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BIPOLAR FUZZY GRAPHS BASED ON THE PRODUCT OPERATOR

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Abstract. From both theoretical and experimental perspectives, bipolar fuzzy set theory serves as a foundation for bipolar cognitive modeling and multi-agent decision analysis, where the product operator may be preferred over the min operator in some scenarios. In this paper, we discuss the basic properties of operations on product bipolar fuzzy graphs (PBFGs)(bipolar fuzzy graphs based on the product operator) such as direct product, Cartesian product, strong product, lexicographic product, union, ring sum and join. Also we define the notion of complement of PBFGs and investigate its properties. Moreover, application of PBFG theory is presented in multi-agent decision making.

Keywords: Bipolar fuzzy set, product operator, product bipolar fuzzy graph, decision making.

AMS Subject Classification: 05C99.

1. Introduction

In many domains of information processing, bipolarity is a core feature to be considered. Positive information represents what is possible or preferred, while negative information represents what is forbidden or surely false. If the information is moreover endowed with vagueness and imprecision, then bipolar fuzzy sets (BFSs) constitute an appropriate knowledge representation framework. The BFS [22, 23] is an extension of fuzzy set [21] characterized by a positive membership and negative membership degree. The positive membership degree (0, 1] of an element in a BFS indicates that it partially meets the associated property, the element's negative membership degree $[-1, 0)$ indicates that it partially fulfils the inferred contradictory characteristic, while membership degree 0 shows that it is irrelevant to the property [9]. This area has recently stimulated research in several directions such as applications in preference modeling, argumentation, knowledge representation, cooperative games and multi-criteria decision analysis.

Graph representations are widely used in various domains for dealing with structural information, including operations research, systems analysis, networks, pattern recognition, image interpretation, and economics. However, some aspects of a graph theoretic problem

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may be uncertain in many cases. The vehicle travel time or vehicle capacity on a road network, for example, may not be known precisely. In such instances, it is logical to use fuzzy set theory to cope with the uncertainty. To put it another way, fuzzy graphs are developed to describe structures of object connections in which the existence of a concrete object and the relationship between two objects are matters of degree. Fuzzy graphs have a wide range of applications including control theory, neural networks, information theory, expert systems, cluster analysis, medical diagnosis, database theory, network optimization, and decision making. Rosenfeld [18] investigated fuzzy relations on fuzzy sets and constructed the structure of fuzzy graphs using max and min operations to provide analogs of various core graph theoretical notions. Bhattacharya [6] provided some remarks regarding fuzzy graphs. Mordeson and Peng [11] proposed the idea of strong fuzzy graphs and outlined various operations on fuzzy graphs. Bhutani and Battou [7] further looked at operations on fuzzy graphs that preserved the M-strong characteristic. Mordeson and Peng [11] first presented the complement of a fuzzy graph, which Sunitha and Vijayakumar [19] later improved. Akram [1] pioneered the concept of BFGs and further provide its applications in decision making [2, 3]. The notion of PBFGs was proposed by Rashmanlou et al. [17] and then modified by Naz at el. [12]. Ghorai and Pal [8] discussed different types of PBFGs. Naz et al. [4, 5, 13, 14, 15, 16] developed some new graph models in generalized fuzzy circumstances along its interesting applications in decision making. In this research paper, we study basic properties of operations on graphs in the context of bipolar fuzzy setting based on product operator.

This paper is organized as follows: Section 2 provides background information regarding BFSs and BFGs. In Section 3, we define some operations on PBFGs such as direct product, Cartesian product, strong product, lexicographic product, union, ring sum, join and complement of PBFGs. In Section 4, we provide an application of PBFGs in multi-agent decision making. Finally, in Section 5, we draw conclusions.

2. Preliminaries

In this section, we recall some basic concepts which are necessary for this paper.

Definition 2.1. [22] *A BFS S on a non-empty set $\aleph$ is an object having the following form*

$$S = \{(r, \tau_S^P(r), \tau_S^N(r)) \mid r \in \aleph\}$$

which is characterized by a positive membership function τ_S^P and a negative membership function τ_S^N, where

$$\tau_S^P : \aleph \to [0,1],\ r \in \aleph \to \tau_S^P(r) \in [0,1],$$
$$\tau_S^N : \aleph \to [-1,0],\ r \in \aleph \to \tau_S^N(r) \in [-1,0].$$

If $\tau_S^P(r) \neq 0$ and $\tau_S^N(r) = 0$, then r is regarded as having only positive satisfaction for S. If $\tau_S^P(r) = 0$ and $\tau_S^N(r) \neq 0$, then r does not satisfy the property of S but somewhat satisfies the counter property of S. Finally if $\tau_S^P(r) \neq 0$ and $\tau_S^N(r) \neq 0$, then the membership function of the property overlaps that of its counter property over some portion of $\aleph$.

Definition 2.2. [22] *A mapping $T = (\tau_T^P, \tau_T^N) : \aleph \times \aleph \to [0,1] \times [-1,0]$ is said to be a bipolar fuzzy relation on a non-empty set $\aleph$.*

By introducing the concept of BFSs into the theory of graphs, Akram [1] put forward the notion of the BFGs using min and max operators as follows:

Definition 2.3. [1] *A bipolar fuzzy graph with a finite set $\aleph$ as the underlying set is a pair $\Im = (S, T)$, where $S = (\tau_S^P, \tau_S^N)$ is a BFS in $\aleph$ and $T = (\tau_T^P, \tau_T^N)$ is a bipolar fuzzy relation in $\Re$ such that*

$$\tau_T^P(rs) \leq \min\{\tau_S^P(r), \tau_S^P(s)\} \text{ and } \tau_T^N(rs) \geq \max\{\tau_S^N(r), \tau_S^N(s)\} \text{ for all } r, s \in \aleph.$$

where S is the bipolar fuzzy vertex set of $\Im$ and T is the bipolar fuzzy edge set of $\Im$.

