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ASSESSING THE PERFORMANCE OF INSULATING FLUIDS VIA POINT OF STATISTICAL INFERENCE VIEW

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ABSTRACT. In this paper, the statistical inference is used in order to study the performance or aging of the insulating fluids. Transformer oil is used as an example of insulating fluids. The insulation property of insulating oil or transformer oil is lost by consumption. Breakdown tests are performed to check oil's efficiency, but the cost of these tests is not inexpensive. Hence this statistical study aims to reduce the cost of these tests by applying statistical inference approaches to censored data. The Type-II Gumbel distribution fits well real-life data which contains failure times to breakdown of an insulating fluid between electrodes. The Type-II hybrid censored scheme is proposed to assess the study and also to reduce the cost of breakdown tests in practical tests.

Keywords: Statistical inference, insulating fluids, reliability theory, transformers oil, censored data.

AMS Subject Classification: 62F10, 62F15, 62N01.

1. INTRODUCTION

Insulating oil is very important in maintaining the reliable operation of power transformers since the majority of these transformers depends on insulating oils as liquid dielectrics. The dielectric strength of these insulating oil, which is affected mainly by the presence of pollutants such as water or acids, is also known as the breakdown voltage. The amount of water in the insulating oil, measured in ppm, extremely affects the value of the breakdown voltage for this oil as the electric field forces the water droplets to be drawn to places where field intensity is high. The breakdown voltage is the amount of voltage required to induce a spark between two semi-elliptical electrodes immersed in the insulating oil to be tested. To measure the breakdown voltage of insulating oil, several samples are tested using an apparatus consisting of a container, where the oil is poured, with two semi-elliptical steel electrodes at a distance 2.5 mm from each other. The voltage is then increased gradually until a spark is noticed between the two electrodes at which the voltage is recorded. Each sample is tested six times and the breakdown voltage is then the average value of the six values. Several research articles have been made to study breakdown voltage and other

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physical and chemical properties of various types of insulating oils such as mineral oil, synthetic esters, vegetable oils, and silicon fluid, see [17]. Due to deterioration over time, insulating oil has to be tested periodically to ensure that its insulation, cooling, and many other features are still acceptable. Testing requires sending several samples to a laboratory to take results which may cost a lot of money.

In our study, we will perform a statistical model using censored samples in order to predict the lifetime of these insulating oils. The conventional Type-I and Type-II censoring schemes are the most widely discussed censoring schemes in reliability theory. Type-I censoring scheme forces the test to end at a pre-determined time (T_0) while Type-II censoring scheme ends the test directly after the failure of a fixed number of items (R) which means that the time of the test is controlled in Type-I censoring scheme but the efficiency may be low and the test may be ended without getting failures, whereas Type-II censoring scheme guarantees to obtain R failures, but the time taken to end the test is random due to uncertainty of the R^{th} item failure. As a result, a more flexible censoring scheme is needed for life-testing experiments.

The hybrid censoring scheme (HCS) is a mixture of Type-I and Type-II censoring schemes which provides a more flexible and administratively convenient life-testing procedure. Assume we are going to test n identical units. Then either the test will be ended after the failure of a pre-determined number R out of n units, or after reaching a pre-specified time T. Let $Y_{i:n}$ denotes the i-th ordered failure item, then either the test is terminated at time $T_1 = \min\{Y_{R:n}, T\}$ or at time $T_2 = \max\{Y_{R:n}, T\}$. In Type-I HCS, T_1 refers to the time of ending an experiment for testing, while T_2 refers to the time of ending an experiment in Type-II HCS. Many authors discussed the estimation of the unknown parameters for various probability distributions in case of Type-I hybrid censored data, see [5] and [11]. In addition to it, [3], [8] and [2] can also be referred for the estimation of the parameters under Type-II HCS. In this paper, the Type-II Gumbel distribution, introduced by German mathematician Emil Gumbel (1891-1911) in 1958, is considered as a good model for the failure times of the insulating fluids. The probability density function (PDF) and the cumulative distribution function (CDF) are defined for the type -II Gumbel distribution of the random variable X as follows

$$f(x) = \alpha\beta x^{-(\alpha+1)} e^{-\beta x^{-\alpha}}, \ x > 0, \ \alpha, \beta > 0. \tag{1}$$

and

$$F(x) = e^{-\beta x^{-\alpha}}, \quad x > 0, \ \alpha, \beta > 0, \tag{2}$$

while the reliability and hazard function of type -II Gumbel distribution , respectively, are given by

$$S(t) = 1 - e^{-\beta t^{-\alpha}}; t \geq 0 \tag{3}$$

and

$$h(t) = \frac{\alpha\beta x^{-(\alpha+1)} e^{-\beta x^{-\alpha}}}{1 - e^{-\beta t^{-\alpha}}}; t \geq 0. \tag{4}$$

