TWMS J. App. and Eng. Math. V.12, N.2, 2022, pp. 670-680

NONUNIFORM p - TIGHT WAVELET FRAMES ON POSITIVE HALF LINE

O. AHMAD[1], N. AHMAD[1], §

Abstract. Wavelet frames have gained considerable popularity during the past decade, primarily due to their substantiated applications in diverse and widespread fields of engineering and science. Tight wavelet frames provide representations of signals and images where repetition of the representation is favored and the ideal reconstruction property of the associated filter bank algorithm, as in the case of orthonormal wavelets is kept. The main objective of this paper is to introduce a notion of nonuniform wavelet system in $L^2(\mathbb{R}^+)$ and provide a complete characterization of such systems to be tight nonuniform wavelet frames in $L^2(\mathbb{R}^+)$ by using Walsh-Fourier transform.

Keywords: Wavelet frame, Walsh functions, Walsh-Fourier transform.

AMS Subject Classification: 42C15, 42C40, 43A70, 42A38.

1. Introduction

In real life application, all signals are not obtained from uniform shifts; therefore, there is a natural question regarding analysis and decompositions of these types of signals by a stable mathematical tool. Gabardo and Nashed [13] filled this gap by the concept of nonuniform multiresolution analysis and nonuniform wavelets based on the theory of spectral pairs for which the associated translation set $\Lambda = \{0, r/N\} + 2\,\mathbb{Z}$ is no longer a discrete subgroup of $\mathbb{R}$ but a spectrum associated with a certain one-dimensional spectral pair and the associated dilation is an even positive integer related to the given spectral pair. In the framework of mathematical analysis and linear algebra, redundant representations are obtained by analysing vectors with respect to an overcomplete system.Then the obtained vectors are interpreted using the frame theory as introduced by Duffin and Schaeffer [11] and recently studied at depth, see [9] and the compressive list of references therein. Most commonly used coherent/structured frames are wavelet, Gabor, and wave-packet frames which are a mixture type of wavelet and Gabor frames [9]. Frames provide a useful model to obtain signal decompositions in cases where redundancy, robustness, over-sampling, and irregular sampling ploy a role. Today, the theory of frames has become an interesting

[1] Department of Mathematics, National Institute of Technology, Srinagar-190006, Jammu and Kashmir, India.
e-mail: siawoahmad@gmail.com ORCID: https://orcid.org/0000-0002-9903-2656.
e-mail: neyaz@nitsri.net; ORCID: https://orcid.org/0000-0001-7557-1404.
§ Manuscript received: March 19, 2020; accepted: April 10, 2020.
TWMS Journal of Applied and Engineering Mathematics, Vol.12, No.2

and fruitful field of mathematics with abundant applications in signal processing, image processing, harmonic analysis, Banach space theory, sampling theory, wireless sensor networks, optics, filter banks, quantum computing, and medicine. Recall that a countable collection $\{f_k : k \in \mathbb{Z}\}$ in an infinite-dimensional separable Hilbert space H is called a *frame* if there exist positive constants A and B such that

$$A\|f\|^2 \le \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \le B\|f\|^2, \tag{1.1}$$

holds for every $f \in \mathcal{H}$ and we call the optimal constants A and B the lower frame bound and the upper frame bound, respectively. If we only require the second inequality to hold in (1.1), then $\{f_k : k \in \mathbb{Z}\}$ is called a *Bessel collection.* A frame is tight if $A = B$ in (1.1) and if $A = B = 1$ it is called a *Parseval frame* or a *normalized tight frame.*

Wavelet frames are one of structured frames which are obtained by translating and dilating a finite number of functions. Wavelet frames are different from the orthonormal wavelets because of redundancy. By sacrificing orthonormality and allowing redundancy, wavelet frames become much easier to construct than the orthonormal wavelets. An important problem in practice is therefore to determine conditions on the wavelet function, dilation and translation parameters so that the corresponding wavelet system forms a frame. In her famous book, Daubechies [10] proved the first result on the necessary and sufficient conditions for wavelet frames, and then, Chui and Shi [8] gave an improved result. After about ten years, Casazza and Christenson [7] proved a stronger version of Daubechies sufficient condition for wavelet frames in $L^2(\mathbb{R})$. Recently, Ahmad and his collaborators in the series of papers [2, 3, 4, 20, 21, 22, 23, 24] investigated wavelet and Gabor frames and obtained many interested results.

During last two decades there is a substantial body of work that has been concerned with the wavelet and Gabor frames on positive half line. Kozyrev [15] found a compactly supported p-adic wavelet basis for $L^2(\mathbb{Q}_p)$ which is an analog of the Haar basis. It turns out that these wavelets are eigenfunctions of some p-adic pseudodifferential operators in [17]. Such property used to solve p-adic pseudodifferential equations which are needed for some physical problems. Khrennikov et al. [16] developed a method to find explicitly the solution for a wide class of evolutionary linear pseudo-differential equations. Farkov [12] indicated several differences between the constructed wavelets in Walsh analysis and the classical wavelets, and characterized all compactly supported refinable functions on the Vilenkin group G_p with $p \ge 2$. Manchanda et al. [18] introduced the vector-valued wavelet packets and obtained their properties and orthogonality formulas. Albeverio et al. [5] presented a complete characterization of scaling functions generating an p-MRA, suggested a method for constructing sets of wavelet functions, and proved that any set of wavelet functions generates a p-adic wavelet frame. More Recently, Zhang [25] characterize the shift-invariant Bessel sequences, frame sequences and Riesz sequences in $L^2(\mathbb{R}^+)$ and give a characterization of dual wavelet frames using Walsh-Fourier transform. Motivated and inspired by the above work, we introduce the notion of tight nonuniform wavelet frames generated by Walsh functions and obtain their complete characterization.