Definition 2.4. [20] *Let $WA : \mathbb{R}^n \to \mathbb{R}$. If $WA_w(\gamma_1, \gamma_2, \ldots, \gamma_n) = \sum_{i=1}^{n} \varkappa_i \gamma_i$, then the function WA is called a weighted averaging (WA) operator, where $\varkappa = (\varkappa_1, \varkappa_2, \ldots, \varkappa_n)^T$ is weight vector of $(\gamma_1, \gamma_2, \ldots, \gamma_n)^T$, $\varkappa_i \in [0, 1] (i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} \varkappa_i = 1$.*

3. Operations on PBFGs

In this section, we define some basic operations on graphs under bipolar fuzzy environment based on the product operator and investigate its properties.

Definition 3.1. [13] *A PBFG with a finite set $\aleph$ as the underlying set is a pair $\Im = (S, T)$, where $S = (\tau_S^P, \tau_S^N)$ is a BFS in $\aleph$ and $T = (\tau_T^P, \tau_T^N)$ is a BFS in $\widetilde{\aleph^2}$ such that*

$$\tau_T^P(rs) \leq \tau_S^P(r)\tau_S^P(s), \ \tau_T^N(rs) \geq -|\tau_S^N(r)||\tau_S^N(s)| \text{ for all } rs \in \widetilde{\aleph^2} \tag{1}$$

$$\text{and } \tau_T^P(rs) = \tau_T^N(rs) = 0 \text{ for all } rs \in \widetilde{\aleph^2} - \Re, \tag{2}$$

where S is the bipolar fuzzy vertex set of $\Im$ and T is the bipolar fuzzy edge set of $\Im$.

Example 3.1. *Consider a PBFG (see Fig. 1) over $\aleph = \{e, f, g, h\}$ defined by*

$$S = \left\langle \left(\frac{e}{0.5}, \frac{f}{0.6}, \frac{g}{0.8}, \frac{h}{0.9} \right), \left(\frac{e}{-0.7}, \frac{f}{-0.4}, \frac{g}{-0.5}, \frac{h}{-0.3} \right) \right\rangle,$$
$$T = \left\langle \left(\frac{ef}{0.3}, \frac{fg}{0.1}, \frac{eg}{0.2}, \frac{gh}{0.6}, \frac{eh}{0}, \frac{fh}{0} \right), \left(\frac{ef}{-0.1}, \frac{fg}{-0.2}, \frac{eg}{-0.3}, \frac{gh}{-0.1}, \frac{eh}{0}, \frac{fh}{0} \right) \right\rangle.$$

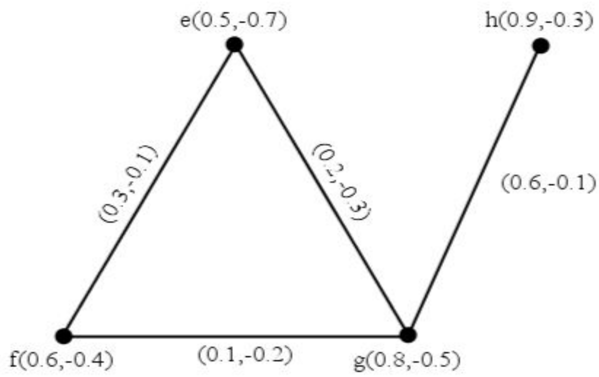

Figure 1. PBFG.

Definition 3.2. *Let $\Im_1 = (S_1, T_1)$ and $\Im_2 = (S_2, T_2)$ be two PBFGs of the graphs $\mathbb{G}_1 = (\aleph_1, \Re_1)$ and $\mathbb{G}_2 = (\aleph_2, \Re_2)$, respectively. The direct product of $\Im_1$ and $\Im_2$ is denoted by $\Im_1 \times \Im_2 = (S_1 \times S_2, T_1 \times T_2)$ and defined as:*

(i): $\begin{cases} (\tau_{S_1}^P \times \tau_{S_2}^P)(r_1, r_2) = \tau_{S_1}^P(r_1)\tau_{S_2}^P(r_2) \\ (\tau_{S_1}^N \times \tau_{S_2}^N)(r_1, r_2) = -|\tau_{S_1}^N(r_1)||\tau_{S_2}^N(r_2)| \quad \text{for all } (r_1, r_2) \in \aleph_1 \times \aleph_2, \end{cases}$

(ii): $\begin{cases} (\tau^P_{T_1} \times \tau^P_{T_2})((r_1, r_2)(s_1, s_2)) = \tau^P_{T_1}(r_1 s_1)\tau^P_{T_2}(r_2 s_2) \\ (\tau^N_{T_1} \times \tau^N_{T_2})((r_1, r_2)(s_1, s_2)) = -|\tau^N_{T_1}(r_1 s_1)||\tau^N_{T_2}(r_2 s_2)| \quad \text{for all } r_1 s_1 \in \widetilde{\aleph^2_1}, \text{ for all } r_2 s_2 \in \widetilde{\aleph^2_2}. \end{cases}$