Recently, many authors have contributed to statistical methodology and characterization of Gumbel Type-II distribution. For example, [15] discussed some properties of Gumbel distribution. [7] considered Bayesian analysis of Gumbel Type-II distribution under doubly censored samples using different loss functions. In this paper, we will give a proposal to the classical and Bayesian estimation procedures for the unknown parameters of the Type-II Gumbel distribution under the Type-II HCS. The rest of this paper is organized as follows: In Section 2 the maximum likelihood estimates (MLEs) of the parameters under consideration are obtained in addition to the corresponding approximate confidence intervals (ACIs). Section 3 is devoted to the Bayesian estimation and the MCMC approach. The simulation study is presented in Section 4 to assess the quality of obtained estimators. A real data set is analyzed in Section 5 for illustration. At the end of this paper, conclusions are given in Section 6.

2. Maximum likelihood estimation

Deriving the estimators of parameters depends basically on the log-likelihood functions. There are many advantage for the maximum likelihood estimators such as asymptotically minimum variance, asymptotically normally distributed, satisfaction of the invariant property and asymptotically unbiased, see [1]. We can observe one of the following two types of censored data under Type-II HCS:

Case I: $\{y_{1:n} < y_{2:n} < .... < y_{R:n}\}$ if $y_{R:n} > T$.

Case II: $\{y_{1:n} < ... < y_{R:n} < y_{R+1:n} < ... < y_{m:n} < T\}$ if $T > y_{R:n}$ and the m-th failure took place before T, $R \leq m \leq n$.

For case I, the likelihood function is given by

$$\begin{aligned} L_1\left(\alpha,\beta \mid data\right) = c_1\alpha^R\beta^R\left(1-e^{-\beta y_{R:n}^{-\alpha}}\right)^{n-R} \\ \times\prod_{i=1}^{R} y_{i:n}^{-(\alpha+1)}e^{-\beta y_{i:n}^{-\alpha}}, \end{aligned} \tag{5}$$

where $c_1 = \dfrac{n!}{(n-R)!}$, while for case II, it is given by

$$\begin{aligned} L_2\left(\alpha,\beta \mid data\right) = c_2\alpha^m\beta^m\left(1-e^{-\beta T^{-\alpha}}\right)^{n-m} \\ \times\prod_{i=1}^{m} y_{i:n}^{-(\alpha+1)}e^{-\beta y_{i:n}^{-\alpha}}, \end{aligned} \tag{6}$$

where $c_2 = \dfrac{n!}{(n-m)!}$.

Combining the two likelihood functions as follows

$$\begin{aligned} L\left(\alpha,\beta \mid data\right) = c\alpha^H\beta^H\left(1-e^{-\beta u^{-\alpha}}\right)^{n-H} \\ \times\prod_{i=1}^{H} y_{i:n}^{-(\alpha+1)}e^{-\beta y_{i:n}^{-\alpha}}, \end{aligned} \tag{7}$$

where $c = \dfrac{n!}{(n-H)!}$ and H denotes the number of failures; $u = y_{R:n}$ if $H = R$ and $u = T$ if $H > R$.

