}^{Q'}(q'_i)_\ell}, \quad \Phi_6(\ell) = \frac{\prod_{i=1}^{R}(r_i)_\ell}{\prod_{i=1}^{R'}(r'_i)_\ell},$$

$$\Psi_1(\ell) = \frac{\prod_{i=1}^{S}(s_i)_\ell}{\prod_{i=1}^{S'}(s'_i)_\ell}, \quad \Psi_2(\ell) = \frac{\prod_{i=1}^{T}(t_i)_\ell}{\prod_{i=1}^{T'}(t'_i)_\ell}, \quad \Psi_3(\ell) = \frac{\prod_{i=1}^{U}(u_i)_\ell}{\prod_{i=1}^{U'}(u'_i)_\ell}, \quad \Psi_4(\ell) = \frac{\prod_{i=1}^{W}(w_i)_\ell}{\prod_{i=1}^{W'}(w'_i)_\ell}$$

in general quadruple series identity (29), applying some algebraic properties of Pochhammer symbols and after simplification, we obtain:

$$F_M^{(4)}\left[\begin{array}{l} (a_A) ::: (b_B) ; (d_D) ; (e_E) ; (g_G) :: (h_H) ; (m_M) ; (n_N) ; \\ (a'_{A'}) ::: (b'_{B'}) ; (d'_{D'}) ; (e'_{E'}) ; (g'_{G'}) :: (h'_{H'}) ; (m'_{M'}) ; (n'_{N'}) ; \end{array}\right.$$

$$\left.\begin{array}{l} (p_P) ; (q_Q) ; (r_R) : (s_S) ; (t_T) ; (u_U) ; (w_W) ; \\ (p'_{P'}) ; (q'_{Q'}) ; (r'_{R'}) : (s'_{S'}) ; (t'_{T'}) ; (u'_{U'}) ; (w'_{W'}) ; \end{array} \frac{\alpha x}{t^m}, \frac{\beta y}{t}, \mu ut, \frac{\lambda z}{t^r}\right]$$

$$= \sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}\sum_{\ell,s=0}^{m\ell+rs\leq j+n^\star} \frac{\prod_{i=1}^{A}(a_i)_{2j+2n^\star+n-(m-1)\ell-(r-1)s}}{\prod_{i=1}^{A'}(a'_i)_{2j+2n^\star+n-(m-1)\ell-(r-1)s}} \frac{\prod_{i=1}^{B}(b_i)_{2j+2n^\star+n-(m-1)\ell-rs}}{\prod_{i=1}^{B'}(b'_i)_{2j+2n^\star+n-(m-1)\ell-rs}}$$

$$\times \frac{\prod_{i=1}^{D}(d_i)_{2j+2n^\star+n-m\ell-(r-1)s}}{\prod_{i=1}^{D'}(d'_i)_{2j+2n^\star+n-m\ell-(r-1)s}} \frac{\prod_{i=1}^{E}(e_i)_{j+n^\star+n+s+\ell}}{\prod_{i=1}^{E'}(e'_i)_{j+n^\star+n+s+\ell}} \frac{\prod_{i=1}^{G}(g_i)_{j+n^\star-(m-1)\ell-(r-1)s}}{\prod_{i=1}^{G'}(g'_i)_{j+n^\star-(m-1)\ell-(r-1)s}}$$

$$\times \frac{\prod_{i=1}^{H}(h_i)_{j+n^\star-(m-1)\ell-rs}}{\prod_{i=1}^{H'}(h'_i)_{j+n^\star-(m-1)\ell-rs}} \frac{\prod_{i=1}^{M}(m_i)_{\ell+j+n^\star+n}}{\prod_{i=1}^{M'}(m'_i)_{\ell+j+n^\star+n}} \frac{\prod_{i=1}^{N}(n_i)_{\ell+s}}{\prod_{i=1}^{N'}(n'_i)_{\ell+s}} \frac{\prod_{i=1}^{P}(p_i)_{2j+2n^\star+n-m\ell-rs}}{\prod_{i=1}^{P'}(p'_i)_{2j+2n^\star+n-m\ell-rs}}$$

$$\times \frac{\prod_{i=1}^{Q}(q_i)_{j+n^\star-m\ell-(r-1)s}}{\prod_{i=1}^{Q'}(q'_i)_{j+n^\star-m\ell-(r-1)s}} \frac{\prod_{i=1}^{R}(r_i)_{j+n^\star+n+s}}{\prod_{i=1}^{R'}(r'_i)_{j+n^\star+n+s}} \frac{\prod_{i=1}^{S}(s_i)_{\ell}}{\prod_{i=1}^{S'}(s'_i)_{\ell}} \frac{\prod_{i=1}^{T}(t_i)_{j+n^\star-m\ell-rs}}{\prod_{i=1}^{T'}(t'_i)_{j+n^\star-m\ell-rs}}$$

