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CUBIC B-SPLINE COLLOCATION METHOD FOR COUPLED SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS WITH VARIOUS BOUNDARY CONDITIONS

KHIMYA AMLANI[1], MANOJ KUMAR[2], ANAND SHUKLA[3], §

Abstract. This paper is concerned with collocation approach using cubic B-spline to solve coupled system of boundary value problems with various boundary conditions. The collocation equations are methodically derived using cubic B splines, for problems with Dirichlet data and an iterative method with assured convergence is described to solve the resulting system of algebraic equations. Problems with Cauchy or mixed boundary condition have been converted into series of Dirichlet problems using the bisection method. Nonlinear problem is linearized using quasilinearization to be handled by our method. Fourth order equation is converted into a coupled second order equations and solved by the proposed method . Several illustrative examples are presented with their error norms and order of convergence.

Keywords: Collocation method, Cubic B-spline, Quasilinearization, Bisection method, Boundary Value Problem.

AMS Subject Classification: 83-02, 99A00.

1. Introduction

Many problems in physics, chemistry, biology and engineering science are modelled mathematically by system of ordinary differential equations. Most realistic system of differential equations do not have exact analytic solution hence approximate and numerical techniques are being demanded. Various numerical methods have been developed for solution of such systems of boundary value problems. The existence and uniqueness of solution of system of second order boundary value problem including the approximation of solution via finite difference equations are addressed in [1, 2, 3]. For Nonlinear system of second order BVPs, J.Lu [7] proposed variational iteration method , M.Dehghan and M. Lakestani

[1] Applied Science and Humanities Department, Sardar Vallabhbhai Patel Institute of Technology, Vasad-388 306, Gujarat, India.
e-mail: khimya_amlani@rediffmail.com; ORCID: https://orcid.org/0000-0003-0034-2388.
[2] Department of Mathematics, Motilal Nehru National Institute of Technology, Allahabad-211004 (U.P.), India.
e-mail: manoj@mnnit.ac.in; ORCID: https://orcid.org/0000-0001-5330-0160.
[3] Department of Mathematics, Wollega University, Ethiopia.
e-mail: anandshukla86@gmail.com; ORCID: https://orcid.org/0000-0003-2300-5376.
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[10] solved by cubic B-spline scaling functions, Saadatmandi et al. [11] employed a homotopy perturbation method,Geng and Cui [5] presented analytical solution in the form of series by reproducing kernel space, Malik Zaka Ullah [25] studied efficient multi-step iterative method. S.A. Khuri and A.Safy [13], [27] presented numerical methods based on spline collocation to solve the linear and nonlinear system of second order boundary value problems. K.N.S. Kasi [14] used Galerkin method for the coupled system of boundary value problems. N. Caglar and H. Caglar [12] proposed cubic B-spline collocation method to solve the linear system. M. El-Gamel [18] used sinc-collocation method, M. Dehghan, A. Nikpour [22] solved the system of second order boundary value problems using the local radial basis functions based differential quadrature collocation method. Kumar et al. [20] solved singularly perturbed non−linear elliptic BVPs using finite element method Y. Gupta et al. [17] solved system of singularly Perturbed Problems using B-spline. S. Kutluay [28] discussed the solution of Modified Burger's Equation by cubic B-spline collocation method. M. Munguia [26] described cubic B-spline in approximating the solution of boundary value problems. Polynomials as well as non-polynomials splines of various degrees have been used to solve BVPs[6, 15, 16] .

