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SOLUTION OF THE TWO-DIMENSIONAL TELEGRAPH EQUATION VIA THE LOCAL RADIAL BASIS FUNCTIONS METHOD

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Abstract. In the present study, we utilize the local meshless method for solving second order hyperbolic partial differential equation in two dimensions. First we apply the Crank-Nicolson difference scheme for the time derivative and the local radial basis functions (LRBFs) collocation method for the spatial derivative. The local approach breaks down the problem domain into subdomains and results small matrix system for each data. Some numerical examples are included to verify the computational efficiency of the proposed method.

Keywords: Radial basis function, Scattered data, Local RBF, Numerical partial differential equations

AMS Subject Classification: 83-02, 99A00

1. Introduction

Meshless (mesh-free) methods, as numerical techniques are used to analyze a wide range of industrial and engineering applications. In these methods, only scattered nodes are required to approximate unknown functions [1]. RBF is an interpolation method used for scattered data approximation. The coefficient matrix of global RBF functions will be full and large and usually it is ill-conditioned so to construct the approximation function by local radial basis functions, the only points are used fallen within local influence domain. Therefore, much less computational work and the well-conditioned interpolation matrix are advantages of LRBFs. While a great number of investigations have been developed to estimate solution of partial differential equations, including FDM [7], FEM [8], BEM [9] and FVM [10], researchers' attention has recently drawn to RBF. Ronald Hardy first introduced RBF methods in 1968 [13]. Then, Richard Franke [15], Charles Micchelli [14] and Edward Kansa respectively in 1979, 1986 and 2000 extended the RBF's theory. Various advantages for RBF method have been reported frequently in previous studies.
In the present study we are dealing with the linear second order hyperbolic partial differential equation in two space dimensions [2]

$$\frac{\partial^2 u}{\partial t^2} + 2\alpha(x,y)\frac{\partial u}{\partial t} + \beta^2(x,y)u = A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial y^2} + f(x,y,t), \tag{1}$$

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where $\alpha(x,y) > 0, \beta(x,y) > 0, A(x,y) > 0$ and $B(x,y) > 0$ are known variable coefficients. Eq. (1) represents the two dimensional telegraph equation when α and β are constant and $\alpha > \beta > 0$, $A = B = 1$.
The telegraph equation has been repeatedly studied in previous works. Telegraph equations are commonly used in the study of wave propagation of electric signals [32] and also in wave phenomena [33]. In [17], polynomial differential quadrature method was used to solve two dimensional hyperbolic telegraph equation under Dirichlet and Neumann boundary conditions. Dehghan and Shokri introduced additional meshless method for solving this equation using radial basis functions of thin plate splines at collocation points [20,5,3]. In [31], the advantages of local weak and strong forms of meshless method were combined. In this method, local Petrov–Galerkin weak form was applied only to the nodes onto the Neumann boundaries while a meshless collocation method based on the strong form of the equation was applied on the interior nodes and those located on the Dirichlet boundaries. All these attempts resulted in acceptable accuracy of the solutions. Recently, the numerical schemes via the operational matrices of differentiation and integration have received considerable attention for solving PDEs [34-36].
This paper includes 4 sections as follows: In Section 2: a preliminary of RBF's approximation technique and explanation of implementing the local radial basis functions (LRBFs) are introduced. In Section3: time discrete scheme utilizing Crank-Nicolson finite difference scheme is used. In Section 4: some test problems are presented to demonstrate the efficiency of the scheme. Section 5 includes conclusion.

2. A brief review of radial basis functions (RBFs)

Radial basis functions (RBFs) method is an efficient and a truly mesh free technique for interpolation of multidimensional scattered data for solving partial differential equations (PDEs). The local RBF method utilizes only the nodal points in the influence domain of in computing step, in order to have a sparse and better-conditioned linear system [2,17,20,21,22,23]. The most widely used RBFs are:

Multiquadric: $\sqrt{1+(\varepsilon r)^2}$
Inverse Multiquadric: $\frac{1}{\sqrt{1+(\varepsilon r)^2}}$
Inverse quadric: $\frac{1}{1+(\varepsilon r)^2}$
Generalized multiquadric: $(1+(\varepsilon r)^2)^\beta$
Gaussian: $e^{-(\varepsilon r)^2}$
A function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is called to be radial if there exists a univariate function
$\phi : [0,\infty) \to \mathbb{R}$, such that
$\Phi(x) = \varphi(r)$. Where $r = \|x\|$ and $\|.\|$ is some norm on $\mathbb{R}^d$, usually the Euclidean norm. The approximation of a solution $u(x)$, may be written as a linear combination of N radial functions as:

$$u(x) \simeq \sum_{j=1}^{N} \lambda_j \varphi(\boldsymbol{x}, \boldsymbol{x_j}) + \psi(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Omega \subset \mathbb{R}^d, \tag{2}$$

where N is the number of data points, $x = (x_1, x_2, ..., x_d)$, d stands for the dimension of the problem, $\lambda' s$ are coefficients to be determined and φ is the radial basis function. In this article, we use multiquadrics functions, $\varphi(x, x_j) = \varphi(r_j) = \sqrt{c^2 + r_j^2}$ where $r_j = \|x - x_j\|$ is the Eucliden norm. Equation (2) may be written without the additional balancing polynomial ψ . If so, φ must be unconditionally positive definite in order to guarantee the