Then the log-likelihood function will be written as follows

$$\begin{aligned} \ln\ L(\alpha,\beta \mid data) = l(\alpha,\beta \mid data) = \ln c + H\ \ln\alpha + H\ \ln\beta + \\ -(\alpha+1)\sum_{i=1}^{H}\ln y_{i:n} - \beta\sum_{i=1}^{H} y_{i:n}^{-\alpha} \\ +(n-H)\ln\left(1-e^{-\beta u^{-\alpha}}\right), \end{aligned} \tag{8}$$

yielding to the likelihood equations for α and β respectively as follows

$$\frac{H}{\alpha} - \sum_{i=1}^{H}\ln y_{i:n} + \beta\sum_{i=1}^{H} y_{i:n}^{-\alpha}\ln y_{i:n} + \frac{\beta(H-n)e^{-\beta u^{-\alpha}}u^{-\alpha}\ln u}{\left(1-e^{-\beta u^{-\alpha}}\right)} = 0, \tag{9}$$

and

$$\frac{H}{\beta} - \sum_{i=1}^{H} y_{i:n}^{-\alpha} + \frac{(n-H)e^{-\beta u^{-\alpha}}u^{-\alpha}}{\left(1-e^{-\beta u^{-\alpha}}\right)} = 0. \tag{10}$$

Getting an exact solution for the nonlinear simultaneous equations (9) and (10) in two unknown vaiables α and β is too difficult. Hence, solving numerically, for example using Newton Raphson, will help in finding an approximate solution. The algorithm for Newton Raphson is described briefly in [6]. Finally, the estimates of α and β are the MLEs of the parameters and will be denoted as $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$.

Moreover, the MLEs of $S(t)$ and $h(t)$ can be obtained, using the invariance property of the MLEs, by replacing α and β and by $\hat{\alpha}$ and $\hat{\beta}$ as follows

$$\hat{S}(t) = 1 - e^{-\hat{\beta}t^{-\hat{\alpha}}} \text{and } \hat{h}(t) = \frac{\hat{\alpha}\hat{\beta}t^{-(\hat{\alpha}+1)}e^{-\hat{\beta}t^{-\hat{\alpha}}}}{1-e^{-\hat{\beta}t^{-\hat{\alpha}}}}. \tag{11}$$

2.1. **Approximate confidence intervals.** The asymptotic variances and covariances of the MLEs, $\hat{\alpha}$ and $\hat{\beta}$ are given by the entries of the inverse of the Fisher information matrix $I_{ij} = E\left\{-\left[\partial^2\ell(\Phi)/\partial\phi_i\ \partial\phi_j\right]\right\}$, where $i,j = 1,2$ and $\Phi = (\phi_1,\phi_2) = (\alpha,\beta)$. Unfortunately, obtaining exact closed forms for the above expectations are is not easy.
By dropping the expectation operator E, we obtain the observed Fisher information matrix $\hat{I}_{ij} = \left\{-\left[\partial^2\ell(\Phi)/\partial\phi_i\ \partial\phi_j\right]\right\}_{\Phi=\hat{\Phi}}$, which is then for constructing confidence intervals for the parameters. The second partial derivatives of the log-likelihood function for the observed Fisher information matrix is the same as that for the entries, and it can easily obtained. Therefore, the observed information matrix will given as follows

$$\hat{I}(\alpha,\beta) = \begin{pmatrix} -\frac{\partial^2\ell}{\partial\alpha^2} & -\frac{\partial^2\ell}{\partial\alpha\partial\beta} \\ -\frac{\partial^2\ell}{\partial\beta\partial\alpha} & -\frac{\partial^2\ell}{\partial\beta^2} \end{pmatrix}_{(\alpha,\beta)=(\alpha,\beta)}. \tag{12}$$

Hence, using the inverse of the observed information matrix $\hat{I}(\alpha,\beta)$, we can obtain the approximate (or observed) asymptotic variance-covariance matrix $\left[\hat{V}\right]$ for the MLEs which is equivalent to

$$\left[\hat{V}\right] = \hat{I}^{-1}(\alpha,\beta) = \begin{pmatrix} \widehat{Var}(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta}) \\ cov(\hat{\alpha},\hat{\beta}) & \widehat{Var}\left(\hat{\beta}\right) \end{pmatrix}. \tag{13}$$

Under some regularity conditions, it is well known that (α, β) is approximately distributed as multivariate normal with mean (α, β) and covariance matrix $I^{-1}(\alpha, \beta)$, (see [13]), Then, the $100(1-\gamma)\%$ two sided confidence intervals of α and β will be given as follows

$$\widehat{\alpha} \pm Z_{\frac{\gamma}{2}} \sqrt{\widehat{Var}(\hat{\alpha})} \text{ and } \widehat{\beta} \pm Z_{\frac{\gamma}{2}} \sqrt{\widehat{Var}\left(\widehat{\beta}\right)}. \tag{14}$$

where $Z_{\frac{\gamma}{2}}$ is the percentile of the standard normal distribution with right-tail probability $\frac{\gamma}{2}$.