This article is devoted to collocation method using cubic B-spline for solving the coupled system of ordinary differential equations.

$$\begin{aligned} U'' &= F(x, U, U', V, V') \\ V'' &= G(x, U, U', V, V') \end{aligned} \tag{1}$$

With boundary conditions

$$U(a) = \alpha,\ U(b) = \beta,\ V(a) = \gamma,\ V(b) = \delta \tag{2}$$

Some physical problems while formulated mathematically the boundary conditions can be of the type

$$U(a) = \alpha,\ U(b) = \beta,\ V(a) = \gamma,\ U'(b) = \delta \tag{3}$$

The fourth order problem $W^{(iv)} = H(x, W, W', W'', W''')$ is usually solved by quintic spline but in our work we have rewritten the problem as system of two second order ordinary differential equations as

$$\begin{aligned} W'' &= U \\ U'' &= H(x, W, W', U, U') \end{aligned} \tag{4}$$

and uses the cubic B-spline to approximate the solution.

Present study is close to N. Caglar and H. Caglar[12] only in the sense of presenting the equations arising by the application of cubic B-spline collocation method. Large number of equations arises in this collocation and solved directly by them. But in this paper we have derived the equations by collocation method in the same manner and develop iterative method for solving them with assured convergence. Further, unlike the earlier efforts, our method of presenting the equations and solution procedure can be used for problems with various boundary conditions.

The organization of paper is as follows. Section 2 describes basic cubic B-spline properties necessary for collocation method. Section 3 presents cubic B-spline colloction method for linear system and iterative method for solving the algebraic system in the form of an algorithm. Error norms and order of convergence are discussed in section 4. In section 5, equation(1) is linearized by quasilinearization technique and solved using the method developed in section 3 and problems with various conditions has been handeled. In section 6 the linear problem with cauchy boundary condition given in equation(3) is considered and well known simple Bisection method comes handy to solve this kind of problem under our framework of section 3. Finally in section 7, the following problem arising in fluid dynamics[23]

$$F^{(iv)}(z) - m^2 F^{''}(z) + ReF(z)F^{'''}(z) = 0 \tag{5}$$

with boundary conditions

$$F(0) = 0, \; F(1) = 1, \; F^{'}(1) = 0, \; F^{''}(0) = 0$$

is treated efficiently by cubic B-spline collocation method by writing as two second order equations followed by the use of quasilinearization technique to linearize it and then solved as in section 6.

2. Cubic B-spline Collocation Method

We present the prelimnaries of collocation approach using cubic B-spline for the numerical solution of coupled system of boundary value problem given in (1) as described in [13, 14, 22].

Subdivide the interval [a,b] and choose piecewise uniform grid points represented by $\prod : x_0 < x_1 < x_2 < ... < x_n$, such that $x_0 = a, x_n = b$ and h is the piecewise uniform spacing. Let $S_3(\prod)$ be the space of cubic spline functions over the partition $\prod$. We can define the cubic B−spline basis functions $B_j(x)$, for $j = -1, 0, 1, ..., n+1$, for $S_3(\prod)$ after including two more points on each side of the partition $(\prod)$. Thus the partition $\prod$ becomes

$\prod : x_{-2} < x_{-1} < x_0 < ... < x_n < x_{n+1} < x_{n+2}$

Now, the cubic B-spline basis function is defined as ,

$$B_j(x) = \frac{1}{h^3} \begin{cases} (x - x_{i-2})^3 & \text{if } x \in [x_{i-2}, x_{i-1}] \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3 & \text{if } x \in [x_{i-1}, x_i] \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3 & \text{if } x \in [x_i, x_{i+1}] \\ (x_{i+2} - x)^3 & \text{if } x \in [x_{i+1}, x_{i+2}] \\ 0 & Otherwise \end{cases}$$

It can easily be verified that each of the functions $B_j(x)$ is twice continuously differentiable on the entire interval [4, 9, 21].
The cubic B-spline collocation method starts by dividing the interval into n equal parts where $h = (b - a)/n$, and approximating the underlying function as

$$U(x) = \sum_{j=-1}^{n+1} a_j \; B_j(x) \qquad and \qquad V(x) = \sum_{j=-1}^{n+1} b_j \; B_j(x) \tag{6}$$

where $a_j' s$ and $b_j' s$ are unknown co-efficients to be determined by the collocation method using the boundary conditions. The values of $B_j(x)$, $B_j'(x)$ and $B_j''(x)$ at different knots are given below. (From [19])