Proposition 3.1. *The direct product of two PBFGs is a PBFG.*

Proof. Let $\Im_1$ and $\Im_2$ be two PBFGs of $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively. Since all conditions for $S_1 \times S_2$ are obvious. So, we verify only conditions for $T_1 \times T_2$.
Consider $r_1 s_1 \in \widetilde{\aleph^2_1}, r_2 s_2 \in \widetilde{\aleph^2_2}$. Then

$$\begin{aligned} (\tau^P_{T_1} \times \tau^P_{T_2})((r_1, r_2)(s_1, s_2)) &= \tau^P_{T_1}(r_1 s_1)\tau^P_{T_2}(r_2 s_2) \leq (\tau^P_{S_1}(r_1)\tau^P_{S_1}(s_1))(\tau^P_{S_2}(r_2)\tau^P_{S_2}(s_2)) \\ &= (\tau^P_{S_1}(r_1)\tau^P_{S_2}(r_2))(\tau^P_{S_1}(s_1)\tau^P_{S_2}(s_2)) \\ &= (\tau^P_{S_1} \times \tau^P_{S_2})((r_1, r_2))(\tau^P_{S_1} \times \tau^P_{S_2})((s_1, s_2)), \\ (\tau^N_{T_1} \times \tau^N_{T_2})((r_1, r_2)(s_1, s_2)) &= -|\tau^N_{T_1}(r_1 s_1)||\tau^N_{T_2}(r_2 s_2)| \geq -(|\tau^N_{S_1}(r_1)||\tau^N_{S_1}(s_1)|)(|\tau^N_{S_2}(r_2)||\tau^N_{S_2}(s_2)|) \\ &= -(|\tau^N_{S_1}(r_1)||\tau^N_{S_2}(r_2)|)(|\tau^N_{S_1}(s_1)||\tau^N_{S_2}(s_2)|) \\ &= -|(\tau^N_{S_1} \times \tau^N_{S_2})((r_1, r_2))||(\tau^N_{S_1} \times \tau^N_{S_2})((s_1, s_2))|. \end{aligned}$$

Hence proved. □

Definition 3.3. *Let $\Im_1$ and $\Im_2$ be two PBFGs of $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively. The Cartesian product of $\Im_1$ and $\Im_2$ is denoted by $\Im_1 \,\square\, \Im_2$ and defined as:*

(i): $\begin{cases} (\tau^P_{S_1} \,\square\, \tau^P_{S_2})(r_1, r_2) = \tau^P_{S_1}(r_1)\tau^P_{S_2}(r_2) \\ (\tau^N_{S_1} \,\square\, \tau^N_{S_2})(r_1, r_2) = -|\tau^N_{S_1}(r_1)||\tau^N_{S_2}(r_2)| \quad \text{for all } (r_1, r_2) \in \aleph_1 \times \aleph_2, \end{cases}$

(ii): $\begin{cases} (\tau^P_{T_1} \,\square\, \tau^P_{T_2})((r, r_2)(r, s_2)) = \tau^P_{S_1}(r)\tau^P_{T_2}(r_2 s_2) \\ (\tau^N_{T_1} \,\square\, \tau^N_{T_2})((r, r_2)(r, s_2)) = -|\tau^N_{S_1}(r)||\tau^N_{T_2}(r_2 s_2)| \quad \text{for all } r \in \aleph_1, \text{ for all } r_2 s_2 \in \widetilde{\aleph^2_2}, \end{cases}$

(iii): $\begin{cases} (\tau^P_{T_1} \,\square\, \tau^P_{T_2})((r_1, t)(s_1, t)) = \tau^P_{S_2}(t)\tau^P_{T_1}(r_1 s_1) \\ (\tau^N_{T_1} \,\square\, \tau^N_{T_2})((r_1, t)(s_1, t)) = -|\tau^N_{S_2}(t)||\tau^N_{T_1}(r_1 s_1)| \quad \text{for all } t \in \aleph_2, \text{ for all } r_1 s_1 \in \widetilde{\aleph^2_1}. \end{cases}$

Remark 3.1. *The Cartesian product of two PBFGs is not necessarily a PBFG. A counterexample is shown as follows:*

Example 3.2. *We consider two PBFGs $\Im_1$ and $\Im_2$ as shown in Fig. 2, where* $S_1 = \left\langle \left(\frac{e}{0.3}, \frac{f}{0.5}\right), \left(\frac{e}{-0.6}, \frac{f}{-0.8}\right) \right\rangle$, $T_1 = \left\langle \frac{ef}{0.1}, \frac{ef}{-0.1} \right\rangle$, $S_2 = \left\langle \left(\frac{g}{0.4}, \frac{h}{0.7}\right), \left(\frac{g}{-0.2}, \frac{h}{-0.5}\right) \right\rangle$ *and* $T_2 = \left\langle \frac{gh}{0.2}, \frac{gh}{-0.1} \right\rangle$.
Then we get