solvability of the resulting system (e.g., Gaussian or inverse multiquadrics). More specifically, when φ is conditionally positive definite, i.e., has a polynomial growth to infinity, then ψ is usually required. Examples are thin plate splines and multiquadrics.
If P_q^d denotes the space of d-variate polynomials of order not exceeding q, and the polynomials $P_1, ..., P_m$ be the basis of P_q^din $\mathbb{R}^d$, then the polynomial $\psi(x)$ in Eq. (2), is usually written as:

$$\psi(\boldsymbol{x}) = \sum_{i=1}^{m} \xi_i P_i(\boldsymbol{x}), \tag{3}$$

where $m = (q-1+d)!/(d!(q-1)!)$.
To determinate the coefficients $(\lambda_1, ..., \lambda_N)$ and $(\zeta_1, ..., \zeta_m)$, the collocation method is used, where, in addition to the N equations obtained from (2) at the N points, another m extra equations are required. This is insured by the m extra side conditions for (2),

$$\sum_{j=1}^{N} \lambda_j P_i(\boldsymbol{x_j}) = 0 \quad i = 1, ..., m. \tag{4}$$

If so, for linear partial differential operator L in (2),Lu can be approximated by

$$Lu(x) \simeq \sum_{j=1}^{N} \lambda_j L\varphi(\boldsymbol{x}, \boldsymbol{x_j}) + L\psi(\boldsymbol{x}). \tag{5}$$

Now having (N-3) interpolation points, then $u^n(x, y)$ can be approximated by

$$u^n(x, y) \simeq \sum_{j=1}^{N-3} \lambda_j^n \varphi(r_j) + \lambda_{N-2}^n x + \lambda_{N-1}^n y + \lambda_N^n. \tag{6}$$

To determine the interpolation coefficients λ_j, $(j = 1, 2, \dots, N)$, the collocation method is used at every point (x_i, y_i), $i = 1, 2, ..., N-3$. Thus we have

$$u^n(x_i, y_i) \approx \sum_{j=1}^{N-3} \lambda_j^n \varphi(r_{ij}) + \lambda_{N-2}^n x_i + \lambda_{N-1}^n y_i + \lambda_N^n, \tag{7}$$

where

$$\varphi(r_{ij}) = \sqrt{c^2 + (x_i - x_j)^2 + (y_i - y_j)^2} \quad (MQRBF). \tag{8}$$

The additional conditions are also written as:

$$\sum_{j=1}^{N-3} \lambda_j^n = \sum_{j=1}^{N-3} \lambda_j^n x_j = \sum_{j=1}^{N-3} \lambda_j^n y_j = 0, \tag{9}$$

where $u^n = [u_1^n\, u_2^n\, u_{3.}^n ... u_{N-3}^n\, 0\, 0\, 0]^T$, $[\lambda]^n = [\lambda_1^n\, \lambda_2^n ... \lambda_N^n]^T$ and $A = [a_{ij}, 1 \le i, j \le N]$ is given by

$$A = \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1(N-3)} & x_1 & y_1 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \varphi_{(N-3)1} & \cdots & \varphi_{(N-3)(N-3)} & x_{N-3} & y_{N-3} & 1 \\ x_1 & \cdots & x_{N-3} & 0 & 0 & 0 \\ y_1 & \cdots & y_{N-3} & 0 & 0 & 0 \\ 1 & \cdots & 1 & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

Then

$$[u]^n = A[\lambda]^n, \tag{11}$$

so

$$[\lambda]^n = A^{-1}[u]^n. \tag{12}$$

2.1. **The local RBF.** In LRBF, the procedure is the same as RBF except for that instead of using all N points through the domain, only n points of the influence domain of the point x is used. Hence, the interpolation LRBF $u^n(x)$ is obtained as:

$$u^n(x,y) \approx \sum_{j=1}^{m-3} \lambda_j^n \varphi(r_j) + \lambda_{m-2}^n x + \lambda_{m-1}^n y + \lambda_m^n, \tag{13}$$

Accordingly,

$$[U]^n = A[\Lambda]^n. \tag{14}$$

can be computed if the scattered points in the influence domain are assumed to be distinct. We only use the nodal points, where the procedure is exactly the same as in RBFs in the computing step, noting that every shape function is obtained in the local sense.

Some attractive advantages of the proposed method which make it compareble with other schemes for solving PDEs are as follows:

1. The method requires no domain elements.
2. The method is simple and also computationally attractive.
3. The method could be easily extended to higher dimension problems.
4. The scheme while using much fewer volume computing in compared with other methods, obtains more accurate results.
5. The scheme is more flexible for solving various partial differential equations.

2.2. **Error analysis.** Consider the finite-dimensional subspaces $S_N \subset H^1(\Omega)$ of the form [28,29]

$$S_N = span\{\Phi(0 - x_1), ..., \Phi(0 - x_N)\} + \mathcal{P}_m^d \tag{15}$$

where $\Phi : \mathcal{R}^d \to \mathcal{R}$ is a radial basis function, $\mathcal{P}_m^d$ denotes the space of polynomials of degree less than m and $X = \{x_1, ..., x_N\} \subseteq \Omega$ is a set of distinct nodes.