$$B_j(x_k) = \begin{cases} 4 & \text{if } j = k \\ 1 & \text{if } j-k = \pm 1 \\ 0 & \text{if } j-k = \pm 2 \end{cases} \tag{7}$$

$$B'_j(x_k) = \begin{cases} 0 & \text{if } j = k \\ \pm\frac{3}{h} & \text{if } j-k = \pm 1 \\ 0 & \text{if } j-k = \pm 2 \end{cases} \tag{8}$$

$$B''_j(x_k) = \begin{cases} -\frac{12}{h^2} & \text{if } j = k \\ \frac{6}{h^2} & \text{if } j-k = \pm 1 \\ 0 & \text{if } j-k = \pm 2 \end{cases} \tag{9}$$

3. Cubic B-spline Collocation Method For Linear Problem With Dirichlet Boundary Conditions

We consider the linear system [12]

$$\begin{aligned} U'' &= p_1(x)\ U' + p_2(x)\ U + p_3(x) + p_4(x)\ V + p_5(x)\ V' \\ V'' &= q_1(x)\ V' + q_2(x)\ V + q_3(x) + q_4(x)\ U + q_5(x)\ U' \end{aligned} \tag{10}$$

along with boundary conditions

$$U(a) = \alpha,\ U(b) = \beta,\ V(a) = \gamma,\ V(b) = \delta$$

Substituting (6) in (10) and collocating at x = x_k.

$$\begin{aligned} &\sum_{j=-1}^{n+1} a_j\ B''_j(x_k) - p_1(x) \sum_{j=-1}^{n+1} a_j\ B'_j(x_k) - p_2(x) \sum_{j=-1}^{n+1} a_j\ B_j(x_k) - p_4(x) \\ &\qquad \times \sum_{j=-1}^{n+1} b_j\ B_j(x_k) - p_5(x) \sum_{j=-1}^{n+1} b_j\ B'_j(x_k) = p_3(x), \\ &\sum_{j=-1}^{n+1} b_j\ B''_j(x_k) - q_1(x) \sum_{j=-1}^{n+1} b_j\ B'_j(x_k) - q_2(x) \sum_{j=-1}^{n+1} b_j\ B_j(x_k) - q_4(x) \\ &\qquad \times \sum_{j=-1}^{n+1} a_j\ B_j(x_k) - q_5(x) \sum_{j=-1}^{n+1} a_j\ B'_j(x_k) = q_3(x) \\ &\qquad\qquad k = 0, 1, 2, ..., n \end{aligned} \tag{11}$$

By substituting the expressions of $B_j(x_k)$, $B'_j(x_k)$ and $B''_j(x_k)$ from (7), (8) and (9) respectively at x = x_k we obtain,

$$a_{k-1}TP_1(x_k) + a_kTP_2(x_k) + a_{k+1}TP_3(x_k) - b_{k-1}TP_4(x_k) - b_kTP_5(x_k) - b_{k+1}TP_6(x_k) = p_3(x_k)$$
$$b_{k-1}TQ_1(x_k) + b_kTQ_2(x_k) + b_{k+1}TQ_3(x_k) - a_{k-1}TQ_4(x_k) - a_kTQ_5(x_k) - a_{k+1}TQ_6(x_k) = q_3(x_k)$$
$$k = 0, 1, 2, ..., n. \tag{12}$$

where,

$$\begin{aligned}
TP_1(x_k) &= \frac{6}{h^2} + \frac{3}{h}p_1(x_k) - p_2(x_k)\\
TP_2(x_k) &= \frac{-12}{h^2} - 4p_2(x_k)\\
TP_3(x_k) &= \frac{6}{h^2} - \frac{3}{h}p_1(x_k) - p_2(x_k)\\
TP_4(x_k) &= p_4(x_k) - \frac{3}{h}p_5(x_k)\\
TP_5(x_k) &= 4p_4(x_k)\\
TP_6(x_k) &= p_4(x_k) + \frac{3}{h}p_5(x_k)
\end{aligned}$$