In addition; constructing the asymptotic confidence interval (CI) of the reliability and hazard functions, functions in the parameters α and β, require finding their variances. Using the delta method, which is a general technique for determining CIs for functions of MLEs, see [10], the approximate estimates of the variance of $\hat{S}(t)$ and $\hat{h}(t)$ calculated. Accordingly, the variance of $\hat{S}(t)$ and $\hat{h}(t)$ respectively will be approximated as follows

$$\hat{\sigma}^2_{\hat{S}(t)} = \left[\nabla\hat{S}(t)\right]^T \left[\hat{V}\right] \left[\nabla\hat{S}(t)\right] \text{ and } \hat{\sigma}^2_{\hat{h}(t)} = \left[\nabla\hat{h}(t)\right]^T \left[\hat{V}\right] \left[\nabla\hat{h}(t)\right],$$

where $\nabla\hat{S}(t)$ and $\nabla\hat{h}(t)$ are, respectively, the gradient of $\hat{S}(t)$ and $\hat{h}(t)$ with respect to α and β.

Then, the $100(1-\gamma)\%$ two sided confidence intervals of $S(t)$ and $h(t)$ can be given by

$$\hat{S}(t) \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{\sigma}^2_{\hat{S}(t)}} \text{and } \hat{h}(t) \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{\sigma}^2_{\hat{h}(t)}}. \tag{15}$$

3. Bayesian approach for estimation and prediction

In this section, we will obtain Bayesian estimates of the unknown parameters α and β in addition to some lifetime parameters $S(t)$ and $h(t)$ against the squared error and LINEX loss functions. The prior knowledge about the parameters are represented by independent informative prior distributions. The parameters α and β are assumed to be independent and follow the gamma prior distributions as follows:

$$\begin{aligned} &\pi_1(\alpha) \propto \alpha^{a_1-1} e^{-b_1\alpha} &&, \ \alpha > 0, a_1 > 0, b_1 > 0, \\ &\pi_2(\beta) \propto \beta^{a_2-1} e^{-b_2\beta} &&, \ \beta > 0, a_2 > 0, b_2 > 0, \end{aligned} \tag{16}$$

where the hyper-parameters a_i and $b_i, i = 1, 2$, are assumed to be known, and chosen to reflect the prior belief about the unknown parameters. Many authors like [12] and [4] established the Bayesian estimation for their parameter models based on informative gamma priors. Using Bayes' theorem, we can combine the likelihood function (7) with the priors (16) to obtain the posterior distribution of the parameters α and β denoted by $\pi^*(\alpha, \beta)$ as follows

$$\pi^*(\alpha, \beta) = \frac{\pi_1(\alpha)\ \pi_2(\beta)\ L(\alpha, \beta \mid \text{data})}{\int_0^\infty \int_0^\infty \pi_1(\alpha)\ \pi_2(\beta)\ L(\alpha, \beta \mid \text{data})\ d\alpha d\beta}. \tag{17}$$

The squared error loss (SEL) function, a symmetrical loss function that assigns equal losses to both overestimation and underestimation, is a commonly used loss function. For an estimator $\hat{\phi}$ that estimates the parameter ϕ, the SEL function will be given as

$$L\left(\phi, \hat{\phi}\right) = \left(\hat{\phi} - \phi\right)^2,$$

Therefore, $g(\alpha, \beta)$, which is Bayes estimate of any function of α and β under the SEL function, is given by

$$\hat{g}_{BS}(\alpha, \beta) = E_{\alpha,\beta,\lambda|\text{data}}\left(g(\alpha, \beta)\right),$$

where

$$E_{\alpha,\beta,\lambda|\underline{x}}\left(g\left(\alpha,\beta\right)\right)=\frac{\int\limits_0^\infty\int\limits_0^\infty g\left(\alpha,\beta\right)\ \pi_1\left(\alpha\right)\ \pi_2\left(\beta\right)\ L(\alpha,\beta\mid \text{data})d\alpha d\beta}{\int\limits_0^\infty\int\limits_0^\infty \pi_1\left(\alpha\right)\ \pi_2\left(\beta\right)\ L(\alpha,\beta\mid \text{data})\ d\alpha d\beta}. \tag{18}$$