The RBF interpolant u_I to a function $u \in C(R^d)$ on the set X is represented by

$$u_I = \sum_{j=1}^{N} \alpha_j \Phi\left(\mathbf{x} - \mathbf{x}_j\right) + p\left(\mathbf{x}\right) \tag{16}$$

where p is a polynomial of degree less than m. It is shown that [2] there always exists an u_I which satisfies the condition

$$u_I\left(\mathbf{x}_j\right) = u\left(\mathbf{x}_j\right),\ 1 \le j \le N, \tag{17}$$

$$\sum_{j=1}^{N} \alpha_j p(\mathbf{x}_j) = 0,\ p \ runs \ through \ a \ basis \ of \mathcal{P}_m^d. \tag{18}$$

Now consider the RBF Φ whose Fourier transform $\hat{\Phi}$ has the property

$$\hat{\Phi}\left(w\right) \sim (1 + \|\,|w|\,\|^{-2\beta}). \tag{19}$$

We denote the global data density

$$h = h_{X,\Omega} \equiv \sup_{x \in \Omega} \min_{1 \le j \le \mathrm{N}} \|\mathbf{x} - \mathbf{x}_j\|_2. \tag{20}$$

Lemma 2.1. *Assume $u \in H^k(\Omega), \Phi$ satisfies (20) with $\beta \geq k > d/2 + m$. Let S_N be given by (15). Then there exists a function $s \in S_N$ such that for $x \in \omega$, the estimate*

$$\|u - s\|_m \leq Ch^{k-m}\|u\|_k, \quad \boldsymbol{x} \in \Omega, \tag{21}$$

is valid if is sufficiently small [4].
As a conclusion, increasing the number of nodes in numerical method has no significant effect on the error and the proposed scheme will be of high algebraic convergence rates when the method utilizes infinitely smooth conditionally strictly positive definite functions.

2.3. **Aim of this paper.** *In this work we use the local radial basis functions collocation technique to overcome the drawback of global radial basis functions that when using multiquadric basis functions produces ill-conditioning in the matrix system.*

3. Time discrete schemes

We start considering the following linear second order hyperbolic partial differential equation in two space dimensions

$$\frac{\partial^2 u}{\partial t^2} + 2\alpha(x,y)\frac{\partial u}{\partial t} + \beta^2(x,y)u = A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial y^2} + f(x,y,t), \tag{22}$$

with $u(x,y,t)$ in the region $\Omega = \{(x,y), L_x^0 < x < L_x^1, L_y^0 < y < L_y^1\}, \quad t > 0$ where that $\alpha(x,y), \beta(x,y), A(x,y)$ and $B(x,y)$ are known variable coefficients.
The initial conditions associated with Eq.(22) will be assumed to be as:

$$u(x,y,0) = f(x,y), \quad (x,y) \in \Omega, \tag{23}$$

with the initial velocity

$$\frac{\partial u}{\partial t}(x,y,0) = g(x,y), \quad (x,y) \in \Omega, \tag{24}$$

Also the Dirichlet boundary conditions are assumed to be given by:

$$u(x,y,t) = h(x,y), \quad (x,y) \in \partial\Omega \quad, and \;\; t > 0. \tag{25}$$

First, we discretize Eq.(22) by virtue of the following θ-weighted scheme[6],

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{(\delta t)^2} + 2\alpha\,(x,y)\,\frac{u^{n+1} - u^{n-1}}{2\delta t} + \;\theta\;\left(\beta^2\,(x,y)\,u^{n+1} - A\,(x,y)\,u_{xx}^{n+1} - B\,(x,y)\,u_{yy}^{n+1}\right)$$
$$+\,(1-\theta)\;\left(\beta^2\,(x,y)\,u^n - A\,(x,y)\,u_{xx}^n - B\,(x,y)\,u_{yy}^n\right) = f^{n+1}\,(x,y) \tag{26}$$

where $u^n = u(x,y,t^n)$, $t^n = t^{n-1} + \delta t, \delta t$ is the time step size and $0 \leq \theta \leq 1$. Simplifying Eq. (26) gives

$$(1 + \alpha\delta t + \theta(\delta t)^2\beta^2)u^{n+1} - \theta(\delta t)^2\;\left(Au_{xx}^{n+1} + Bu_{yy}^{n+1}\right) = (2 - \;(1-\theta)\,(\delta t)^2\beta^2)u^n$$
$$+\,(1-\theta)\,(\delta t)^2\;\left(Au_{xx}^n + Bu_{yy}^n\right) + (\alpha\delta t - 1)\,u^{n-1}(\delta t)^2 f^{n+1}. \tag{27}$$

4. Numerical experiments

In this section we solve several numerical examples for different values of t, with the proposed method and compute the following error norms: [21]

$$L_2 = \sqrt{\delta x \sum_{i=1}^{N}\left(U_i^{exact} - U_i^{numerical}\right)^2} \tag{28}$$

$$L_\infty = \max_i |U_i^{exact} - U_i^{numerical}| \tag{29}$$

$$RMS = \sqrt{\sum_{i=1}^{N} \frac{1}{N}\left(U_i^{exact} - U_i^{numerical}\right)^2} \tag{30}$$

Remark: We solve these problems with the method just presented with shape parameter $c = 1$ at different final times $t = 1, 3, 6$, and $c = 2.4$ for the first example. We also show errors obtained for the presented method with the time step size $\delta t = 0.001$ for three test problems at different final times with $N = 421$ interpolation points and $m = 5$ the number of selected nearest neighboring points for a specified collocation point. We compare the mentioned method with the global RBF method used in [2]. The results obtained from the local method are acceptable in comparison with the global method, given that only a limited number of points in the sub-domain are used and the resulting matrices are not ill- conditioned. Moreover, according to results, reported to the tables, we can see that as time goes ahead, the results obtained via LRBFs method get better than GRBF, i.e, LRBFs method seems superior to GRBFs method.