$$\times \frac{\prod_{i=1}^{U}(u_i)_{j+n^\star+n}}{\prod_{i=1}^{U'}(u'_i)_{j+n^\star+n}} \frac{\prod_{i=1}^{W}(w_i)_{s}}{\prod_{i=1}^{W'}(w'_i)_{s}} \frac{(\alpha x)^{\ell}\,(\beta y)^{j+n^\star-m\ell-rs}\,(\mu u)^{j+n^\star+n}\,(\lambda z)^{s}}{\ell!\,(j+n^\star-m\ell-rs)!\,(j+n^\star+n)!\,s!}\, t^n$$

$$= \sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty} \frac{\prod_{i=1}^{A}(a_i)_{2j+2n^\star+n}}{\prod_{i=1}^{A'}(a'_i)_{2j+2n^\star+n}} \frac{\prod_{i=1}^{B}(b_i)_{2j+2n^\star+n}}{\prod_{i=1}^{B'}(b'_i)_{2j+2n^\star+n}} \frac{\prod_{i=1}^{D}(d_i)_{2j+2n^\star+n}}{\prod_{i=1}^{D'}(d'_i)_{2j+2n^\star+n}} \frac{\prod_{i=1}^{E}(e_i)_{j+n^\star+n}}{\prod_{i=1}^{E'}(e'_i)_{j+n^\star+n}} \frac{\prod_{i=1}^{G}(g_i)_{j+n^\star}}{\prod_{i=1}^{G'}(g'_i)_{j+n^\star}}$$

$$\times \frac{\prod_{i=1}^{H}(h_i)_{j+n^\star}}{\prod_{i=1}^{H'}(h'_i)_{j+n^\star}} \frac{\prod_{i=1}^{M}(m_i)_{j+n^\star+n}}{\prod_{i=1}^{M'}(m'_i)_{j+n^\star+n}} \frac{\prod_{i=1}^{P}(p_i)_{2j+2n^\star+n}}{\prod_{i=1}^{P'}(p'_i)_{2j+2n^\star+n}} \frac{\prod_{i=1}^{Q}(q_i)_{j+n^\star}}{\prod_{i=1}^{Q'}(q'_i)_{j+n^\star}} \frac{\prod_{i=1}^{R}(r_i)_{j+n^\star+n}}{\prod_{i=1}^{R'}(r'_i)_{j+n^\star+n}} \frac{\prod_{i=1}^{T}(t_i)_{j+n^\star}}{\prod_{i=1}^{T'}(t'_i)_{j+n^\star}}$$

$$\times \frac{\prod_{i=1}^{U}(u_i)_{j+n^\star+n}}{\prod_{i=1}^{U'}(u'_i)_{j+n^\star+n}} \frac{(\beta y)^{j+n^\star}\,(\mu u)^{j+n^\star+n}}{(j+n^\star)!\,(j+n^\star+n)!}$$

$$\times \sum_{\ell,s=0}^{m\ell+rs\leq j+n^\star} \frac{\prod_{i=1}^{A'}(1-a'_i-2j-2n^\star-n)_{(m-1)\ell+(r-1)s}}{\prod_{i=1}^{A}(1-a_i-2j-2n^\star-n)_{(m-1)\ell+(r-1)s}} \frac{\prod_{i=1}^{B'}(1-b'_i-2j-2n^\star-n)_{(m-1)\ell+rs}}{\prod_{i=1}^{B}(1-b_i-2j-2n^\star-n)_{(m-1)\ell+rs}}$$

$$\times \frac{\prod_{i=1}^{D'}(1-d'_i-2j-2n^\star-n)_{m\ell+(r-1)s}}{\prod_{i=1}^{D}(1-d_i-2j-2n^\star-n)_{m\ell+(r-1)s}} \frac{\prod_{i=1}^{E}(e_i+j+n^\star+n)_{\ell+s}}{\prod_{i=1}^{E'}(e'_i+j+n^\star+n)_{\ell+s}} \frac{\prod_{i=1}^{G'}(1-g'_i-j-n^\star)_{(m-1)\ell+(r-1)s}}{\prod_{i=1}^{G}(1-g_i-j-n^\star)_{(m-1)\ell+(r-1)s}}$$