One can note that it is difficult to solve the multiple integrals in (18) analytically due to the complexity of the likelihood function given in (7). As a result, the Bayes estimate of α, β, $S\left(t\right)$ and $h(t)$ can be computed using the MCMC approximation method which is used to generate samples from the joint posterior density function in (17) and also to construct the associated credible intervals. The joint posterior distribution will then be written as:

$$\pi^*\left(\alpha,\beta\right)\propto\alpha^{H+a_1-1}\beta^{H+a_2-1}e^{-\alpha b_1-\beta b_2}\left(1-e^{-\beta u^{-\alpha}}\right)^{n-H}$$
$$\times\prod_{i=1}^{H}y_{i:n}^{-(\alpha+1)}e^{-\beta y_{i:n}^{-\alpha}}. \tag{19}$$

The conditional posterior distributions for α,β and λ are

$$\pi_1^*\left(\alpha\mid\beta,\lambda,\text{data}\right)\propto\alpha^{H+a_1-1}e^{-\alpha b_1}\left(1-e^{-\beta u^{-\alpha}}\right)^{n-H}$$
$$\times\prod_{i=1}^{H}y_{i:n}^{-(\alpha+1)}e^{-\beta y_{i:n}^{-\alpha}}, \tag{20}$$

and

$$\pi_2^*\left(\beta\mid\alpha,\lambda,\text{data}\right)\propto\beta^{H+a_2-1}\left(1-e^{-\beta u^{-\alpha}}\right)^{n-H}\ e^{-\beta\left(b_2+\sum_{i=1}^{H}y_{i:n}^{-\alpha}\right)} \tag{21}$$

It is well observed that the conditional posteriors of α and β in Equations , (20) and (21) are not known distributions, so it is not appropriate to use Gibbs sampling and a better choice to implement the MCMC approach is to use the Metropolis-Hasting (M-H) sampler. The following is the algorithm that illustrates the process of the Metropolis–Hastings within Gibbs sampling:

(1) Start with initial guess $\left(\alpha^{(0)},\beta^{(0)}\right)$.

(2) Set $j=1$.

(3) Using the following M-H algorithm, generate $\alpha^{(j)}$ and $\beta^{(j)}$ from $\pi_1^*\left(\alpha^{(j-1)}\mid\beta^{(j-1)},\text{data}\right)$ and

$\pi_2^*\left(\beta^{(j-1)}\mid\alpha^{(j)},\text{data}\right)$ with the normal proposal distributions

$$N\left(\alpha^{(j-1)},var\left(\alpha\right)\right)\text{ and }N\left(\beta^{(j-1)},var\left(\alpha\right)\right).$$

(i) Generate proposal α^* from $N\left(\alpha^{(j-1)},var\left(\alpha\right)\right)$ and β^* from $N\left(\beta^{(j-1)},var\left(\alpha\right)\right)$

(ii) Evaluate the acceptance probabilities

$$\eta_\alpha=\min\left[1,\frac{\pi_1^*\left(\alpha^*\mid\beta^{(j-1)},\text{data}\right)}{\pi_1^*\left(\alpha^{(j-1)}\mid\beta^{(j-1)},\text{data}\right)}\right],$$
$$\eta_\beta=\min\left[1,\frac{\pi_2^*\left(\beta^*\mid\alpha^{(j)},\text{data}\right)}{\pi_2^*\left(\beta^{(j-1)}\mid\alpha^{(j)},\text{data}\right)}\right].$$

(iii) Generate a $u_1$and u_2 from a Uniform $(0,1)$ distribution.

(iv) If $u_1<\eta_\alpha$, accept the proposal and set $\alpha^{(j)}=\alpha^*$, else set $\alpha^{(j)}=\alpha^{(j-1)}$.

(v) If $u_2<\eta_\beta$, accept the proposal and set $\beta^{(j)}=\beta^*$, else set $\beta^{(j)}=\beta^{(j-1)}$.