The paper is structured as follows. In Section 2, we discuss the notations and basic definitions of Walsh-Fourier analysis. Section 3 is devoted to main results of this paper.

2. Preliminaries on Walsh-Fourier Analysis

As usual, let $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ and $\mathbb{N} = \mathbb{Z}^+ - \{0\}$. Denote by $[x]$ the integer part of x. Let p be a fixed natural number greater than 1. For $x \in \mathbb{R}^+$ and any positive integer j, we set

$$x_j = [p^j x] (\operatorname{mod} p), \qquad x_{-j} = [p^{1-j} x] (\operatorname{mod} p), \tag{2.1}$$

where $x_j, x_{-j} \in \{0, 1, \dots, p-1\}$. Clearly, x_j and x_{-j} are the digits in the p-expansion of x:

$$x = \sum_{j<0} x_{-j} p^{-j-1} + \sum_{j>0} x_j p^{-j}.$$

Moreover, the first sum on the right is always finite. Besides,

$$[x] = \sum_{j<0} x_{-j} p^{-j-1}, \qquad \{x\} = \sum_{j>0} x_j p^{-j},$$

where $[x]$ and $\{x\}$ are, respectively, the integral and fractional parts of x.

Consider on $\mathbb{R}^+$ the addition defined as follows:

$$x \oplus y = \sum_{j<0} \zeta_j p^{-j-1} + \sum_{j>0} \zeta_j p^{-j},$$

with $\zeta_j = x_j + y_j (\operatorname{mod} p)$, $j \in \mathbb{Z} \backslash \{0\}$, where $\zeta_j \in \{0, 1, \dots, p-1\}$ and x_j, y_j are calculated by (2.1). Clearly, $[x \oplus y] = [x] \oplus [y]$ and $\{x \oplus y\} = \{x\} \oplus \{y\}$. As usual, we write $z = x \ominus y$ if $z \oplus y = x$, where $\ominus$ denotes subtraction modulo p in $\mathbb{R}^+$.

Let $\varepsilon_p = \exp(2\pi i/p)$, we define a function $r_0(x)$ on $[0, 1)$ by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/p) \\ \varepsilon_p^\ell, & \text{if } x \in \left[\ell p^{-1}, (\ell+1) p^{-1}\right), \quad \ell = 1, 2, \dots, p-1. \end{cases}$$

The extension of the function r_0 to $\mathbb{R}^+$ is given by the equality $r_0(x+1) = r_0(x), \forall\, x \in \mathbb{R}^+$. Then, the system of *generalized Walsh functions* $\{w_m(x) : m \in \mathbb{Z}^+\}$ on $[0, 1)$ is defined by

$$w_0(x) \equiv 1 \quad \text{and} \quad w_m(x) = \prod_{j=0}^{k} \left(r_0(p^j x)\right)^{\mu_j}$$

where $m = \sum_{j=0}^{k} \mu_j p^j$, $\mu_j \in \{0, 1, \dots, p-1\}$, $\mu_k \neq 0$. They have many properties similar to those of the Haar functions and trigonometric series, and form a complete orthogonal system. Further, by a Walsh polynomial we shall mean a finite linear combination of Walsh functions. For $x, y \in \mathbb{R}^+$, let

$$\chi(x, y) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j y_{-j} + x_{-j} y_j)\right), \tag{2.2}$$

where x_j, y_j are given by equation (2.1).

We observe that

$$\chi\left(x, \frac{m}{p^n}\right) = \chi\left(\frac{x}{p^n}, m\right) = w_m\left(\frac{x}{p^n}\right), \qquad \forall\, x \in [0, p^n),\ m, n \in \mathbb{Z}^+,$$

and

$$\chi(x \oplus y, z) = \chi(x, z)\, \chi(y, z), \quad \chi(x \ominus y, z) = \chi(x, z)\, \overline{\chi(y, z)},$$

where $x, y, z \in \mathbb{R}^+$ and $x \oplus y$ is p-adic irrational. It is well known that systems $\{\chi(\alpha, .)\}_{\alpha=0}^{\infty}$ and $\{\chi(\cdot, \alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2[0,1]$ (See [14, 19]).

The *Walsh-Fourier transform* of a function $f \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^+} f(x)\, \overline{\chi(x, \xi)}\, dx, \tag{2.3}$$

where $\chi(x, \xi)$ is given by (2.2). The Walsh-Fourier operator $F : L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$, $Ff = \hat{f}$, extends uniquely to the whole space $L^2(\mathbb{R}^+)$. The properties of the Walsh-Fourier transform are quite similar to those of the classic Fourier transform (see [14, 19]). In particular, if $f \in L^2(\mathbb{R}^+)$, then $\hat{f} \in L^2(\mathbb{R}^+)$ and

$$\left\| \hat{f} \right\|_{L^2(\mathbb{R}^+)} = \left\| f \right\|_{L^2(\mathbb{R}^+)}. \tag{2.4}$$