$$\times \frac{\prod_{i=1}^{H'}(1-h'_i-j-n^\star)_{(m-1)\ell+rs}}{\prod_{i=1}^{H}(1-h_i-j-n^\star)_{(m-1)\ell+rs}} \frac{\prod_{i=1}^{N}(n_i)_{\ell+s}}{\prod_{i=1}^{N'}(n'_i)_{\ell+s}} \frac{\prod_{i=1}^{P'}(1-p'_i-2j-2n^\star-n)_{m\ell+rs}}{\prod_{i=1}^{P}(1-p_i-2j-2n^\star-n)_{m\ell+rs}}$$

$$\times \frac{\prod_{i=1}^{Q'}(1-q'_i-j-n^\star)_{m\ell+(r-1)s}}{\prod_{i=1}^{Q}(1-q_i-j-n^\star)_{m\ell+(r-1)s}} \frac{\prod_{i=1}^{T'}(1-t'_i-j-n^\star)_{m\ell+rs}}{\prod_{i=1}^{T}(1-t_i-j-n^\star)_{m\ell+rs}} (-j-n^\star)_{m\ell+rs}$$

$$\times \frac{\prod_{i=1}^{M}(m_i+j+n^\star+n)_\ell}{\prod_{i=1}^{M'}(m'_i+j+n^\star+n)_\ell} \frac{\prod_{i=1}^{S}(s_i)_\ell}{\prod_{i=1}^{S'}(s'_i)_\ell} \frac{\prod_{i=1}^{R}(r_i+j+n^\star+n)_s}{\prod_{i=1}^{R'}(r'_i+j+n^\star+n)_s} \frac{\prod_{i=1}^{W}(w_i)_s}{\prod_{i=1}^{W'}(w'_i)_s}$$

$$\times(-1)^{[(A-A'+B-B'+G-G'+H-H')(m-1)+(D-D'+P-P'+Q-Q'+T-T'+1)m]\ell} \frac{\left(\frac{\alpha x}{(\beta y)^m}\right)^\ell}{\ell!}$$

$$\times(-1)^{[(A-A'+D-D'+G-G'+Q-Q')(r-1)+(B-B'+H-H'+P-P'+T-T'+1)r]s} \frac{\left(\frac{\lambda z}{(\beta y)^r}\right)^s}{s!} t^n. \tag{36}$$

On using definition of the Srivastava and Daoust double hypergeometric function (1) in the r.h.s. of equation (36), we obtain assertion (35).

□

Remark 3.1. *Taking* $A = B = D = E = G = H = M = N = P = Q = R = A' = B' = D' = E' = G' = H' = M' = N' = P' = Q' = R' = 0$ *in equation* (35)*, we deduce the following consequence of Theorem 3.1:*

Corollary 3.1. *For* $S \leq S'+1$, $T \leq T'+1$, $U \leq U'+1$, $W \leq W'+1$ *and* $t \neq 0$*, the following generating function (in terms of the product of four generalized hypergeometric functions of one variable) for the Srivastava-Daoust double hypergeometric function holds true:*

$${}_SF_{S'}\left[\begin{array}{ll}(s_S) ; & \\ & \frac{\alpha x}{t^m} \\ (s'_{S'}) ; & \end{array}\right] {}_TF_{T'}\left[\begin{array}{ll}(t_T) ; & \\ & \frac{\beta y}{t} \\ (t'_{T'}) ; & \end{array}\right] {}_UF_{U'}\left[\begin{array}{ll}(u_{U'}) ; & \\ & \mu ut \\ (u'_{U'}) ; & \end{array}\right] {}_WF_{W'}\left[\begin{array}{ll}(w_W) ; & \\ & \frac{\lambda z}{t^r} \\ (w'_{W'}) ; & \end{array}\right]$$

$$= \sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty} \frac{(\beta y)^{j+n^\star}(\mu u)^{j+n^\star+n} \prod_{i=1}^{T}(t_i)_{j+n^\star} \prod_{i=1}^{U}(u_i)_{j+n^\star+n}}{(j+n^\star)!\,(j+n^\star+n)! \prod_{i=1}^{T'}(t'_i)_{j+n^\star} \prod_{i=1}^{U'}(u'_i)_{j+n^\star+n}}$$

$$\times F^{1+T':S;\;W}_{\;T\;\;:S';W'}\left(\begin{array}{lll}[1-(t'_{T'})-j-n^\star:m,r],\ [-j-n^\star:m,r] & : & [(s_S):1] \;; \\ [1-(t_T)-j-n^\star:m,r] & : & [(s'_{S'}):1] \;; \end{array}\right.$$

$$\left.\begin{array}{ll}[(v_V):1] \;; & \\ & (-1)^{m(T-T'+1)}\frac{(\alpha x)}{(\beta y)^m},\ (-1)^{r(T-T'+1)}\frac{(\lambda z)}{(\beta y)^r} \\ [(w_W):1] \;; & \end{array}\right) t^n, \tag{37}$$

where m and r are positive integers and $n^\star$ is defined equation (30).