$(\mu^P_{T_1} \square \mu^P_{T_2})((e, g)(e, h)) = 0.06, (\mu^N_{T_1} \square \mu^N_{T_2})((e, g)(e, h)) = -0.06, (\mu^P_{T_1} \square \mu^P_{T_2})((f, g)(f, h)) = 0.1,$
$(\mu^N_{T_1} \square \mu^N_{T_2})((f, g)(f, h)) = -0.08, (\mu^P_{T_1} \square \mu^P_{T_2})((e, g)(f, g)) = 0.04, (\mu^N_{T_1} \square \mu^N_{T_2})((e, g)(f, g)) = -0.02,$
$(\mu^P_{T_1} \square \mu^P_{T_2})((e, h)(f, h)) = 0.07, (\mu^N_{T_1} \square \mu^N_{T_2})((e, h)(f, h)) = -0.05.$
It is easy to see that,

$$\begin{aligned} (\mu^P_{T_1} \square \mu^P_{T_2})((e, g)(e, h)) = 0.06 &\nleq 0.03 = (\mu^P_{S_1} \,\square\, \mu^P_{S_2})((e, g))(\mu^P_{S_1} \,\square\, \mu^P_{S_2})((e, h)), \\ (\mu^N_{T_1} \square \mu^N_{T_2})((e, g)(e, h)) = -0.06 &\ngeq -0.04 = (\mu^N_{S_1} \,\square\, \mu^N_{S_2})((e, g))(\mu^N_{S_1} \,\square\, \mu^N_{S_2})((e, h)), \\ (\mu^P_{T_1} \square \mu^P_{T_2})((f, g)(f, h)) = 0.1 &\nleq 0.07 = (\mu^P_{S_1} \,\square\, \mu^P_{S_2})((f, g))(\mu^P_{S_1} \,\square\, \mu^P_{S_2})((f, h)), \\ (\mu^N_{T_1} \square \mu^N_{T_2})((f, g)(f, h)) = -0.08 &\ngeq -0.06 = (\mu^N_{S_1} \,\square\, \mu^N_{S_2})((f, g))(\mu^N_{S_1} \,\square\, \mu^N_{S_2})((f, h)), \\ (\mu^P_{T_1} \square \mu^P_{T_2})((e, g)(f, g)) = 0.04 &\nleq 0.02 = (\mu^P_{S_1} \,\square\, \mu^P_{S_2})((e, g))(\mu^P_{S_1} \,\square\, \mu^P_{S_2})((f, g)). \end{aligned}$$

Therefore, $\Im_1 \,\square\, \Im_2$ is not a PBFG.

Definition 3.4. *If a bipolar fuzzy membership degree is attached from [-1, 1] to each edge of a PBFG $\Im$ of a graph $\mathbb{G}$ and each vertex is crisply in $\Im$, then $\Im$ is called a product bipolar fuzzy edge graph.*

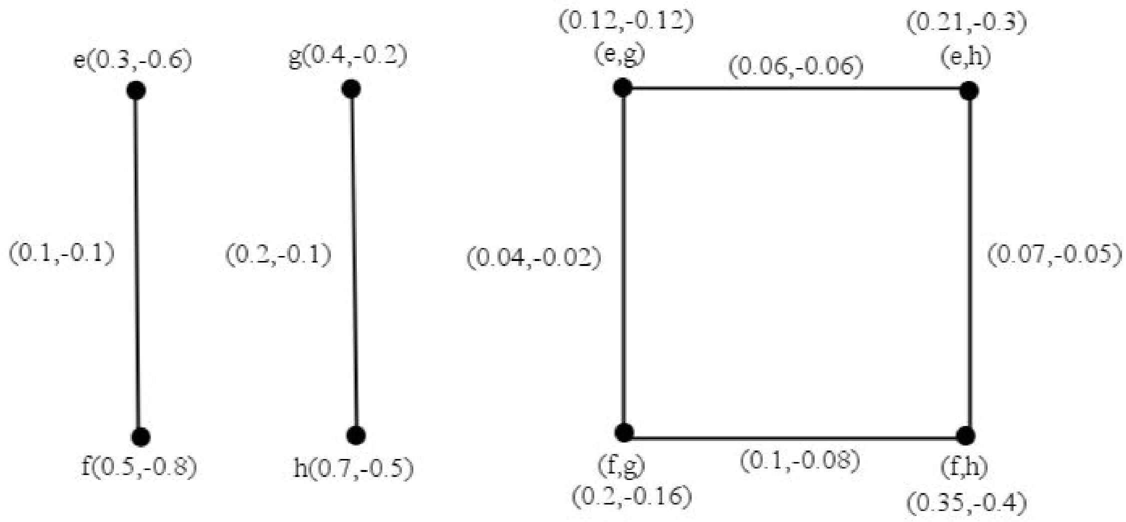

FIGURE 2. Graphs $\Im_1$, $\Im_2$ and their Cartesian product $\Im_2 \,\Box\, \Im_2$.

Proposition 3.2. *Let $\Im_1$ and $\Im_2$ be two product bipolar fuzzy edge graphs of the graphs $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively. The Cartesian product $\Im_1 \,\Box\, \Im_2$ of $\Im_1$ and $\Im_2$ is a product bipolar fuzzy edge graph of $\mathbb{G}_1 \,\Box\, \mathbb{G}_2$.*