Moreover, if $f \in L^2[0, 1]$, then we can define the Walsh-Fourier coefficients of f as

$$\hat{f}(n) = \int_0^1 f(x)\overline{w_n(x)}\, dx. \tag{2.5}$$

The series $\sum_{n \in \mathbb{Z}^+} \hat{f}(n) w_n(x)$ is called the *Walsh-Fourier series* of f. Therefore, from the standard L^2-theory, we conclude that the Walsh-Fourier series of f converges to f in $L^2[0, 1]$ and Parseval's identity holds:

$$\left\| f \right\|_2^2 = \int_0^1 \left| f(x) \right|^2 dx = \sum_{n \in \mathbb{Z}^+} \left| \hat{f}(n) \right|^2. \tag{2.6}$$

By p-adic interval $I \subset \mathbb{R}^+$ of range n, we mean intervals of the form

$$I = I_n^k = \left[kp^{-n}, (k+1)p^{-n} \right),\ k \in \mathbb{Z}^+.$$

The p-adic topology is generated by the collection of p-adic intervals and each p-adic interval is both open and closed under the p-adic topology (see [14]). The family $\left\{ [0, p^{-j}) : j \in \mathbb{Z} \right\}$ forms a fundamental system of the p-adic topology on $\mathbb{R}^+$. Therefore, the generalized Walsh functions $w_j(x), 0 \leq j \leq p^n - 1$, assume constant values on each p-adic interval I_n^k and hence continuous on these intervals. Thus, $w_j(x) = 1$ for $x \in I_n^0$.

Let $E_n(\mathbb{R}^+)$ be the space of p-adic entire functions of order n, that is, the set of all functions which are constant on all p-adic intervals of range n. Thus, for every $f \in E_n(\mathbb{R}^+)$, we have

$$f(x) = \sum_{k \in \mathbb{Z}^+} f(p^{-n}k) \chi_{I_n^k}(x), \quad x \in \mathbb{R}^+. \tag{2.7}$$

Clearly each Walsh function of order up to p^{n-1} belongs to $E_n(\mathbb{R}^+)$. The set $E(\mathbb{R}^+)$ of p-adic entire functions on $\mathbb{R}^+$ is the union of all the spaces $E_n(\mathbb{R}^+)$. It is clear that $E(\mathbb{R}^+)$ is dense in $L^p(\mathbb{R}^+), 1 \leq p < \infty$ and each function in $E(\mathbb{R}^+)$ is of compact support. Thus, we consider the following set of functions

$$E^0(\mathbb{R}^+) = \left\{ f \in E(\mathbb{R}^+) :\ \hat{f} \in L^\infty(\mathbb{R}^+) \ \textit{and supp}\ f \subset \mathbb{R}^+ \setminus \{0\} \right\}. \tag{2.8}$$

3. Characterization of Nonuniform Tight wavelet Frames in $L^2(\mathbb{R}^+)$

Given an integer $N \geq 1$ and an odd integer r with $1 \leq r \leq 2N-1, r$ and N are relatively prime, we consider the translation set Λ^+ as

$$\Lambda^+ = \left\{0, \frac{r}{N}\right\} + \mathbb{Z}^+. \tag{3.1}$$

When $N > 1$, the dilation factor of N ensures that

$$N\Lambda^+ \subset \mathbb{Z}^+ \subset \Lambda^+.$$

Definition 3.1. *For a given $\psi \in L^2(\mathbb{R}_+)$, a system of the form*

$$W(\psi, j, \lambda) = \left\{\psi_{j,\lambda} =: N^{j/2}\psi\big(N^j x \ominus \lambda\big);\ j \in \mathbb{Z},\ \lambda \in \Lambda^+\right\}. \tag{3.2}$$

is called the *nonuniform wavelet system* in $L^2(\mathbb{R}^+)$.

On taking Fourier transform, the system (3.2) can be rewritten as

$$\hat{\psi}_{j,\lambda}(\xi) = N^{-j/2}\hat{\psi}\big(N^{-j}\xi\big)\overline{w_\lambda\big(N^{-j}\xi\big)}. \tag{3.3}$$

Definition 3.2. *The wavelet system $W(\psi, j, \lambda)$ defined by (3.2) is called a nonuniform wavelet frame for $L^2(\mathbb{R}^+)$, if there exist constants A and $B, 0 < A \leq B < \infty$ such that for all $\varphi \in L^2(\mathbb{R}^+)$*

$$A\big\|f\big\|_2^2 \leq \sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda_+}\big|\langle\varphi, \psi_{j,\lambda}\rangle\big|^2 \leq B\big\|f\big\|_2^2. \tag{3.4}$$

In order to prove the main result to be presented in this section, we need the following lemma whose proof can be found in [**?**].

Lemma 3.1 *Let $f \in E^0(\mathbb{R}^+)$ and $\psi \in L^2(\mathbb{R}^+)$. If $ess\sup_{\xi\in[1,N]}\sum_{j\in\mathbb{Z}}|\hat{\psi}\left(N^{-j}\xi\right)|^2 < \infty$, then*

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda^+}|\langle f, \psi_{j,\lambda}\rangle|^2 = \int_{\mathbb{R}^+}\left|\hat{f}(\xi)\right|^2\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(N^{-j}\xi\right)\right|^2 d\xi + R_\psi(f) \tag{3.5}$$

where

$$\begin{aligned} R_\psi(f) &= \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^+}\overline{\hat{f}(\xi)}\hat{\psi}\left(N^{-j}\xi\right)\left\{\sum_{\ell=0}^{N-1}\hat{f}\left(\xi \oplus N^j\ell\right)\overline{\hat{\psi}\left(N^{-j}\xi \oplus \ell\right)}\right\}d\xi \\ &= \sum_{j\in\mathbb{Z}}\sum_{\ell=0}^{N-1}\int_K\overline{\hat{f}(\xi)}\hat{\psi}\left(N^{-j}\xi\right)\hat{f}\left(\xi \oplus N^j\ell\right)\overline{\hat{\psi}\left(N^{-j}\xi \oplus \ell\right)}d\xi. \end{aligned} \tag{3.6}$$

Furthermore, the iterated series in (3.6) is absolutely convergent.