Example 4.1. *In this example we consider the second order hyperbolic Eq.(22) with $\alpha = \beta = A = B = 1$.*

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f\left(x, y, t\right), \tag{31}$$

with the analytical solution

$$u(x, y, t) = e^{x+y-t}, \ \ 0 \le x, y \le 2, t > 0. \tag{32}$$

the initial conditions are given by

$$\begin{cases} u\left(x, y, 0\right) = e^{x+y}, \ 0 \le x \ , \ y \le 2 \\ u_t\left(x, y, 0\right) = -e^{x+y}, \ 0 \le x \ , \ y \le 2 \end{cases} \tag{33}$$

We extract the right-hand side function $f(x, y, t)$ and boundary conditions from the exact solution.

$$\begin{cases} u(0, y, t) = e^{y-t} \\ u(2, y, t) = e^{2+y-t} \\ u(x, 0, t) = e^{x-t} \\ u(x, 2, t) = e^{2+x-t} \end{cases} \qquad t > 0. \tag{34}$$

And

$$f(x, y, t) = -2e^{x+y-t} \tag{35}$$

First, we discretize Eq.(31) according to θ-weighted scheme and assume that $\theta = 1/2$.

$$\begin{aligned} &\frac{u^{n+1} - 2u^n + u^{n-1}}{(\delta t)^2} + \frac{u^{n+1} - u^{n-1}}{2\delta t} + \ \frac{1}{2}\ \left(u^{n+1} - u_{xx}^{n+1} - u_{yy}^{n+1}\right) \\ &+ \ \frac{1}{2}\ \left(u^n - u_{xx}^n - u_{yy}^n\right) = f^{n+1}(x, y), \end{aligned} \tag{36}$$

where $u^n = u(x, y, t^n), t^n = t^{n-1} + \delta t$, δt is the time step size and

$$\begin{aligned} &(1 + \delta t + \frac{1}{2}(\delta t)^2)u^{n+1} - \frac{1}{2}(\delta t)^2\ \left(u_{xx}^{n+1} + u_{yy}^{n+1}\right) \\ &= (2 - \ \frac{1}{2}(\delta t)^2)u^n + \frac{1}{2}(\delta t)^2\ \left(u_{xx}^n + u_{yy}^n\right) + \left(\delta t - 1\right)u^{n-1} + (\delta t)^2 f^{n+1}\left(x, y\right). \end{aligned} \tag{37}$$

We consider the approximation

$$u^n(x_i, y_i) \simeq \sum_{j=1}^{N-3} \lambda_j^n \varphi(r_{ij}) + \lambda_{N-2}^n x_i + \lambda_{N-1}^n y_i + \lambda_N^n. \tag{38}$$

Substituting (38) in the governing equation (37), we have

$$\left(1 + \delta t + \frac{1}{2}(\delta t)^2\right)\left(\sum_{j=1}^{N-3} \lambda_j^{n+1} \varphi(r_{ij}) + \lambda_{N-2}^{n+1} x_i + \lambda_{N-1}^{n+1} y_i + \lambda_N^{n+1}\right) -$$
$$\frac{1}{2}(\delta t)^2 \left(\sum_{j=1}^{N-3} \lambda_j^{n+1}(\varphi_{xx}(r_{ij}) + \varphi_{yy}(r_{ij}))\right) = \left(2 - \frac{1}{2}(\delta t)^2\right)\left(\sum_{j=1}^{N-3} \lambda_j^n(\varphi_{xx}(r_{ij}) + \varphi_{yy}(r_{ij}))\right) +$$
$$\frac{1}{2}(\delta t)^2 \left(\sum_{j=1}^{N-3} \lambda_j^n \varphi(r_{ij}) + \lambda_{N-2}^n x_i + \lambda_{N-1}^n y_i + \lambda_N^n\right) + (\delta t - 1)$$
$$\left(\sum_{j=1}^{N-3} \lambda_j^{n-1} \varphi(r_{ij}) + \lambda_{N-2}^{n-1} x_i + \lambda_{N-1}^{n-1} y_i + \lambda_N^{n-1}\right) + (\delta t)^2 f^{n+1}(x_i, y_i) \tag{39}$$

For initial time step, we have

$$\begin{aligned} &(1 + \delta t + \frac{1}{2}(\delta t)^2)u^1 - \frac{1}{2}(\delta t)^2 \left(u_{xx}^1 + u_{yy}^1\right) \\ &= (2 - \frac{1}{2}(\delta t)^2)u^0 + \frac{1}{2}(\delta t)^2 \left(u_{xx}^0 + u_{yy}^0\right) + (\delta t - 1)\, u^{-1} + (\delta t)^2 f^1(x, y) \end{aligned} \tag{40}$$

To approximate $u^{-1}(x, y)$, we utilize fram,

$$\frac{u^1 - u^{-1}}{2\delta t} = -e^{x+y} \Rightarrow u^{-1}(x, y) = u^1(x, y) + 2(\delta t)\; e^{x+y}. \tag{41}$$