Proof. Consider $r \in \aleph_1, r_2s_2 \in \widetilde{\aleph_2^2}$. Then

$$\begin{aligned}
(\tau_{T_1}^P \,\Box\, \tau_{T_2}^P)((r,r_2)(r,s_2)) &= \tau_{S_1}^P(r)\tau_{T_2}^P(r_2s_2) \leq \tau_{S_1}^P(r)(\tau_{S_2}^P(r_2)\tau_{S_2}^P(s_2)) \\
&= (\tau_{S_1}^P(r)\tau_{S_2}^P(r_2))(\tau_{S_1}^P(r)\tau_{S_2}^P(s_2)) \\
&= (\tau_{S_1}^P \,\Box\, \tau_{S_2}^P)((r,r_2))(\tau_{S_1}^P \,\Box\, \tau_{S_2}^P)((r,s_2)), \\
(\tau_{T_1}^N \,\Box\, \tau_{T_2}^N)((r,r_2)(r,s_2)) &= -|\tau_{S_1}^N(r)||\tau_{T_2}^N(r_2s_2)| \geq -|\tau_{S_1}^N(r)|(|\tau_{S_2}^N(r_2)||\tau_{S_2}^N(s_2)|) \\
&= -(|\tau_{S_1}^N(r)||\tau_{S_2}^N(r_2)|)(|\tau_{S_1}^N(r)||\tau_{S_2}^N(s_2)|) \\
&= -|(\tau_{S_1}^N \,\Box\, \tau_{S_2}^N)((r,r_2))||(\tau_{S_1}^N \,\Box\, \tau_{S_2}^N)((r,s_2))|.
\end{aligned}$$

Consider $t \in \aleph_2, r_1s_1 \in \widetilde{\aleph_1^2}$. Then

$$\begin{aligned}
(\tau_{T_1}^P \,\Box\, \tau_{T_2}^P)((r_1,t)(s_1,t)) &= \tau_{T_1}^P(r_1s_1)\tau_{S_2}^P(t) \leq (\tau_{S_1}^P(r_1)\tau_{S_1}^P(s_1))\tau_{S_2}^P(t) \\
&= (\tau_{S_1}^P(r_1)\tau_{S_2}^P(t))(\tau_{S_1}^P(s_1)\tau_{S_2}^P(t)) \\
&= (\tau_{S_1}^P \,\Box\, \tau_{S_2}^P)((r_1,t))(\tau_{S_1}^P \,\Box\, \tau_{S_2}^P)((s_1,t)), \\
(\tau_{T_1}^N \,\Box\, \tau_{T_2}^N)((r_1,t)(s_1,t)) &= -|\tau_{T_1}^N(r_1s_1)||\tau_{S_2}^N(t)| \geq -(|\tau_{S_1}^N(r_1)||\tau_{S_1}^N(s_1)|)|\tau_{S_2}^N(t)| \\
&= -(|\tau_{S_1}^N(r_1)||\tau_{S_2}^N(t)|)(|\tau_{S_1}^N(s_1)||\tau_{S_2}^N(t)|) \\
&= -|(\tau_{S_1}^N \,\Box\, \tau_{S_2}^N)((r_1,t))||(\tau_{S_1}^N \,\Box\, \tau_{S_2}^N)((s_1,t))|.
\end{aligned}$$

Hence proved. □

Definition 3.5. *Let $\Im_1$ and $\Im_2$ be two PBFGs. The strong product of these two PBFGs is denoted by $\Im_1 \boxtimes \Im_2$ and defined as:*

(i): $\begin{cases} (\tau_{S_1}^P \boxtimes \tau_{S_2}^P)(r_1,r_2) = \tau_{S_1}^P(r_1)\tau_{S_2}^P(r_2) \\ (\tau_{S_1}^N \boxtimes \tau_{S_2}^N)(r_1,r_2) = -|\tau_{S_1}^N(r_1)||\tau_{S_2}^N(r_2)| \quad \text{for all } (r_1,r_2) \in \aleph_1 \times \aleph_2, \end{cases}$

$$\textbf{(ii):} \begin{cases} (\tau^P_{T_1} \boxtimes \tau^P_{T_2})((r,r_2)(r,s_2)) = \tau^P_{S_1}(r)\tau^P_{T_2}(r_2s_2) \\ (\tau^N_{T_1} \boxtimes \tau^N_{T_2})((r,r_2)(r,s_2)) = -|\tau^N_{S_1}(r)||\tau^N_{T_2}(r_2s_2)| \quad \text{for all } r \in \aleph_1, \text{ for all } r_2s_2 \in \widetilde{\aleph_2^2}, \end{cases}$$

$$\textbf{(iii):} \begin{cases} (\tau^P_{T_1} \boxtimes \tau^P_{T_2})((r_1,t)(s_1,t)) = \tau^P_{S_2}(t)\tau^P_{T_1}(r_1s_1) \\ (\tau^N_{T_1} \boxtimes \tau^N_{T_2})((r_1,t)(s_1,t)) = -|\tau^N_{S_2}(t)||\tau^N_{T_1}(r_1s_1)| \quad \text{for all } t \in \aleph_2, \text{ for all } r_1s_1 \in \widetilde{\aleph_1^2}, \end{cases}$$

$$\textbf{(iv):} \begin{cases} (\tau^P_{T_1} \boxtimes \tau^P_{T_2})((r_1,r_2)(s_1,s_2)) = \tau^P_{T_1}(r_1s_1)\tau^P_{T_2}(r_2s_2) \\ (\tau^N_{T_1} \boxtimes \tau^N_{T_2})((r_1,r_2)(s_1,s_2)) = -|\tau^N_{T_1}(r_1s_1)||\tau^N_{T_2}(r_2s_2)| \quad \text{for all } r_1s_1 \in \widetilde{\aleph_1^2}, \text{ for all } r_2s_2 \in \widetilde{\aleph_2^2}. \end{cases}$$

Proposition 3.3. *The strong product of two product bipolar fuzzy edge graphs is a product bipolar fuzzy edge graph.*