and ,

$$\begin{aligned}
TQ_1(x_k) &= \frac{6}{h^2} + \frac{3}{h}q_1(x_k) - q_2(x_k)\\
TQ_2(x_k) &= \frac{-12}{h^2} - 4q_2(x_k)\\
TQ_3(x_k) &= \frac{6}{h^2} - \frac{3}{h}q_1(x_k) - q_2(x_k)\\
TQ_4(x_k) &= q_4(x_k) - \frac{3}{h}q_5(x_k)\\
TQ_5(x_k) &= 4q_4(x_k)\\
TQ_6(x_k) &= q_4(x_k) + \frac{3}{h}q_5(x_k)
\end{aligned}$$

at $x = x_0$

$$U(x_0) = \alpha \quad \& \quad V(x_0) = \gamma$$

$$\sum_{j=-1}^{n+1} a_j\ B_j(x_0) = \alpha \quad \& \quad \sum_{j=-1}^{n+1} b_j\ B_j(x_0) = \gamma$$

$$a_{-1} = \alpha - 4a_0 - a_1 \quad \& \quad b_{-1} = \gamma - 4b_0 - b_1$$

Eliminating a_{-1} and b_{-1} from the equations of (11) with $k = 0$ we obtain,

$$\begin{aligned}
a_0(\ T\ P_2 - 4\ T\ P_1) + a_1(T\ P_3 -\ T\ P_1) + b_0(4\ T\ P_4 - T\ P_5) + b_1(\ T\ P_4 - T\ P_6)\\
= p_3(\ x_0) - \alpha\ T\ P_1 + \gamma\ T\ P_4\\
b_0(\ T\ Q_2 - 4\ T\ Q_1) + b_1(T\ Q_3 -\ T\ Q_1) + a_0(4\ T\ Q_4 - T\ Q_5) + a_1(\ T\ Q_4 - T\ Q_6)\\
= q_3(\ x_0) - \alpha\ T\ Q_1 + \gamma\ T\ Q_4
\end{aligned} \tag{13}$$

at $x = x_n$

$$U(x_n) = \beta \qquad \& \qquad V(x_n) = \delta$$

$$\sum_{j=-1}^{n+1} a_j \ B_j(x_n) = \beta \quad \& \quad \sum_{j=-1}^{n+1} b_j \ B_j(x_n) = \delta$$

$$a_{n+1} = \beta - 4a_n - a_{n-1} \ \& \ \ b_{n+1} = \delta - 4b_n - b_{n-1}$$

Eliminating a_{n+1} and b_{n+1} from the equations of (11) we obtain,

$$\begin{aligned} a_{n-1}(\ T\ P_1 - T\ P_3) + a_n(\ T\ P_2 - 4\ T\ P_3) + b_{n-1}(T\ P_6 -\ T\ P_4) + b_n(4\ T\ P_6 -\ T\ P_5) \\ = p_3(x_n) - \beta\ T\ P_3 + \delta\ T\ P_6 \\ b_{n-1}(\ T\ Q_1 - T\ Q_3) + b_n(\ T\ Q_2 - 4\ T\ Q_3) + a_{n-1}(T\ Q_6 -\ T\ Q_4) + a_n(4\ T\ Q_6 -\ T\ Q_5) \\ = q_3(x_n) - \beta\ T\ Q_3 + \delta\ T\ Q_6 \end{aligned} \tag{14}$$

Equation (12) for $k = 1, 2, \ldots, n-1$ and boundary equations (13) and (14) together can be written in the matrix vector form as

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} = \begin{bmatrix} \tilde{c} \\ \tilde{d} \end{bmatrix}$$

$$A_1\ \tilde{a} + A_2\ \tilde{b} = \tilde{c}.$$
$$A_3\ \tilde{a} + A_4\ \tilde{b} = \tilde{d}.$$