4. Applications

In this section, we consider certain special cases of our main result.

I. Taking $x=0$, $T=T'=0$, $\beta=-1$, $y\to x$; $U=U'=0$, $\mu=u=1$; $W=W'=0$, $\lambda=1$, $r=2$, $z\to y$ in equation (37), we get

$$\exp\left(t-\frac{x}{t}+\frac{y}{t^2}\right)=\sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}\frac{(-x)^{j+n^\star}}{(j+n^\star)!\,(j+n^\star+n)!}\ {}_2F_0\left[\begin{array}{ccc}\frac{-j-n^\star}{2},\ \frac{-j-n^\star+1}{2} & ; & \\ & & \frac{4y}{x^2}\\ — & ; & \end{array}\right]t^n, \tag{38}$$

which in view of equation (9) can be represented as:

$$\exp\left(t-\frac{x}{t}+\frac{y}{t^2}\right)=\sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}\frac{(i\sqrt{y})^{j+n^\star}}{(j+n^\star)!\,(j+n^\star+n)!}\ H_{j+n^\star}\left(\frac{ix}{2\sqrt{y}}\right)t^n. \tag{39}$$

Comparing equation (39) with the generating function (13) of Dattoli *et al.*, we find that the 2-variable Hermite-Tricomi functions ${}_HC_n(x,y)$ have the following series definition in terms of the Hermite polynomials $H_n(x)$:

$${}_HC_n(x,y)=\sum_{j=0}^{\infty}\frac{(i\sqrt{y})^{j+n^\star}}{(j+n^\star)!\,(j+n^\star+n)!}\ H_{j+n^\star}\left(\frac{ix}{2\sqrt{y}}\right), \tag{40}$$

which is the modified form of equation (20) and $n^\star$ is defined by equation (30).

II. Taking $S=0$, $S'=1$, $s'_1=1$, $\alpha=(-1)^m$; $T=T'=0$, $\beta=-1$, $U=U'=0$, $\mu=u=1$, $W=W'=0$, $\lambda=(-1)^r$ in equation (37), we get

$$C_0\left[(-1)^{m-1}\frac{x}{t^m}\right]\exp\left[\left(t-\frac{y}{t}\right)+(-1)^r\frac{z}{t^r}\right]=\sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}\frac{(-y)^{j+n^\star}}{(j+n^\star)!\,(j+n^\star+n)!}$$

$$\times F^{1:0;0}_{0:1;0}\left(\begin{array}{ccccc}[-j-n^\star:m,r] & : & — & ; & — ;\\ & & & & \frac{x}{(-y)^m},\ \frac{z}{(-y)^r}\\ — & : & [1:1] & ; & — ;\end{array}\right)t^n. \tag{41}$$

Comparing equation (41) with the generating function (23) of Khan *et al.*, we find that the Laguerre-Gould-Hopper-Tricomi functions ${}_{LH^{(m,r)}}C_n(x,y,z)$ have the following series definition in terms of the Srivastava-Daoust hypergeometric function:

$${}_{LH^{(m,r)}}C_n(x,y,z)=\sum_{j=0}^{\infty}\frac{(-y)^{j+n^\star}}{(j+n^\star)!\,(j+n^\star+n)!}$$

$$\times F^{1:0;0}_{0:1;0}\left(\begin{array}{ccccc}[-j-n^\star:m,r] & : & — & ; & — ;\\ & & & & \frac{x}{(-y)^m},\ \frac{z}{(-y)^r}\\ — & : & [1:1] & ; & — ;\end{array}\right), \tag{42}$$

which is the modified form of the equation (25) and $n^\star$ is defined by equation (30).

5. Conclusions

In this article, we have established the modified forms of the series definitions of the hybrid Tricomi functions of Dattoli *et. al.* and Khan *et al.* given by equations (20) and (25), respectively. The modifications are in the sense that we have introduced the concept of $n^\star$ which justify the authenticity of $(n+j)!$ present in the denominator of equations (20) and (25).

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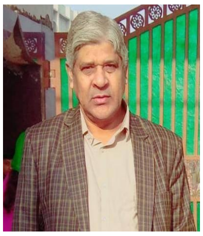

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