Definition 3.6. *Let $\Im_1$ and $\Im_2$ be two PBFGs of $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively. The lexicographic product of these two PBFGs is denoted by $\Im_1 \circ \Im_2$ and defined as follows:*

$$\textbf{(i):} \begin{cases} (\tau^P_{S_1} \circ \tau^P_{S_2})(r_1,r_2) = \tau^P_{S_1}(r_1)\tau^P_{S_2}(r_2) \\ (\tau^N_{S_1} \circ \tau^N_{S_2})(r_1,r_2) = -|\tau^N_{S_1}(r_1)||\tau^N_{S_2}(r_2)| \quad \text{for all } (r_1,r_2) \in \aleph_1 \times \aleph_2, \end{cases}$$

$$\textbf{(ii):} \begin{cases} (\tau^P_{T_1} \circ \tau^P_{T_2})((r,r_2)(r,s_2)) = \tau^P_{S_1}(r)\tau^P_{T_2}(r_2s_2) \\ (\tau^N_{T_1} \circ \tau^N_{T_2})((r,r_2)(r,s_2)) = -|\tau^N_{S_1}(r)||\tau^N_{T_2}(r_2s_2)| \quad \text{for all } r \in \aleph_1, \text{ for all } r_2s_2 \in \widetilde{\aleph_2^2}, \end{cases}$$

$$\textbf{(iii):} \begin{cases} (\tau^P_{T_1} \circ \tau^P_{T_2})((r_1,t)(s_1,t)) = \tau^P_{S_2}(t)\tau^P_{T_1}(r_1s_1) \\ (\tau^N_{T_1} \circ \tau^N_{T_2})((r_1,t)(s_1,t)) = -|\tau^N_{S_2}(t)||\tau^N_{T_1}(r_1s_1)| \quad \text{for all } t \in \aleph_2, \text{ for all } r_1s_1 \in \widetilde{\aleph_1^2}, \end{cases}$$

$$\textbf{(iv):} \begin{cases} (\tau^P_{T_1} \circ \tau^P_{T_2})((r_1,r_2)(s_1,s_2)) = \tau^P_{S_2}(r_2)\tau^P_{S_2}(s_2)\tau^P_{T_1}(r_1s_1) \\ (\tau^N_{T_1} \circ \tau^N_{T_2})((r_1,r_2)(s_1,s_2)) = -|\tau^N_{S_2}(r_2)||\tau^N_{S_2}(s_2)||\tau^N_{T_1}(r_1s_1)| \quad \text{for all } r_1s_1 \in \widetilde{\aleph_1^2}, \ r_2 \neq s_2. \end{cases}$$

Proposition 3.4. *The lexicographic product of two product bipolar fuzzy edge graphs is a product bipolar fuzzy edge graph.*

Proof. From the proof of Proposition 3.2, it follows that:

$$\begin{aligned} (\tau^P_{T_1} \circ \tau^P_{T_2})((r,r_2)(r,s_2)) &\leq (\tau^P_{S_1} \circ \tau^P_{S_2})((r,r_2))(\tau^P_{S_1} \circ \tau^P_{S_2})((r,s_2)), \\ (\tau^N_{T_1} \circ \tau^N_{T_2})((r,r_2)(r,s_2)) &\geq -|(\tau^N_{S_1} \circ \tau^N_{S_2})((r,r_2))||(\tau^N_{S_1} \circ \tau^N_{S_2})((r,s_2))| \text{ for all } r \in \aleph_1, r_2s_2 \in \widetilde{\aleph_2^2}, \\ (\tau^P_{T_1} \circ \tau^P_{T_2})((r_1,t)(s_1,t)) &\leq (\tau^P_{S_1} \circ \tau^P_{S_2})((r_1,t))(\tau^P_{S_1} \circ \tau^P_{S_2})((s_1,t)), \\ (\tau^N_{T_1} \circ \tau^N_{T_2})((r_1,t)(s_1,t)) &\geq -|(\tau^N_{S_1} \circ \tau^N_{S_2})((r_1,t))||(\tau^N_{S_1} \circ \tau^N_{S_2})((s_1,t))| \text{ for all } t \in \aleph_2, r_1s_1 \in \widetilde{\aleph_1^2}. \end{aligned}$$

Suppose that $r_1s_1 \in \widetilde{\aleph_1^2}$, $r_2 \neq s_2$. Then

$$\begin{aligned} (\tau^P_{T_1} \circ \tau^P_{T_2})((r_1,r_2)(s_1,s_2)) &= \tau^P_{S_2}(r_2)\tau^P_{S_2}(s_2)\tau^P_{T_1}(r_1s_1) \leq \tau^P_{S_2}(r_2)\tau^P_{S_2}(s_2)(\tau^P_{S_1}(r_1)\tau^P_{S_1}(s_1)) \\ &= (\tau^P_{S_1}(r_1)\tau^P_{S_2}(r_2))(\tau^P_{S_1}(s_1)\tau^P_{S_2}(s_2)) \\ &= (\tau^P_{S_1} \circ \tau^P_{S_2})((r_1,r_2))(\tau^P_{S_1} \circ \tau^P_{S_2})((s_1,s_2)), \\ (\tau^N_{T_1} \circ \tau^N_{T_2})((r_1,r_2)(s_1,s_2)) &= -|\tau^N_{S_2}(r_2)||\tau^N_{S_2}(s_2)||\tau^N_{T_1}(r_1s_1)| \geq -|\tau^N_{S_2}(r_2)||\tau^N_{S_2}(s_2)|(|\tau^N_{S_1}(r_1)||\tau^N_{S_1}(s_1)|) \\ &= -(|\tau^N_{S_1}(r_1)||\tau^N_{S_2}(r_2)|)(|\tau^N_{S_1}(s_1)||\tau^N_{S_2}(s_2)|) \\ &= -|(\tau^N_{S_1} \circ \tau^N_{S_2})((r_1,r_2))||(\tau^N_{S_1} \circ \tau^N_{S_2})((s_1,s_2))|. \end{aligned}$$