(4) Compute the reliability function, hazard function as

$$\begin{cases} S^{(j)}(t) = 1 - e^{-\beta^{(j)} t^{-\alpha^{(j)}}}; & t \geq 0 \\ h^{(j)}(t) = \dfrac{\alpha^{(j)} \beta^{(j)} t^{-(\alpha^{(j)}+1)} e^{-\beta^{(j)} t^{-\alpha^{(j)}}}}{1 - e^{-\beta^{(j)} t^{-\alpha^{(j)}}}}; & t \geq 0 \end{cases}.$$

(5) Set $j = j + 1$.

(6) Repeat Steps $(3) - (5)$ N times and obtain $\alpha^{(i)}, \beta^{(i)}, S^{(i)}(t)$ and $h^{(i)}(t), i = 1, 2, ...N$. The first M simulated varieties will be discarded to guarantee the convergence and the removal of the effect of the selection of initial values. Then the selected samples are $\alpha^{(j)}, \beta^{(j)}, S^{(j)}(t)$ and $h^{(j)}(t), j = M+1, ...N$, for sufficiently large N, form an approximate posterior samples which can be used to develop the Bayesian inferences.

The proposed distributions are chosen to be normal distributions as proposals for generating samples in Metropolis-Hasting (M-H) algorithm, as one of the assumptions to apply MCMC is that the proposed distribution should be symmetric, see [14]. The accepted function involved in Metropolis-Hastings (MH) algorithm guarantees that the proposed distribution is the target posterior that we are interested in, see [9].

Based on SEL, the approximate Bayes estimates of $\varphi = \alpha, \beta, S(t)$ or $h(t)$ is given by

$$\hat{\varphi}_{BS} = \frac{1}{N - M} \sum_{j=M+1}^{N} \varphi^{(j)}, \tag{22}$$

The credible intervals (CRIs) of α and β , $S(t)$ and $h(t)$ can be computed by sorting $\alpha^{(i)}, \beta^{(i)}, S^{(i)}(t)$ and $h^{(i)}(t)$,
$i = M + 1, 2, ...N$. Then the $100(1 - \vartheta)\%$ CRIs of $\varphi = \alpha, \beta, S(t)$ or $h(t)$ will be $\left(\varphi_{(N\ \vartheta/2)}, \varphi_{(N\ (1-\vartheta/2))}\right)$.

4. Simulation study

In this section a simulation study is performed utilizing 1000 Type-II hybrid censored samples in order to compare the estimators of parameters and some lifetime parameters reliability function and hazard function of the type -II Gumbel distribution. The samples are generaed from type -II Gumbel distribution, with initial values $\alpha_\circ = 0.06$ and $\beta_\circ = 1.3$. The comparison between the different methods of the resulting estimators of α, β,$S(t)$ and $h(t)$, at $t = 1000000$, has been considered in their mean square error (MSE) which is computed, for $k = 1, 2, 3, 4$ ($\psi_1 = \alpha,\ \psi_2 = \beta,\ \psi_3 = S(10^6), \psi_4 = h(10^6)$), as $MSE(\psi_k) = \frac{1}{M} \sum_{i=1}^{M} \left(\hat{\psi}_k^{(i)} - \psi_k\right)^2$, where $M = 1000$ is the number of simulated samples. Another criterion is used to compare the 95% CIs obtained by using asymptotic distributions of the MLEs and CRIs. The comparison of them is made in terms of the average confidence interval lengths (ACLs) and coverage probability (CP). The CP of a confidence interval is the proportion of the time that the interval contains the initial value of interest. The hyperparameters for the informative priors are chosen as follows: $a_1 = a_2 = b_1 = b_2 = 0.001$. The results of esimate parameters and their MSE are shown in Table 1, while the results of ACL and CP of 95% CIs are shown in Table 2.

From the results, the following notices can be observed:

(1) It is observed that as the values of n, R and T increase, the MSEs decrease and Bayes estimates have the smallest MSEs for α,β and $S(t)$.

(2) From Table 2, it can be seen that the CRIs give more accurate results than the ACIs, for different values of n, R and T .
(3) The estimated values for $h(t)$ are zero in all cases and hence it is not stated in Tables 1 and 2.

Table **1** : Comparison between the average of MLEs and Bayesian estimates according to the MSE.