3.1. **Iterative Algorithm for solving the system.** We solve this system by an iterative method presented stepwise as under
First we assume the initial approximation for $\tilde{b}$.
Step 1: Solve the tridiagonal system $A_1\ \tilde{a} = \tilde{c} - A_2\ \tilde{b}$.
Step 2: Solve the tridiagonal system $A_4\ \tilde{b} = \tilde{d} - A_3\ \tilde{a}$ with $\tilde{a}$ obtained from step 1.
Step 3: Check the difference brtween starting $\tilde{b}$ and $\tilde{b}$ of step 2. If the difference is not negligible go to step 1 with latest $\tilde{b}$.
This iterative method can be viewed as iteration defined by:

$$\tilde{a} = \tilde{e} + \mathbf{A}\tilde{a}$$

where,

$$\mathbf{A} = A_1^{-1}\ A_2\ A_4^{-1}\ A_3 \qquad and \qquad \tilde{e} = A_1^{-1}(\tilde{c} -\ A_2\ A_4^{-1}\tilde{d})$$

It may be noted that the matrices A_1 and A_4 are diagonally dominant and hence nonsingular.
To discuss the convergence of this iterative method, we consider the elements of $\mathbf{A}$. Elements of A_2 and A_3 contain terms with $\frac{1}{h}$and elements of A_1 and A_4 contain terms with $\frac{1}{h^2}$.Thus h^2 appears as a multiplying factor in the matrix $\mathbf{A}$. Norm of the iterative matrix contains h^2 as a factor. As spectral radius is less or equal to the norm for any given matrix, fast convergence is assured. Number of iterations is not more than three in all the problems considered in this paper.

4. Numerical experiments

Order of the method as an approximation to the continuous problem is estimated by the formula given in [24]
Let $u(h_1)$ and $u(h_2)$ be the numerical solution obtained with step size h_1 and h_2 respectively. If $E(h) = |u^{num} - u^{exact}| \approx ch^p$ with step size h then

$$\frac{E(h_1)}{E(h_2)} = \left(\frac{h_1}{h_2}\right)^p$$

$$Hence \qquad p \approx \frac{log(E(h_1)/E(h_2))}{log(h_1/h_2)}$$

This p is estimated as two in some of the examples considered in the paper.(See Ex. 1,2 and 3)
* It may be noted that if the variable under consideration is a polynomial of degree three or more , rate of convergence increases as there will not be any discretization error.(see Ex.2)

Example 1 : Consider the linear system

$$U''(x) + x^2\ U'(x) - x\ U(x) - x\ V'(x) + x^2\ V(x) = f_1(x)$$

$$V''(x) + x\ V'(x) - x^2\ V(x) + U'(x) = 2\ U(x) = f_2(x),\ 0 \le x \le 1$$

With boundary conditions

$$U(0) = \ 0,\ U(1) = \ 1,\ V(0) = \ 1,\ V(1) = \ 1 + e$$

where ,

$$\begin{aligned} f_1(x) &= e^x(1 + x + x^3) - 2x + x^3 \\ f_2(x) &= e^x(-1 + 4x - x^2) - x^3 + x + 2. \end{aligned}$$

The exact solution is

$$u(x) = e^x(x-1) + 1 \ \ and \ \ v(x) = e^x + x.$$

Here, $p_1 = -x^2, p_2 = x, p_3 = f_1, p_4 = -x^2, p_5 = x$

$q_1 = -x, q_2 = x^2, q_3 = f_2, q_4 = -2, q_5 = -1$

Table 1 and Table 2 shows numerically solution and maximum absolute error of present method respectively.