Hence proved. □

Remark 3.2. *In general, the strong product and the lexicographic product of two PBFGs are not PBFGs.*

Definition 3.7. *The union $\Im_1 \cup \Im_2$ of two PBFGs $\Im_1$ and $\Im_2$ is defined as follows:*

$$(\tau^P_{S_1} \cup \tau^P_{S_2})(r) = \begin{cases} \tau^P_{S_1}(r) & \text{if } r \in \aleph_1 - \aleph_2, \\ \tau^P_{S_2}(r) & \text{if } r \in \aleph_2 - \aleph_1, \\ -|\tau^P_{S_1}(r)||\tau^P_{S_2}(r)| & \text{if } r \in \aleph_1 \cap \aleph_2. \end{cases}$$

$$(\tau^N_{S_1} \cup \tau^N_{S_2})(r) = \begin{cases} \tau^N_{S_1}(r) & \text{if } r \in \aleph_1 - \aleph_2, \\ \tau^N_{S_2}(r) & \text{if } r \in \aleph_2 - \aleph_1, \\ \tau^N_{S_1}(r)\tau^N_{S_2}(r) & \text{if } r \in \aleph_1 \cap \aleph_2. \end{cases}$$

$$(\tau^P_{T_1} \cup \tau^P_{T_2})(rs) = \begin{cases} \tau^P_{T_1}(rs) & \text{if } rs \in \Re_1 - \Re_2, \\ \tau^P_{T_2}(rs) & \text{if } rs \in \Re_2 - \Re_1, \\ -|\tau^P_{T_1}(rs)||\tau^P_{T_2}(rs)| & \text{if } rs \in \Re_1 \cap \Re_2. \end{cases}$$

$$(\tau^N_{T_1} \cup \tau^N_{T_2})(rs) = \begin{cases} \tau^N_{T_1}(rs) & \text{if } rs \in \Re_1 - \Re_2, \\ \tau^N_{T_2}(rs) & \text{if } rs \in \Re_2 - \Re_1, \\ \tau^N_{T_1}(rs)\tau^N_{T_2}(rs) & \text{if } rs \in \Re_1 \cap \Re_2. \end{cases}$$

Proposition 3.5. *The union of two PBFGs is a PBFG. The converse holds if $\aleph_1 \cap \aleph_2 = \emptyset$.*

Theorem 3.1. *The union $\Im_1 \cup \Im_2$ of $\Im_1$ and $\Im_2$ is a PBFG of $\mathbb{G}_1 \cup \mathbb{G}_2$ if and only if $\Im_1$ and $\Im_2$ are PBFGs of $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively, where S_1, S_2, T_1 and T_2 are the fuzzy subsets of $\aleph_1, \aleph_2, \Re_1$ and $\Re_2$, respectively and $\aleph_1 \cap \aleph_2 = \emptyset$.*

Proof. Suppose that $\Im_1 \cup \Im_2$ is a PBFG of $\mathbb{G}_1 \cup \mathbb{G}_2$. Let $rs \in \Re_1$, then $rs \notin \Re_2$ and $r, s \in \aleph_1 - \aleph_2$. Thus

$$\tau^P_{T_1}(rs) = (\tau^P_{T_1} \cup \tau^P_{T_2})(rs) \leq (\tau^P_{S_1} \cup \tau^P_{S_2})(r)(\tau^P_{S_1} \cup \tau^P_{S_2})(s) = \tau^P_{S_1}(r)\tau^P_{S_1}(s).$$
$$\tau^N_{T_1}(rs) = (\tau^N_{T_1} \cup \tau^N_{T_2})(rs) \geq -|(\tau^N_{S_1} \cup \tau^N_{S_2})(r)||(\tau^N_{S_1} \cup \tau^N_{S_2})(s)| = -|\tau^N_{S_1}(r)||\tau^N_{S_1}(s)|.$$

Thus $\Im_1$ is a PBFG of $\mathbb{G}_1$. Similarly, we can show that $\Im_2$ is a PBFG of $\mathbb{G}_2$. □

Definition 3.8. *Let $\Im_1$ and $\Im_2$ be two PBFGs of $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively. The ring-sum of $\Im_1$ and $\Im_2$ is denoted by $\Im_1 \oplus \Im_2$ and defined as follows:*

$$(\tau^P_{S_1} \oplus \tau^P_{S_2})(r) = (\tau^P_{S_1} \cup \tau^P_{S_2})(r),\ (\tau^N_{S_1} \oplus \tau^N_{S_2})(r) = (\tau^N_{S_1} \cup \tau^N_{S_2})(r) \text{ if } r \in \aleph_1 \cup \aleph_2,$$

$$(\tau^P_{T_1} \oplus \tau^P_{T_2})(rs) = \begin{cases} \tau^P_{T_1}(rs) & \text{if } rs \in \Re_1 - \Re_2, \\ \tau^P_{T_2}(rs) & \text{if } rs \in \Re_2 - \Re_1, \\ 0 & \text{if } rs \in \Re_1 \cap \Re_2. \end{cases}$$

$$(\tau^N_{T_1} \oplus \tau^N_{T_2})(rs) = \begin{cases} \tau^N_{T_1}(rs) & \text{if } rs \in \Re_1 - \Re_2, \\ \tau^N_{T_2}(rs) & \text{if } rs \in \Re_2 - \Re_1, \\ 0 & \text{if } rs \in \Re_1 \cap \Re_2. \end{cases}$$