The L.H.S of (3.5) converges for all $f \in E^0(\mathbb{R}^+)$ if and only if $\sum_{j\in\mathbb{Z}}|\hat{\psi}\left(N^{-j}\xi\right)|^2$ is locally integrable in $\mathbb{R}^+ \backslash \cup_{j\in\mathbb{Z}} E_j^c$, where E_j is the set of regular points of $\left|\psi\left(N^{-j}\xi\right)\right|^2$, which means that for each $x \in E_j$, we have

$$N^n\int_{\xi - x\in I_n}|\hat{\psi}\left(N^{-j}\xi\right)|^2 d\xi \to |\hat{\psi}\left(N^{-j}\xi\right)|^2 \text{ as } n \to \infty.$$

Then $|E_j^c| = 0$. Thus $|\bigcup_{j\in\mathbb{Z}} E_j^c| = 0$.

Now we state and prove our main result concerning the characterization of the wavelet system $\mathcal{W}(\psi, j, \lambda)$ given by (3.2) to be tight frame for $L^2(\mathbb{R}^+)$.

Theorem 3.2 *The wavelet system $\mathcal{W}(\psi, j, \lambda)$ given by (3.2) is a tight nonuniform wavelet frame for $L^2(\mathbb{R}^+)$ if and only if ψ satisfies*

$$\sum_{j\in\mathbb{Z}} \left|\hat{\psi}\left(N^{-j}\xi\right)\right|^2 = 1, \ \textit{for a.e. } \xi \in [1, N] \tag{3.7}$$

and

$$\sum_{j\in\mathbb{Z}^+} \hat{\psi}\left(N^{-j}\xi\right)\overline{\hat{\psi}\left(N^j(\xi\oplus m)\right)} = 0, \ \textit{for a.e. } \xi \in [1, N], 0 \le m \le N-1. \tag{3.8}$$

Proof. Define

$$S_\psi(m,\xi) = \sum_{k\in\mathbb{Z}^+} \hat{\psi}\left(N^k\xi\right)\overline{\hat{\psi}\left(N^k(\xi\oplus m)\right)}.$$

Assume $f \in E^0(\mathbb{R}^+)$, then for each $\ell \in \mathbb{N}$, there exists $k \in \mathbb{Z}^+$ and a unique $0 \le m \le N-1$ such that $\ell = N^k m$. Since the series in (3.6) is absolutely convergent, we can estimate $R_\psi(f)$ as follows:

$$\begin{aligned}
R_\psi(f) &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^+} \overline{\hat{f}(\xi)}\hat{\psi}\left(N^{-j}\xi\right)\left\{\sum_{\ell\in\mathbb{N}} \hat{f}\left(\xi\oplus N^j\ell\right)\overline{\hat{\psi}\left(N^{-j}\xi\oplus \ell\right)}\right\}d\xi \\
&= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^+} \overline{\hat{f}(\xi)}\hat{\psi}\left(N^{-j}\xi\right)\left\{\sum_{k\in\mathbb{Z}^+}\sum_{m=0}^{N-1} \hat{f}\left(\xi\oplus N^{j+k}m\right)\overline{\hat{\psi}\left(N^{-j}\xi\oplus N^k m\right)}\right\}d\xi \\
&= \int_{\mathbb{R}^+} \overline{\hat{f}(\xi)}\left\{\sum_{k\in\mathbb{Z}^+}\sum_{m=0}^{N-1}\sum_{j\in\mathbb{Z}} \hat{f}\left(\xi\oplus N^{-j}m\right)\hat{\psi}\left(N^{-j-k}\xi\right)\overline{\hat{\psi}\left(N^{-j+k}\xi\oplus N^k m\right)}\right\}d\xi \\
&= \int_{\mathbb{R}^+} \overline{\hat{f}(\xi)}\left\{\sum_{j\in\mathbb{Z}}\sum_{m=0}^{N-1} \hat{f}\left(\xi\oplus N^j m\right)\sum_{k\in\mathbb{Z}^+}\hat{\psi}\left(N^{-j+k}\xi\right)\overline{\hat{\psi}\left(N^k\left(N^{-j}\xi\oplus m\right)\right)}\right\}d\xi \\
&= \int_{\mathbb{R}^+} \overline{\hat{f}(\xi)}\left\{\sum_{j\in\mathbb{Z}}\sum_{m=0}^{N-1} \hat{f}\left(\xi\oplus N^j m\right)S_\psi(m, N^{-j}\xi)\right\}d\xi.
\end{aligned}$$

Let us collect the results we have obtained: If $\psi \in L^2(\mathbb{R}^+)$ and $f \in E^0(\mathbb{R}^+)$, then