5. Cubic B-spline Collocation Method For Nonlinear Problem With Dirichlet Boundary Conditions

Nonlinear system (1) is linearized by quasilinearization technique [8] using initial approximation u and v satisfying the respective boundary conditions.

$$U'' = F(x, u, v, u', v') + (U - u)\frac{\partial F}{\partial u} + (U' - u')\frac{\partial F}{\partial u'} + (V - v)\frac{\partial F}{\partial v} + (V' - v')\frac{\partial F}{\partial v'}$$

$$V'' = G(x, u, v, u', v') + (U - u)\frac{\partial G}{\partial u} + (U' - u')\frac{\partial G}{\partial u'} + (V - v)\frac{\partial G}{\partial v} + (V' - v')\frac{\partial G}{\partial v'}.$$

The unknowns occurring in partial derivatives are replaced by the initial approximations. Thus we get a linear system and this system is solved by the method described in section 3 and 3.1. Three iterations are enough to achieve convergence.

Example 2 : Consider the nonlinear system [13]

$$3u^{''}(x) + 3\ x\ u(x) + 3\ v(x) - u'(x)\ v'(x) = 8 - x$$
$$v^{''}(x) + x^3\ u'(x) + x^2\ u(x) - \frac{x}{2}\ u'(x)\ v'(x) = x^3 + x^2 + 7\ x$$

with boundary conditions

$$u(0) = \ 0,\ u(1) = \ 3,\ v(0) = \ 0,\ v(1) = \ 0$$

The exact solution is

$$u(x) = x^2 + 2\ x \quad \text{and} \quad v(x) = x^3 - x.$$

This nonlinear boundary value problem is converted into a sequence of linear boundary value problem generated by quasilinearization technique with initial approximation $u(x) = 3\ x$ and $v(x) = 0$ satisfying the given boundary conditions. Here,

$$p_1 = \frac{v'}{3},\ p_2 = \ -x,\ p_3 = \frac{(-u'v' + 8 - x)}{3},\ p_4 = -1,\ p_5 = \frac{u'}{3}$$
$$q_1 = \frac{x}{2}\ u', q_2 = 0, q_3 = x^3 + x^2 + 7\ x - \frac{x}{2}\ u'v', q_4 = -x^2, q_5 = x\frac{v'}{2} - x^3$$

Table 3 and Table 4 shows numerically solution and maximum absolute error of present method respectively.

6. Cubic B-spline Collocation Method For Linear Problem with Cauchy Boundary conditions

To use the method discussed in previous sections for the purpose of handling this variant in boundary data, we need to know $V(b)$ rather than the condition $U'(b) = \delta$. By cubic B-spline collocation method $U'(b) = \frac{3}{h}(a_{n-1} - a_{n+1}) = \delta$. Let $g(b, \theta) = \delta - \frac{3}{h}(a_{n-1} - a_{n+1})$.We find the value of θ such that $g(b, \theta) = 0$.To find this value of θ Bisection method is used, starting with two values of θ so that value of the funcion g has of opposite sign. In this way the value of θ can be found upto desired accuracy and we solve the problem, by the method described in section 3 and 3.1

Example 3 : Consider the linear system [18]

$$\begin{aligned} u_1^{''} + (2\ x - 1)\ u_1' + \cos\pi\ xu_2' &= f_1(x) \\ u_2^{''} + xu_1 &= f_2(x), 0 \le x \le 1 \end{aligned}$$

with boundary conditions

$$u_1(0) = \ 0,\ u_1(1) = \ 0,\ u_2(0) = \ 0,\ u_1'(1) = \ 0$$

where,

$$\begin{aligned} f_1(x) &= -\pi^2 \sin\pi\ x + (2\ x - 1)(\pi + 1)\cos\pi\ x \\ f_2(x) &= 2 + \sin\pi\ x \end{aligned}$$

The exact solution is

$$u_1 = \sin \pi \ x \qquad and \qquad u_2 = x^2 - x.$$

Here, $p_1 = 1 - 2x$, $p_2 = 0$, $p_3 = f_1$, $p_4 = 0$, $p_5 = -\cos \pi x$
$q_1 = 0$, $q_2 = 0$, $q_3 = f_2$, $q_4 = -x$, $q_5 = 0$

Table 5 and Table 6 shows numerically solution and maximum absolute error of present method respectively.