So, for $n = 0$ we have

$$\begin{aligned} &(1 + \delta t + \frac{1}{2}(\delta t)^2)u^1 - \frac{1}{2}(\delta t)^2 \left(u_{xx}^1 + u_{yy}^1\right) \\ &= (2 - \frac{1}{2}(\delta t)^2)e^{x+y} + \frac{1}{2}(\delta t)^2 \left(e^{x+y} + e^{x+y}\right) + (\delta t - 1)(u^{-1} + 2(\delta t)e^{x+y}) + (\delta t)^2 - 2e^{x+y}, \end{aligned} \tag{42}$$

$$[u]^n = A[\lambda]^n \;\Rightarrow [\lambda]^n = A^{-1}[u]^n \Rightarrow [\lambda]^1 = A^{-1}[u]^1. \tag{43}$$

The L_2, L_∞ and RMS errors are obtained in Table 1 for $t = 1, 3$, and 6. In Table 2 errors for the problem are shown with the global method [2]. The graph of analytical and estimated functions and absolute error for $t = 3$ and $t = 6$, also analytical and estimated solutions and absolute error with global scheme at $t = 5$, with $dt = 0.001$ and $dx = dy = 0.1$, are demonstrated in Fig.1, Fig.2 and Fig.3 respectively. In table 3 we supposed $c = 2.4$ and the L_2, L_∞ and RMS errors are obtained and absolute error for $t = 3$ are shown in Fig.4.

Table 1. L_2, L_∞ and RMS errors, for constant coefficient problem with $dt = 0.001$, and $dx = dy = 0.1$

t	L_2	L_∞	RMS
1	$1.824e-01$	$2.77e-2$	$8.7e-03$
3	$1.38e-02$	$3.1e-03$	$6.5812e-04$
6	$9.2796e-04$	$1.6232e-04$	$4.4189e-05$

Table 2. L_2, L_∞ and RMS errors, for constant coefficient problem with $dt = 0.001$, and $dx = dy = 0.1$ with the global RBF method

t	L_2	L_∞	RMS
1	$1.6184e-03$	$1.0382e-2$	$4.9441e-04$
3	$4.3229e-04$	$4.3017e-03$	$2.0484e-04$

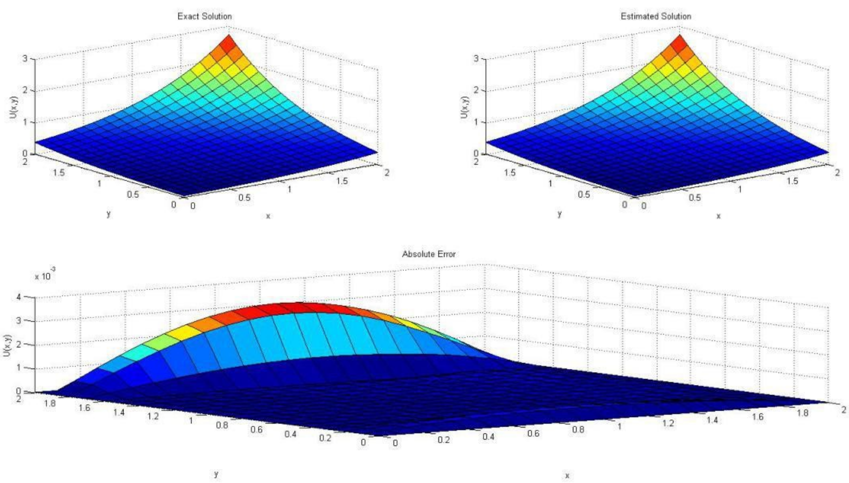

Figure 1. Analytical solution and the absolute error of estimated solutions for all three cases at $t = 3$ with $dt = 0.001$ and $dx = dy = 0.1, u(x, y, t = 3)$, for Example 1.

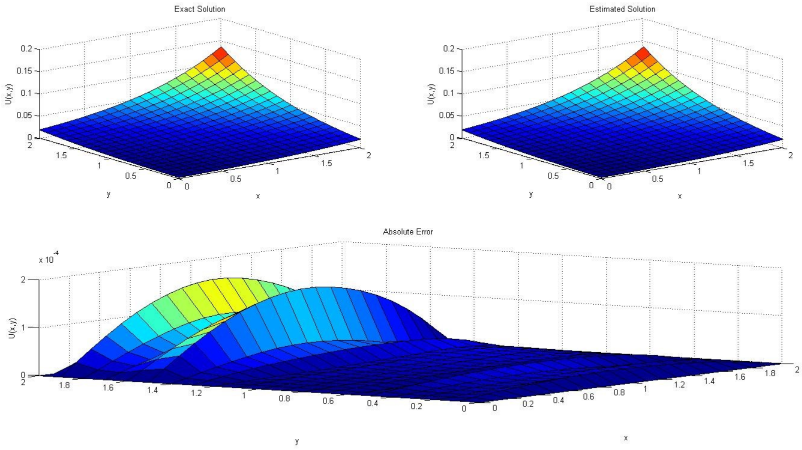

FIGURE 2. Analytical solution and the absolute error of estimated solutions for all three cases at $t = 6$ with $dt = 0.001$ and $dx = dy = 0.1, u(x, y, t = 6)$, for Example 1.