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda^+} |\langle f, \psi_{j,\lambda}\rangle|^2 &= \int_{\mathbb{R}^+} |\hat{f}(\xi)|^2 \sum_{j\in\mathbb{Z}} |\hat{\psi}\left(N^{-j}\xi\right)|^2 d\xi \\
&\quad + \int_{\mathbb{R}^+} \overline{\hat{f}(\xi)}\sum_{j\in\mathbb{Z}}\sum_{m=0}^{N-1} \hat{f}\left(\xi\oplus N^{-j}m\right)S_\psi(m, N^{-j}\xi)d\xi.
\end{aligned} \tag{3.9}$$

The last integrand is integrable and so is the first when $\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(N^{-j}\xi\right)\right|^2$ is locally integrable in $\mathbb{R}^+ \setminus \cup_{j\in\mathbb{Z}} E_j^c$. Further, equation (3.8) implies that

$$S_\psi(m,\xi) = 0 \text{ for all } \ 0 \le m \le N-1.$$

On Combining (3.9) together with (3.7) and (3.8), we obtain

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda^+} |\langle f, \psi_{j,\lambda}\rangle|^2 = \|f\|_2^2, \ \forall \ f \in E^0(\mathbb{R}^+).$$

Since $E^0(\mathbb{R}^+)$ is dense in $L^2(\mathbb{R}^+)$, hence the wavelet system $\mathcal{W}(\psi, j, \lambda)$ given by (3.2) is a tight nonuniform wavelet frame for $L^2(\mathbb{R}^+)$.

Conversely, suppose that the system $\mathcal{W}(\psi, j, \lambda)$ given by (3.2) is a tight nonuniform wavelet frame for $L^2(\mathbb{R}^+)$, then we need to show that the two equations (3.7) and (3.8) are satisfied. Since $\{\psi_{j,\lambda}(x) : j \in \mathbb{Z}, \lambda \in \Lambda^+\}$ is a tight nonuniform wavelet frame for $L^2(\mathbb{R}^+)$, then we have

$$\sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda^+} |\langle f, \psi_{j,\lambda}\rangle|^2 = \|f\|_2^2, \ \forall\ f \in E^0(\mathbb{R}^+). \tag{3.10}$$

Since $\sum_{j\in\mathbb{Z}} |\hat{\psi}\left(N^{-j}\xi\right)|^2$ is locally integrable in $\mathbb{R}^+\setminus \cup_{j\in\mathbb{Z}} E_j^c$. Therefore, for each $\xi_0 \in \mathbb{R}^+\setminus \cup_{j\in\mathbb{Z}} E_j^c$, we consider

$$\hat{f}_1(\xi) = N^{\frac{M}{2}} w_M(\xi - \xi_0)$$

where $f = f_1$ and $w_M(\xi - \xi_0)$ is the Walsh function of $\xi_0 + I_M$. Then, it follows that for $0 \le \ell \le N-1$, $\overline{\hat{f}(\xi)}\hat{f}(\xi \oplus N^{-j}\ell) \equiv 0$, since ξ and $\xi \oplus N^{-j}\ell$ cannot be in $\xi_0 + I_M$ simultaneously and hence, $\|f_1\|_2^2 = 1$. Furthermore, we have

$$1 = \|f_1\|_2^2 = \|\hat{f}_1\|_2^2 = \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda^+} |\langle f, \psi_{j,\lambda}\rangle|^2 = \int_{\xi_0+I_M} \sum_{j\in\mathbb{Z}} N^M \left|\hat{\psi}\left(N^{-j}\xi\right)\right|^2 d\xi + R_\psi(f_1).$$

By letting $M \to \infty$, we obtain

$$1 = \sum_{j\in\mathbb{Z}} \left|\hat{\psi}\left(N^{-j}\xi_0\right)\right|^2 + \lim_{M\to\infty} R_\psi(f_1). \tag{3.11}$$

Now, we proceed to estimate $R_\psi(f_1)$ as:

$$R_\psi(f_1) = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^+} \overline{\hat{f}_1(\xi)}\hat{\psi}\left(N^{-j}\xi\right) \left\{\sum_{\ell\in\mathbb{N}} \hat{f}_1\left(\xi \oplus N^j\ell\right) \overline{\hat{\psi}\left(N^{-j}\xi \oplus \ell\right)}\right\} d\xi$$

$$\begin{aligned} |R_\psi(f_1)| &\le \sum_{j\in\mathbb{Z}} \sum_{\ell\in\mathbb{N}} \int_{\mathbb{R}^+} \left|\hat{f}_1(\xi)\hat{\psi}\left(N^{-j}\xi\right) \hat{f}_1\left(\xi \oplus N^j\ell\right) \hat{\psi}\left(N^{-j}\xi \oplus \ell\right)\right| d\xi \\ &= \sum_{j\in\mathbb{Z}} \sum_{\ell\in\mathbb{N}} N^j \int_{\mathbb{R}^+} \left|\hat{f}_1(N^j\xi)\hat{f}_1\left(N^j(\xi \oplus \ell)\right) \hat{\psi}\left(\xi\right) \hat{\psi}\left(\xi \oplus \ell\right)\right| d\xi. \end{aligned}$$

Note that

$$\left|\hat{\psi}(\xi)\hat{\psi}\left(\xi \oplus \ell\right)\right| \le \frac{1}{2}\left(\left|\hat{\psi}(\xi)\right|^2 + \left|\hat{\psi}\left(\xi \oplus \ell\right)\right|^2\right)$$