7. Cubic B-spline Collocation Method For Nonlinear Problem With Neumann Boundary conditions

Consider the fluid dynamic problem (6) , writing as two second order equations (linearizing with $f = 1.5z - .5z^3$)

$$\begin{aligned} F^{''} &= G \\ G^{''} &= m^2 \ G + Ref \ f^{'''} - ReF \ f^{''} - Ref \ G' \end{aligned}$$

with boundary conditions

$$F(0) = \ 0, \ F(1) = \ 1, \ G(0) = \ 0, \ F'(1) = \ 0$$

This problem arises while describing squeezing flow between two parallel plates in the presence of a magnetic field [23]. The Navier Stoke's equation describing this flow can be reduced to the above equations by introducing the stream function. Here m^2 is the magnetic interaction parameter and Re is the Reynolds number. We solve the nonlinear equation by quasilinearization technique until convergence obtained. And if we know $G(1) = \theta$, problem fits into our formulation. Thus we take two values of θ and solve the problem by the method described in section 6. Here,

$$\begin{aligned} &p_1 = 0, \ p_2 = 0, \ p_3 = 0, \ p_4 = 1, \ p_5 = 0 \\ &q_1 = 0, \ q_2 = m^2 - Re \ f', \ q_3 = Re \ f' \ g', \ q_4 = -Re \ G', \ q_5 = 0 \end{aligned}$$

Numerical results for different Reynolds number and magnetic interaction parameters are obtained by the proposed method and compared with approximate solution $(Table 7, 8, 9)$. Generally it is not possible to find the exact solution of these problems.

8. Conclusion

It is the first time that the algebraic equations arising out collocation method to solve coupled system of two point boundary value problems are solved by an iterative method with assured convergence. Variations of boundary conditions are handled by the same technique using simple method of bisection. Fourth order problem is handled by writing it as two second order equations with high accuracy in the results. The performance of the method has been evaluated by considering test problems and calculating the error norms. The rate of convergence is also calculated and found to be second order convergent. The results are found to be in good agreement with the analytical solution and converges within two or three iterations.Theoretical study of the efficiency of such an approach is desirable.

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Table 1: Numerical Solution obtained by proposed method of Example 1.

	N=10		N=20		N=40	
X	U	V	U	V	U	V
0	0	1	0	1	0	1
0.2	0.02254	1.42119	0.02279	1.42135	0.02286	1.42139
0.4	0.10436	1.89153	0.10477	1.89175	0.10487	1.89181
0.6	0.27056	2.42186	0.27100	2.42205	0.27112	2.42210
0.8	0.55447	3.02540	0.55479	3.02551	0.55487	3.02553
1	1	3.71828	1	3.71828	1	3.71828

Table 2: Maximum Absolute Error and Order of Convergence of Example 1.

N	L_2		L_∞		ROC	
	U	V	U	V	U	V
10	4.34E-04	2.07E-04	5.92E-04	2.95E-04	-	-
20	7.74E-05	3.75E-05	1.52E-04	7.47E-05	1.96	1.98
40	1.35E-05	6.98E-06	3.94E-05	1.88E-05	1.95	1.98

Table 3: Numerical Solution obtained by proposed method of Example 2.

	N=10		N=20		N=40	
X	U	V	U	V	U	V
0	0	0	0	0	0	0
0.2	0.44260	-0.19201	0.44066	-0.19200	0.44030	-0.19200
0.4	0.96199	-0.33602	0.96051	-0.33600	0.96023	-0.33600
0.6	1.56137	-0.38402	1.56035	-0.38400	1.56016	-0.38400
0.8	2.24072	-0.28801	2.24018	-0.28800	2.24008	-0.28800
1	3	0	3	0	3	0

Table 4: Maximum Absolute Error and Order of Convergence of Example 2.