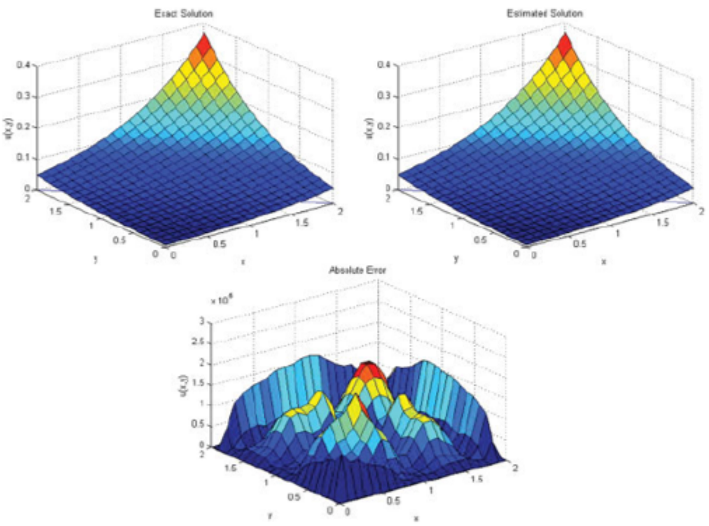

FIGURE 3. Analytical and estimated solutions and absolute error with global RBF at $t = 5$, with$dt = 0.001$ and $dx = dy = 0.1$.

TABLE 3. L_2, L_∞ and RMS errors, for constant coefficient problem with $dt = 0.001$, and $dx = dy = 0.1$ for $c = 2.4$

t	L_2	L_∞	RMS
1	*1.177e-01*	*1.68e-02*	*5.6e-03*
3	*8.7364e-04*	*1.3264e-04*	*4.1602e-05*

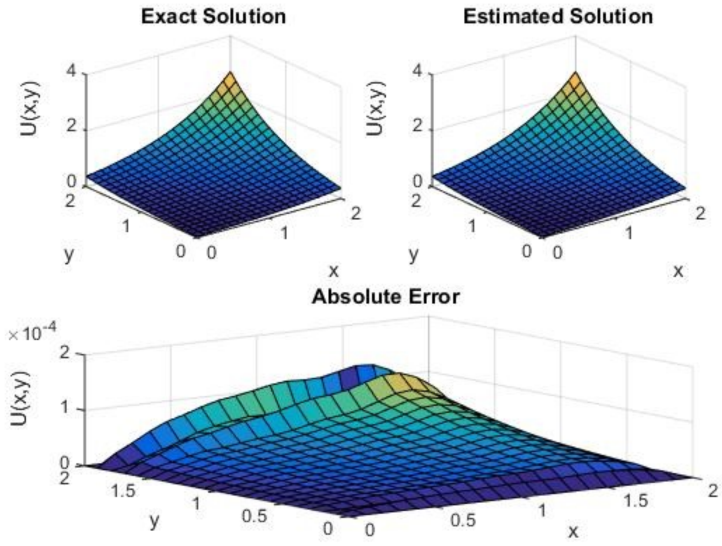

FIGURE 4. Analytical solution and the absolute error of estimated solutions for all three cases at $t = 3$ with $dt = 0.001$ and $dx = dy = 0.1, u(x, y, t = 3)$, for Example 1 with $c = 2.4$.

Example 4.2. *Considering the second order hyperbolic Eq.(22) with $\alpha = \beta = 1$, and $A = B = 1$. The initial conditions are given by*

$$u(x, y, 0) = \sin(x)\sin(y), \quad 0 \le x, y \le \pi, \tag{44}$$

$$u_t(x, y, 0) = -\sin(x)\sin(y), \quad 0 \le x, y \le \pi, \tag{45}$$

and the exact solution is

$$u(x, y, t) = e^{-t}\sin(x)\sin(y), \quad 0 \le x, y \le \pi, t > 0. \tag{46}$$

The L_2, L_∞ and RMS errors are obtained in Table 4 for $t = 1, 3$, and 6, also in Table 5, L_2, L_∞ and RMS errors, for constant coefficient problem with $dt = 0.001$, and $dx = dy = 0.1$ with the global method.The graph of analytical and estimated functions and absolute error for $t = 3$ and t=6 are shown in Fig.5 and Fig.6 respectively. In addition, in Fig.7, analytical and estimated solutions and absolute error at $t = 2$, with $dt = 0.001$ and $dx = dy = \pi/16$, are shown with the global RBF.

TABLE 4. L_2, L_∞ and RMS errors, for constant coefficient problem with $dt = 0.001$, and $dx = dy = 0.1$

t	L_2	L_∞	*RMS*
1	*1.34e-02*	*1.7e-3*	*7.8572e-04*
3	*3.1e-03*	*3.8581e-04*	*1.8155e-04*
6	*1.7036e-04*	*2.1364e-05*	*1.0021e-05*

TABLE 5. L_2, L_∞ and RMS errors, for constant coefficient problem with $dt = 0.001$, and $dx = dy = 0.1$ with the global method.

t	L_2	L_∞	RMS
1	*4.943e-04*	*4.7889e-3*	*2.8170e-04*
2	*4.3304e-04*	*3.1237e-03*	*1.8375e-04*

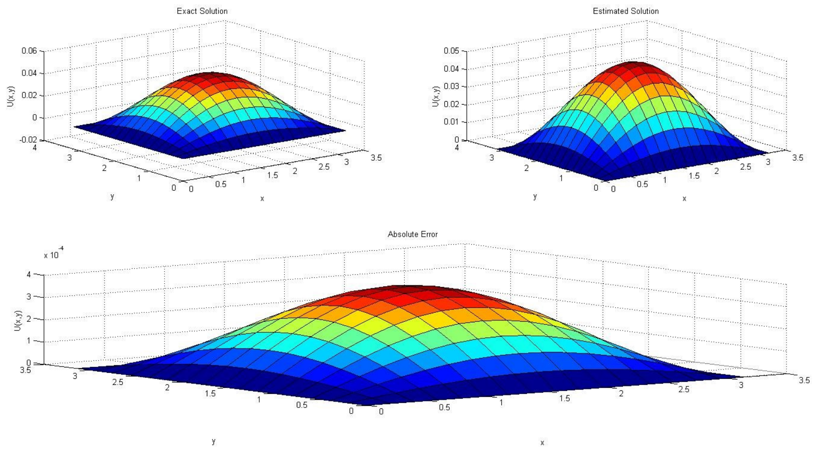

FIGURE 5. Analytical solution and the absolute error of estimated solutions for all three cases at $t = 3$ with $dt = 0.001$ and $dx = dy = 0.1, u(x, y, t = 3)$, for Example 2.