Therefore, we have

$$|R_\psi(f_1)| \le \sum_{j\in\mathbb{Z}} \sum_{\ell\in\mathbb{N}} N^j \int_{\mathbb{R}^+} \left|\hat{f}_1\left(N^j\xi\right) \hat{f}_1\left(N^j(\xi \oplus \ell)\right) ||\hat{\psi}(\xi)\right|^2 d\xi. \tag{3.12}$$

Since $\ell \ne 0, (\ell \in \mathbb{N})$ and $f_1 \in E^0(\mathbb{R}^+)$, there exists a constant $J > 0$ such that

$$\hat{f}_1\left(N^jt\right) \hat{f}_1\left(N^jt \oplus N^j\ell\right)\right) = 0, \ \forall\ |j| > J.$$

On the other hand, for each $|j| \le J$, there exists a constant L such that

$$\hat{f}_1\left(N^jt \oplus N^j\ell\right)\right) = 0, \ \forall\ \ell > L.$$

This means that only finite terms of the series on the R.H.S of (3.12) are non-zero. Consequently, there exits a constant C such that

$$|R_\psi(f_1)| \le C\|\hat{f}_1\|_\infty^2 \|\hat{\psi}\|_2^2 = CN^m \|\hat{\psi}\|_2^2$$

which implies

$$\lim_{M\to\infty} |R_\psi(f_1)| = 0.$$

Hence equation (3.11) becomes

$$\sum_{j\in\mathbb{Z}} \left|\hat{\psi}\left(N^{-j}\xi_0\right)\right|^2 = 1.$$

Finally, we must show that if (3.10) hold for all $f \in E^0(\mathbb{R}^+)$, then equation (3.8) is true. From equalities (3.9), (3.10) and just established equality (3.7), we have

$$\sum_{j\in\mathbb{Z}} \sum_{m=0}^{N-1} \int_{\mathbb{R}^+} \overline{\hat{f}(\xi)} \hat{f}\left(\xi \oplus N^j m\right) S_\psi(m, N^j \xi) d\xi = 0, \ \forall \ f \in E^0(\mathbb{R}^+).$$

By invoking polarization identity, we then have

$$\sum_{j\in\mathbb{Z}} \sum_{m=0}^{N-1} \int_{\mathbb{R}^+} \overline{\hat{f}(\xi)} \hat{g}\left(\xi \oplus N^j m\right) S_\psi(m, N^{-j}\xi)\, d\xi = 0, \ \forall \ f, g \in E^0(\mathbb{R}^+). \tag{3.13}$$

Let us fix $m_0 \in \{0, 1, 2, \cdots, N-1\}$ and $\xi_0 \in \mathbb{R}^+ \setminus \cup_{j\in\mathbb{Z}} E_j^c$ such that neither $\xi_0 \neq 0$ nor $\xi_0 + m_0 \neq 0$. Setting $f = f_1$ and $g = g_1$ such that

$$\hat{f}_1(\xi) = N^{\frac{M}{2}} w_M(\xi - \xi_0) \text{ and } \hat{g}_1(\xi) = \hat{f}_1(\xi \ominus m_0).$$

Then, we have

$$\hat{f}_1(\xi)\hat{g}_1(\xi \oplus m_0) = N^M w_M(\xi - \xi_0). \tag{3.14}$$

Now, equality (3.13) can be written as

$$0 = N^M \int_{\xi_0 + I_M} S_\psi(m_0, \xi) d\xi + I_1,$$

where

$$I_1 = \sum_{\substack{j\in\mathbb{Z} \\ (j,m)\neq(0,m_0)}} \sum_{m=0}^{N-1} \int_{\mathbb{R}^+} \overline{\hat{f}_1(\xi)} \hat{g}_1\left(\xi \oplus N^j m\right) S_\psi(m, N^{-j}\xi) d\xi. \tag{3.15}$$

Since the first summand in (3.14) tends to $S_\psi(m_0, \xi_0)$ as $M \to \infty$. Therefore, we shall prove that

$$\lim_{M\to\infty} I_1 = 0.$$

Since $m \neq 0$, $(m \in \mathbb{N})$ and $f_1, g_1 \in E^0(\mathbb{R}^+)$, there exists a constant $J_0 > 0$ such that

$$\overline{\hat{f}_1(\xi)}\, \hat{g}_1\left(\xi \oplus N^j m\right) = 0 \ \forall \ j > J_0.$$

Therefore, we have

$$I_1 = \sum_{j\le J_0} \sum_{m=0}^{N-1} \int_{\mathbb{R}^+} \overline{\hat{f}_1(\xi)} \hat{g}_1\left(\xi \oplus N^j m\right) S_\psi(m, N^{-j}\xi) d\xi$$

$$|I_1| \le \sum_{j\le J_0} \sum_{m=0}^{N-1} N^j \int_{\mathbb{R}^+} \left|\overline{\hat{f}_1(N^j\xi)} \hat{g}_1\left(N^j(\xi \oplus m)\right)\right| |S_\psi(m, \xi)|\, d\xi.$$