			α		β		$S(t)$	
			MLE	Bayesian	MLE	Bayesian	MLE	Bayesian
n	R	T						
20	12	4	0.0676 (0.0003)	0.0676 (0.0003)	1.2817 (0.0679)	1.2816 (0.068)	0.398 (0.0107)	0.398 (0.0107)
		7	0.0678 (0.0003)	0.0678 (0.0003)	1.274 (0.0692)	1.2741 (0.0691)	0.3963 (0.0116)	0.3963 (0.0116)
	16	4	0.0646 (0.0002)	0.0647 (0.0002)	1.2991 (0.0636)	1.2991 (0.0636)	0.4128 (0.0082)	0.4128 (0.0082)
		7	0.0656 (0.0002)	0.0656 (0.0002)	1.2875 (0.0618)	1.2875 (0.0618)	0.4062 (0.0086)	0.4062 (0.0086)
30	18	8	0.0651 (0.0002)	0.0651 (0.0002)	1.2908 (0.0541)	1.2909 (0.0542)	0.4094 (0.0079)	0.4095 (0.0079)
		12	0.0644 (0.0002)	0.0644 (0.0002)	1.282 (0.0509)	1.2819 (0.0509)	0.4101 (0.0069)	0.4101 (0.0069)
	24	8	0.0638 (0.0001)	0.0638 (0.0001)	1.3123 (0.0466)	1.3123 (0.0465)	0.4193 (0.0056)	0.4193 (0.0056)
		12	0.0634 (0.0001)	0.0634 (0.0001)	1.3002 (0.0483)	1.3001 (0.0483)	0.4178 (0.0056)	0.4177 (0.0056)
50	30	15	0.0626 (0.0001)	0.0626 (0.0001)	1.3115 (0.0353)	1.3115 (0.0354)	0.4243 (0.0041)	0.4242 (0.0041)
		25	0.0633 (0.0001)	0.0633 (0.0001)	1.3123 (0.0361)	1.3124 (0.0362)	0.4217 (0.0046)	0.4217 (0.0046)
	50	15	0.0619 (0.0001)	0.0619 (0.0001)	1.3134 (0.0334)	1.3133 (0.0335)	0.427 (0.0032)	0.4269 (0.0032)
		25	0.0619 (0.0001)	0.0619 (0.0001)	1.3159 (0.0346)	1.3159 (0.0347)	0.4277 (0.0033)	0.4277 (0.0033)

Table 2 : Average lengths of ACIs and CRIs for the estimates with their corresponding coverage probabilities.

			α		β		$s(t)$	
			Non-Bayesian	Bayesian	Non-Bayesian	Bayesian	Non-Bayesian	Bayesian
20	12	4	0.0582 (0.9402)	0.0004 (0.9563)	1.1728 (0.9428)	0.0077 (0.9607)	0.395 (0.9576)	0.0024 (0.9649)
		7	0.0582 (0.9553)	0.0004 (0.9411)	1.1674 (0.9552)	0.0079 (0.9654)	0.3936 (0.9613)	0.0024 (0.9671)
20	16	4	0.0488 (0.9608)	0.0003 (0.9633)	1.1639 (0.9727)	0.0079 (0.9700)	0.3636 (0.9593)	0.0023 (0.947)
		7	0.0494 (0.9684)	0.0003 (0.9409)	1.1545 (0.956)	0.0077 (0.9561)	0.3629 (0.9659)	0.0023 (0.9624)
30	18	8	0.0456 (0.9454)	0.0003 (0.9405)	0.9604 (0.9749)	0.0066 (0.9706)	0.3244 (0.9415)	0.002 (0.9549)
		12	0.045 (0.9506)	0.0003 (0.9666)	0.9548 (0.9572)	0.0064 (0.9716)	0.3254 (0.9406)	0.002 (0.9717)
30	24	8	0.0391 (0.9507)	0.0003 (0.9584)	0.9562 (0.9517)	0.0065 (0.9716)	0.2987 (0.9452)	0.0019 (0.9619)
		12	0.0388 (0.9605)	0.0003 (0.9627)	0.9486 (0.9700)	0.0064 (0.9601)	0.2986 (0.9623)	0.0019 (0.9656)
50	30	15	0.0339 (0.9606)	0.0002 (0.9536)	0.7509 (0.9413)	0.005 (0.9555)	0.2535 (0.9553)	0.0016 (0.9560)
		25	0.0342 (0.9450)	0.0002 (0.9712)	0.7514 (0.9718)	0.005 (0.9741)	0.2532 (0.9577)	0.0015 (0.9531)
50	50	15	0.0269 (0.9730)	0.0002 (0.9467)	0.7367 (0.9688)	0.0049 (0.9711)	0.2235 (0.9427)	0.0014 (0.9725)
		25	0.0269 (0.9631)	0.0002 (0.9496)	0.738 (0.9454)	0.005 (0.9545)	0.2235 (0.9409)	0.0014 (0.9528)