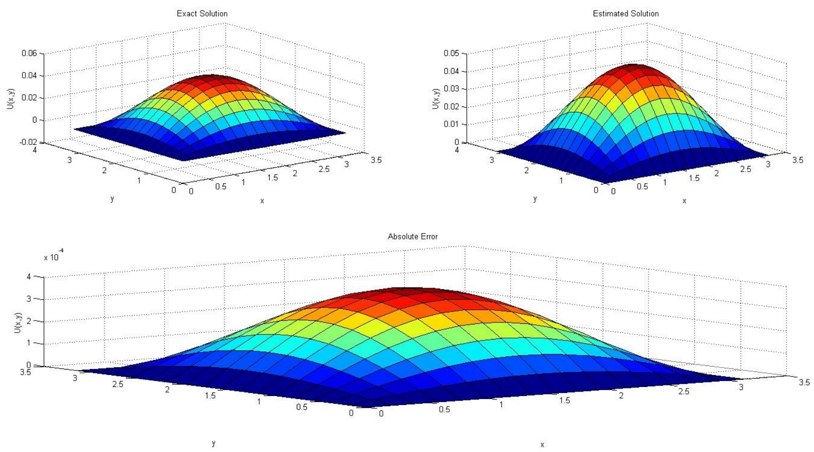

FIGURE 6. Analytical solution and the absolute error of estimated solutions for all three cases at $t = 6$ with $dt = 0.001$ and $dx = dy = 0.1$, for Example 2, with $c = 1$.

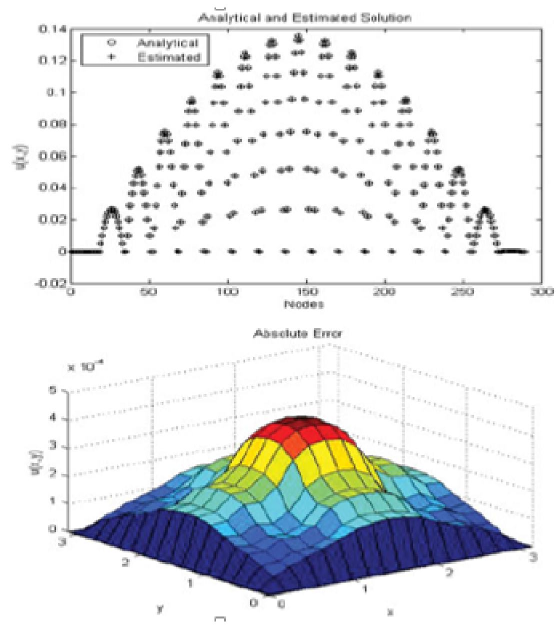

FIGURE 7. Analytical and estimated solutions and absolute error at $t = 2$, with $dt = 0.001$ and $dx = dy = \pi/16$, for Example 2, with the global RBF.

Example 4.3. *As in the previous examples, we have the second order hyperbolic Eq. (22) with* $\alpha = \beta = 1$*, and* $A = B = 1$*. The initial conditions are given by*

$$u(x, y, 0) = x^2 + y^2, \quad 0 \leq x, y \leq 1 \tag{47}$$

$$u_t(x, y, 0) = x^2 + y^2 + 1, \quad 0 \leq x, y \leq 1 \tag{48}$$

and the exact solution is

$$u(x, y, t) = x^2 + y^2 + t, \quad 0 \leq x, y \leq 1, \ t > 0. \tag{49}$$

The right hand side function is

$$f(x, y, t) = -2 + x^2 + y^2 + t \tag{50}$$

The boundary conditions can be obtained from the exact solution. The L_2, L_∞ *and RMS errors are given in Table 6 for* $t = 1$*, 3 and 6 and in Table 7 the results from the global RBF scheme are given. The graph of analytical and estimated solutions and absolute error for* $t = 3$ *and 6 are shown in Fig. 8, Fig.9, and in Fig. 10, analytical and estimated solutions and absolute error at* $t = 10$*, with* $dt = 0.001$ *and* $dx = dy = 0.1$*, with the global RBF. The computational region is* $[0, 1] \times [0, 1]$.