Since

$$2\left|S_\psi(m,\xi)\right| \le \sum_{k\in\mathbb{Z}^+} |\hat{\psi}\left(N^k\xi\right)|^2 + \sum_{k\in\mathbb{Z}^+} \left|\hat{\psi}\left(N^k(\xi\oplus m)\right)\right|^2,$$

hence

$$|I_1| \le I_1^{(1)} + I_1^{(2)}$$

where

$$I_1^{(1)} = \sum_{j\le J_0}\sum_{m=0}^{N-1} N^j \int_{\mathbb{R}^+} \left|\hat{f}_1(N^j\xi)\right| \left|\hat{g}_1\left(N^j(\xi\oplus m)\right)\right| [\tau(\xi)]^2 d\xi,$$

with

$$\int_K [\tau(\xi)]^2 d\xi = \frac{1}{2}\sum_{k\in\mathbb{Z}^+}\int_{\mathbb{R}^+}\left|\hat{\psi}\left(N^k\xi\right)\right|^2 d\xi = \|\hat{\psi}\|_2^2 < \infty,$$

and

$$\begin{aligned} I_1^{(2)} &= \sum_{j\le J_0}\sum_{m=0}^{N-1} N^j \int_{\mathbb{R}^+} \left|\hat{f}_1(N^j\xi)\right| \left|\hat{g}_1\left(N^j(\xi\oplus m)\right)\right| [\tau(\xi\oplus m)]^2 d\xi \\ &= \sum_{j\le J_0}\sum_{m=0}^{N-1} N^j \int_{K} \left|\hat{f}_1(N^j(\eta\ominus m))\right| \left|\hat{g}_1\left(N^j\eta\right)\right| [\tau(\eta)]^2 d\xi. \end{aligned}$$

Thus $I_1^{(2)}$ has the same form as $I_1^{(1)}$ with the roles of $\hat{f}_1$ and $\hat{g}_1$ interchanged. As

$$\hat{f}_1(\xi) = N^{\frac{M}{2}} w_M(\xi-\xi_0),$$

therefore, we deduce that

$$I_1^{(1)} = \sum_{j\le J_0}\sum_{m=0}^{N-1} N^j N^{\frac{M}{2}} \int_{N^j\xi_0+I_{-j+M}} \left|\hat{g}_1\left(N^j(\xi\oplus m)\right)\right| [\tau(\xi)]^2 d\xi.$$

Now, if $\hat{g}_1\left(N^j(\xi\oplus m)\right)\neq 0$, then we must have $N^j\xi + N^j m \in \xi_0 + I_M + m_0$ and $|N^j m| \le N^{-M}$, hence $|m| \le N^{-M-j}$. Thus,

$$\begin{aligned} I_1^{(1)} &= \sum_{j\le J_0} N^j N^{\frac{M}{2}} \int_{N^j\xi_0+I_{-j+M}} [\tau(\xi)]^2 \sum_{m=0}^{N-1} \left|\hat{g}_1\left(N^j(\xi\oplus m)\right)\right| d\xi \\ &\le \sum_{j\le J_0} N^j N^{\frac{M}{2}} \int_{N^j\xi_0+I_{-j+M}} [\tau(\xi)]^2 N^{-M-j} N^{\frac{M}{2}} d\xi \\ &\le \sum_{j\le J_0} \int_{N^j\xi_0+I_{-j+M}} [\tau(\xi)]^2 d\xi \end{aligned} \tag{3.16}$$

For given $\xi_0\neq 0$, we choose

$$N^{J_0} < |\xi_0| = N^{-M}.$$

Then, we obtain

$$N^j\xi_0 + I_{-j+M} \subset I_{-J_0+M}\ \forall\ j\le J_0, \tag{3.17}$$

as $|N^j\xi_0| = N^j N^{-M} \le N^{-M}$ and $I_{-j+M}\subset I_{-J_0+M}$. On the other hand, for any $j_1 < j_2 \le J_0$, we claim that

$$\{N^{j_1}\xi_0 + I_{-j_1+M}\}\cap\{N^{j_2}\xi_0 + I_{-j_2+M}\} = \emptyset. \tag{3.18}$$

In fact, for any $x \in N^{j_1}\xi_0 + I_{-j_1+M}$ and $y \in N^{j_2}\xi_0 + I_{-j_2+M}$, write $x = N^{j_1}\xi_0 + x_1$ and $y = N^{j_2}\xi_0 + y_1$, then $|x-y| = \max\{|N^{j_1}\xi_0 - N^{j_2}\xi_0|,\ |x_1 - y_1|\} = N^{j_2-M} \neq 0$. implies that (3.17) holds. Combining (3.15) - (3.17), we obtain

$$I_1^{(1)} \leq \int_{I_{-J_0+M}} [\tau(\xi)]^2 d\xi \to 0 \text{ as } M \to \infty.$$

This completes the proof of the theorem.

4. Conclusions

Tight wavelet frames provide representations of signals and images where repetition of the representation is favored and the ideal reconstruction property of the associated filter bank algorithm, as in the case of orthonormal wavelets is kept. In this paper we introduce notion of Non-uniform Wavelet frames and provide a characterization of these frames via Walsh-Fourier transform. in $L^2(\mathbb{R}_+)$. Intuitively, frames in $L^2(\mathbb{R}_+)$ can be obtained by projection from ones in $L^2(\mathbb{R})$, while it is not the case for $L^2(\mathbb{R}_+)$ because the projections do not have complete affine structure. This is partially because of the fact that $\mathbb{R}_+$ is not a group in terms of usual addition. $\mathbb{R}_+$ is a group under the operation "$\oplus$" by which the Walsh-Fourier transform is defined.

References

[1] Abdullah, (2016), Affine and quasi-sffine frames on positive half line, J. Math. Ext., 10, (3), pp. 47-61.

[2] Ahmad, O., Sheikh, N. A., (2018), On Characterization of nonuniform tight wavelet frames on local fields, Anal. Theory Appl., 34, pp. 135-146.