N	L_2	L_∞	ROC

	U	V	U	V	U	V
10	1.75E-03	1.45E-05	2.91E-03	2.00E-05	-	-
20	1.60E-03	0	7.04E-04	0	1.2745	-
40	1.00E-04	0	3.30E-04	0	4.41503	-

Table 5 : Numerical solution obtained by present method of Example 3.

	N=10		N=20		N=40	
X	u_1	u_2	u_1	u_2	u_1	u_2
0	0	0	0	0	0	0
0.2	0.58352	-0.16019	0.58671	-0.16005	0.58730	-0.1600
0.4	0.94391	-0.24034	0.94926	-0.24009	0.95107	-0.2400
0.6	0.94391	-0.24037	0.94926	-0.24009	0.95107	-0.2400
0.8	0.58351	-0.16025	0.58671	-0.16006	0.58731	-0.1600
1	0	0	0	0	0	0

Table 6: Maximum Absolute Error and Order of Convergence of Example 3.

N	L_2		L_∞		ROC	
	u_1	u_2	u_1	u_2	u_1	u_2
10	5.26E-03	2.64E-04	7.54E-03	3.70E-04	-	-
20	9.34E-04	4.64E-05	1.90E-03	0.90E-04	1.99	2.04
40	1.07E-04	0	4.85E-04	0	1.97	-

Table 7 : Numerical result of fluid dynamic problem at $m = 1$ and $Re = 1$

X	Present Method			Approximate solution by ([23])
	N=10	N=20	N=40	
0.2	0.29460	0.29452	0.29450	0.29748
0.4	0.56600	0.56587	0.56585	0.570188
0.6	0.79044	0.79031	0.79029	0.79379
0.8	0.94345	0.94337	0.94336	0.94469
1	1	1	1	1

Table 8 : Numerical result of fluid dynamic problem at $m = 20$ and $Re = 1$

X	Present Method			Approximate solution by([23])
	N=10	N=20	N=40	
0.2	0.25140	0.25159	0.25164	0.21078
0.4	0.49696	0.49727	0.49736	0.42156
0.6	0.72589	0.72617	0.72627	0.63232
0.8	0.91334	0.91345	0.91349	0.84205
1	1	1	1	1

Table 9 : Numerical result of fluid dynamic problem at $m = 1$ and $Re = 10$

X	Present Method			Approximate solution by([23])
	N=10	N=20	N=40	
0.2	0.33534	0.33489	0.33482	0.32903
0.4	0.62928	0.62866	0.62857	0.61325
0.6	0.84614	0.84561	0.84554	0.82584
0.8	0.96690	0.96661	0.96657	0.90157
1	1	1	1	1

Khimya Amlani completed her M.Sc(Pure Mathematics) from Maharaja Sayajirao University of Baroda,Gujarat and Ph.D Degree in Mathematics from Veer Narmad South Gujarat University, Surat, Gujarat, India. Her area of research is numerical analysis. Presently she is working as an Assistant Professor at Applied Science and Humanities Department, Sardar Vallabhbahi Patel Institute of Technology, Vasad-388306, Gujarat, India.

Manoj Kumar is presently working as a professor and head at the Department of Mathematics at Motilal Nehru National Institute of Technology Allahabad, Prayagraj − 211004(U.P.) India. His area of research includes, Mathematical Modeling and Numerical Simulation, Finite element method, Neural Network, Adomian decomposition method. He has handled several research projects funded by DST, UGC, CSIR, NBHM, DBT and visited USA, UK, Switzerland, Canada, Italy, Greece, Malaysia, Singapore, Russia and Brazil

Anand Shukla obtained his M.Sc in Mathematics and Scientific Computing and Ph.D Degree in Mathematics from the Department of Mathematics at Motilal Nehru National Institute of Technology Allahabad, Prayagraj-211004 (U.P.) India. His area of area of research is numerical analysis and fluid dynamics. Presently he is working as an assistant professor in the Department of Mathematics at Wollega University, Ethiopia