From Table 2, it is noticed that the average lengths of the approximate confidence intervals for the parameters are very high compared to those in Bayesian case since the credible intervals are established from values generated directly from posterior distributions which are approximately semi-normal according to Metropolis-Hastings (MH) algorithm, also the variations between these values are small as the MCMC approach requires discarding some values for getting accurate results. On the other hand, calculations of the approximate confidence intervals (ACIs) depend on the variance of the maximum likelihood estimates, obtained from the Fisher information Matrix, as we have no method to avoid or discard some singular values to get more accurate results like MCMC approach.

5. Application to real-life data

In this section, the proposed estimation methods are applied to the failure data for times to breakdown of an insulating fluid between electrodes, tested at 34 kilovolts, extracted from ([16], p. 105). The Kolmogorov Smirnov (K-S) distance between the empirical distribution of failure data and CDF of type -II Gumbel distribution is 0.157957 with $P-value$ equals 0.673181. Hence, the type -II Gumbel distribution fits well to the given data.

Table 3: Times (in minutes) to breakdown of an insulating fluid at 34 kilovolts

0.19	0.78	0.96	1.31	2.78	3.16	4.15	4.67	4.85	6.5
7.35	8.01	8.27	12.06	31.75	32.52	33.91	36.71	72.89	

The MLEs of parameters, reliability and hazard functions based on Type-II Hybrid data are presented in Table 4. The Bayes estimates relative to SEL function for the parameters α and β as well as the reliability and hazard functions at $t = 1000000$ minutes, are also displayed in Table 4.

Table 4: Point estimates for the parameters, $S(t)$ and $h(t)$

	MLE	SEL
α	0.111048	0.110278
β	3.85101	3.74402
$S(t)$	0.564123	0.557833
$h(t)$	7.12503×10^{-10}	$7.13541 \times \times 10^{-10}$

From Table 4, the values of estimates are close together which indicates the good performance of the estimators. The 95% ACIs and CRIs for the parameters α and β, the reliability and hazard functions are computed and the results are displayed in Table 5.

Table 5: 95% CIs of α,β,$S(t)$ and $h(t)$

Parameter	MLE	MCMC
α	$[-5.57123, 5.79332]$	$[0.10557, 0.114857]$
β	$[-147.363, 155.065]$	$[3.70068, 3.77925]$
$S(t)$	$[-13.6403, 14.7686]$	$[0.535836, 0.582687]$
$h(t)$	$\left[-4.89432 \times 10^{-8}, 5.03682 \times 10^{-8}\right]$	$\left[6.61103 \times 10^{-10}, 7.63162 \times 10^{-10}\right]$

We note that the value of the hazard function at $t = 1000000$ approaches to zero. In fact, the hazard function is not a density or a probability. However, we can think of it as the probability of failure in an infinitesimally small time period between time t and $t+\Delta t$, $\Delta t \longrightarrow 0$, given that the subject has survived up till time t. In this sense, the hazard is a measure of risk: the greater the hazard between times t and $t + \Delta t$, the greater the risk of failure in this time interval. The probability that the insulated oil completes its mission successfully through duration time, $t = 1000000$ minutes which is equivalent to approximately two years, is about 56%. According to these results, officials and engineers can assign maintenance plans to avoid failure risks.

6. Conclusion

In this study, the classical MLE and Bayesian inference procedures are discussed based on Type-II hybrid censored sample, for the parameters of Type-II Gumbel distribution which fits well to the data representing the breakdown time of an insulating fluid between electrodes. The classical and Bayesian estimations are numerically compared and appropriate comments are finally provided. The computational results show that increasing the sample size improves the performances of all estimators. The study establishes that it is appropriate to use the mentioned distribution to estimate the probability of fluid breakdown in actual use.

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