[3] Ahmad, O., Shah, F. A., Sheikh, N. A., (2018), Gabor frames on non-Archimedean fields, Int. Jour. of Geo. Meth. in Mod. Phy., 15, 1850079, (17 pages).

[4] Ahmad, O., Sheikh, N. A., (2018), Nonuniform wavelet frames on local fields. Jordan Jour. Of Math. And Stat., 11, (1), pp. 51-67.

[5] Albeverio, S., Evdokimov, S., Skopina, M. (2010). p-adic multiresolution analysis and wavelet frames, J. Fourier Anal. Appl., (16), pp. 693-714.

[6] Aldroubi, A., Cabrelli, C., Heil, C., Kornelson, K., Molter, U., (2010), Invariance of a shift-invariant space, J. Fourier Anal. Appl., (16), pp. 60-75.

[7] Casazza, P. G., Christensen, O., (2001), Weyl-Heisenberg frames for subspaces of $L^2(\mathbb{R})$, Proc. Amer. Math. Soc., (129), pp. 145-154

[8] Chui, C. K., Shi, X., (1993), Inequalities of Littlewood-Paley type for frames and wavelets, SIAM J. Math. Anal., (24), pp. 263-277.

[9] Christensen, O. (2016), An Introduction to Frames and Riesz Bases, Second Edition, Birkhäuser, Boston.

[10] Daubechies, I. (1992), Ten Lectures on Wavelets, CBMS-NSF Series in Applied Mathematics, SIAM, Philadelphia.

[11] Duffin, R. J., Shaeffer, A. C., (1952), A class of nonharmonic Fourier series, Trans. Amer. Math. Soc., (72), pp. 341-366.

[12] Farkov, Y. A., (2015), Constructions of MRA-based wavelets and frames in Walsh analysis, Poincare J. Anal. Appl., (2), pp. 13-36.

[13] Gabardo, J. P., Nashed, M., (1998), Nonuniform multiresolution analyses and spectral pairs, J. Funct. Anal., 158, pp. 209-241.

[14] Golubov, B. I., Efimov, A. V., Skvortsov, V. A., (1991), Walsh Series and Transforms: Theory and Applications, (Kluwer, Dordrecht).

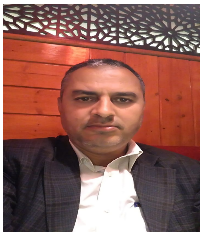
[15] Kozyrev, S. V., (2002), Wavelet analysis as a p-adic spectral analysis, Izv. Akad. Nauk, Ser. Mat., (66), pp. 149-158.

[16] Khrennikov, A. Yu., Shelkovich, V. M., (2009), An infinite family of p-adic non-Haar wavelet bases and pseudo-differential operators, p-Adic Numb. Ultrametr. Anal. Appl., 3, pp. 204-216.

[17] Kozyrev, S. V., (2004), p-adic pseudodifferential operators and p-adic wavelets, Theor. Math. Phys., (138), pp. 1-42.

[18] Manchanda. P., Sharma, V., (2014), Construction of vector valued wavelet packets on $\mathbb{R}^+$ using Walsh-Fourier transform, Indian J. Pure Appl. Math., (45), pp. 539-553.
[19] Schipp, F., Wade, W. R., Simon, P., (1990). Walsh Series: An Introduction to Dyadic Harmonic Analysis, (Adam Hilger, Bristol and New York).
[20] Shah, F. A., Ahmad, O., (2017), Wave packet systems on local fields, Journal of Geometry and Physics, 120, pp. 5-18.
[21] Shah, F. A., Ahmad, O., Rahimi, A., (2018), Frames Associated with Shift Invariant Spaces on Local Fields, Filomat, 32, (9), pp. 3097-3110.
[22] Shah, F. A., Ahmad, O., Jan, H., (2017), Sufficient Conditions for Nonuniform Wavelet Frames on Local Fields, Facta Universtatis, University of Nis, Serbia, 32, (4), pp. 551-563.
[23] Shah, F. A., Ahmad, O., Sheikh, N. A., (2017), Some New Inequalities for Wavelet Frames on Local Fields, Anal. Theory Appl., 33, (2), pp. 134-148.
[24] Shah, F. A., Ahmad, O., Jorgenson, P. E., (2018), Fractional Wave Packet Frames in $L^2(\mathbb{R})$, Jour. of Math. Phy., 59, 073509, doi: 10.1063/1.5047649.
[25] Zhang, Y., (2019), Walsh Shift-Invariant Sequences and p-adic Nonhomogeneous Dual Wavelet Frames in $L^2(\mathbb{R}^+)$, Results Math., 74, (111), 26.

Dr. Owais Ahmad received his Ph.D degree from National Institue of Technology, Srinagar, India. He currently works as a faculty member in the Department of Mathematics, National Institute of Technology, Srinagar, India. His primary research interests span Wavelet Analysis , Quantum Field Theory, Super symmetry, Very special relativity (VSR), Quantization Techniques and Lorentz violating models.



Prof. Neyaz Ahmad received his B.Sc, M.Sc, M.Phil, Ph.D degrees from Aligarh Muslim University. India. He is currently Professor and Head of the Department of Mathematics, National Institute of Technology, Srinagar, India. His area of research includes Wavelet Analysis, Summability Theory and Sequence Spaces. He has published more than 50 research papers in various international journals of high standard..