TABLE 6. L_2, L_∞ and RMS errors, for constant coefficient problem with $dt = 0.001$, and $dx = dy = 0.1$

t	L_2	L_∞	RMS
1	*1.34e-02*	*1.7e-3*	*7.8572e-04*
3	*3.1e-03*	*3.8581e-04*	*1.8155e-04*
6	*1.7036e-04*	*2.1364e-05*	*1.0021e-05*

TABLE 7. L_2, L_∞ and RMS errors, for constant coefficient problem with $dt = 0.001$, and $dx = dy = 0.1$ with the global method

t	L_2	L_∞	RMS
1	*1.8056e-04*	*1.2439e-3*	*1.1309e-04*
3	*1.4585e-04*	*9.9849e-04*	*9.0772e-05*
5	*1.4544e-04*	*1.0265e-03*	*9.3322e-05*

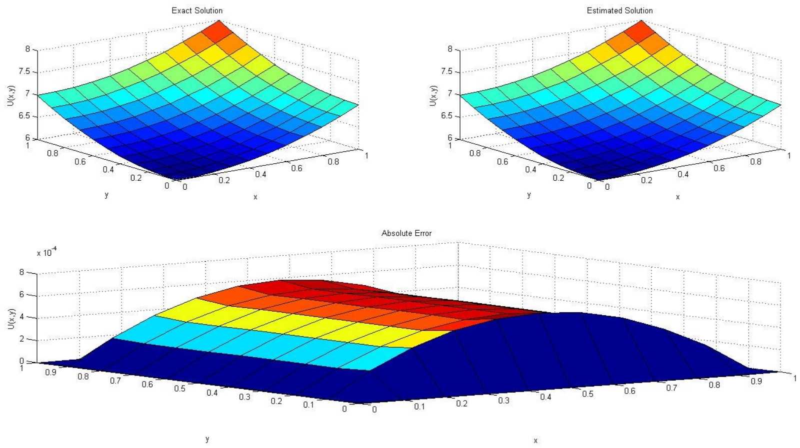

FIGURE 8. Analytical solution and also the absolute error of estimated solutions for all three cases at $t = 3$ with $dt = 0.001$ and $dx = dy = 0.1, u(x, y, t = 3)$, for Example 3.

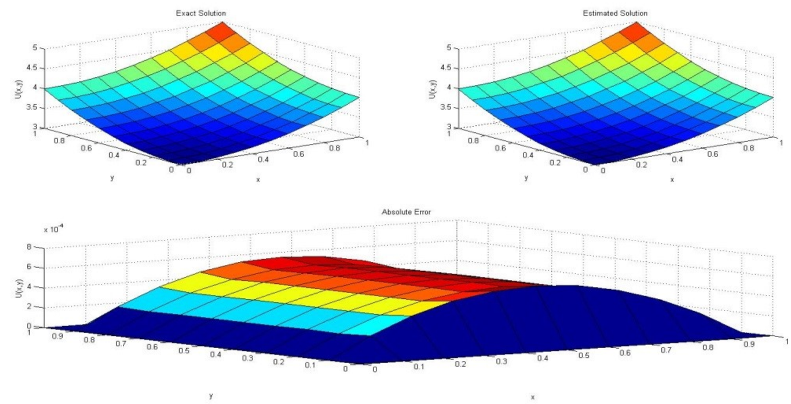

FIGURE 9. Analytical solution and also the absolute error of estimated solutions for all three cases at $t = 6$ with $dt = 0.001$ and $dx = dy = 0.1$, for Example 3.

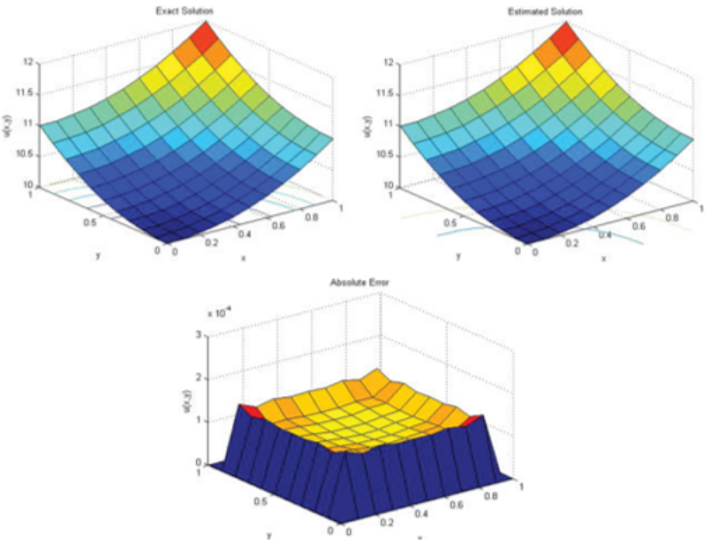

FIGURE 10. Analytical and estimated solutions and absolute error at $t = 10$, with $dt = 0.001$ and $dx = dy = 0.1$, for Example3, with the global RBF.

As shown in examples via local radial basis function we can obtain desired accuracy in comparing with the global method [2] and also on an average desktop computer, the global method cannot be implemented with a large number of centers, due to memory restraints when the dense matrix is formed and every element has to be stored. Using the local method, sparse matrices are stored, and we can obtain desired accuracy.

5. CONCLUSION

This paper deals with solving two-dimensional hyperbolic telegraph equation. The solution method is based on local radial basis functions (LRBFs). Both global and local methods can be used when working with radial basis functions but the coefficient matrix of global RBF will be full and large and usually it is ill-conditioned, because require thousands of data points for centers. The local method overcomes these issues by allowing the user to select small stencils of points [14]. In the approximation function by local radial basis functions, the only points are used fallen within local influence domain. Therefore, much less computational work and the well-conditioned interpolation matrix are obtained. Besides, numerical examples are provided to validate applicability and efficiency of the method. The results demonstrate that by using the proposed method, higher accuracy as well as less computational cost are achieved.

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