Proposition 3.6. *The ring-sum of two PBFGs is a PBFG.*

Definition 3.9. *Let $\Im_1$ and $\Im_2$ be two PBFGs of $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively. The join of $\Im_1$ and $\Im_2$, denoted by $\Im_1 + \Im_2$, is defined as:*

(i): $\begin{cases} (\tau^P_{S_1} + \tau^P_{S_2})(r) = (\tau^P_{S_1} \cup \tau^P_{S_2})(r) \\ (\tau^N_{S_1} + \tau^N_{S_2})(r) = (\tau^N_{S_1} \cup \tau^N_{S_2})(r) \quad \text{for all } r \in \aleph_1 \cup \aleph_2, \end{cases}$

(ii): $\begin{cases} (\tau^P_{T_1} + \tau^P_{T_2})(rs) = (\tau^P_{T_1} \cup \tau^P_{T_2})(rs) \\ (\tau^N_{T_1} + \tau^N_{T_2})(rs) = (\tau^N_{T_1} \cup \tau^N_{T_2})(rs) \quad \text{if } rs \in \Re_1 \cup \Re_2, \end{cases}$

(iii): $\begin{cases} (\tau^P_{T_1} + \tau^P_{T_2})(rs) = \tau^P_{S_1}(r)\tau^P_{S_2}(s) \\ (\tau^N_{T_1} + \tau^N_{T_2})(rs) = -|\tau^N_{S_1}(r)||\tau^N_{S_2}(s)| \quad \text{if } rs \in \Re'. \end{cases}$

Theorem 3.2. *The join $\Im_1 + \Im_2$ of $\Im_1$ and $\Im_2$ is a PBFG of $\mathbb{G}_1 + \mathbb{G}_2$ if and only if $\Im_1$ and $\Im_2$ are PBFGs of $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively, where S_1, S_2, T_1 and T_2 are the fuzzy subsets of $\aleph_1, \aleph_2, \Re_1$ and $\Re_2$, respectively and $\aleph_1 \cap \aleph_2 = \emptyset$.*

Proof. Suppose that $\Im_1 + \Im_2$ is a PBFG. Then from the proof of Theorem 3.1, $\Im_1$ and $\Im_2$ are PBFGs. Conversely, assume that $\Im_1$ and $\Im_2$ are PBFGs. Consider $rs \in \Re_1 \cup \Re_2$. Then the required result follows from Proposition 3.5. Let $rs \in \Re'$. Then

$$(\tau_{T_1}^P + \tau_{T_2}^P)(rs) = \tau_{S_1}^P(r)\tau_{S_2}^P(s) = (\tau_{S_1}^P \cup \tau_{S_2}^P)(r)(\tau_{S_1}^P \cup \tau_{S_2}^P)(s) = (\tau_{S_1}^P + \tau_{S_2}^P)(r)(\tau_{S_1}^P + \tau_{S_2}^P)(s),$$
$$(\tau_{T_1}^N + \tau_{T_2}^N)(rs) = -|\tau_{S_1}^N(r)||\tau_{S_2}^N(s)| = -|(\tau_{S_1}^N \cup \tau_{S_2}^N)(r)||(\tau_{S_1}^N \cup \tau_{S_2}^N)(s)| = -|(\tau_{S_1}^N + \tau_{S_2}^N)(r)||(\tau_{S_1}^N + \tau_{S_2}^N)(s)|.$$

Hence proved. □

Definition 3.10. *The complement of a PBFG $\Im = (S, T)$ of $\mathbb{G} = (\aleph, \Re)$ is a PBFG $\overline{\Im} = (\overline{S}, \overline{T})$, where $\overline{S} = S = (\tau_S^P, \tau_S^N)$ and $\overline{T} = (\overline{\tau_T^P}, \overline{\tau_T^N})$ defined as follows:*

$$\overline{\tau_T^P}(rs) = \begin{cases} \tau_S^P(r)\tau_S^P(s) & \text{if } \tau_T^P(rs) = 0, \\ \tau_S^P(r)\tau_S^P(s) - \tau_T^P(rs) & \text{if } 0 < \tau_T^P(rs) \le 1, \end{cases}$$

$$\overline{\tau_T^N}(rs) = \begin{cases} -|\tau_S^N(r)||\tau_S^N(s)| & \text{if } \tau_T^N(rs) = 0, \\ -|\tau_S^N(r)||\tau_S^N(s)| - \tau_T^N(rs) & \text{if } -1 \le \tau_T^N(rs) < 0. \end{cases}$$

Example 3.3. *Consider a PBFG over $\aleph = \{e, f, g\}$ as shown in Fig. 3, defined by*

$$S = \left\langle \left(\frac{e}{0.7}, \frac{f}{0.3}, \frac{g}{0.6}\right), \left(\frac{e}{-0.5}, \frac{f}{-0.8}, \frac{g}{-0.4}\right) \right\rangle, T = \left\langle \left(\frac{ef}{0.21}, \frac{eg}{0.2}\right), \left(\frac{ef}{-0.4}, \frac{qs}{-0.1}\right) \right\rangle.$$

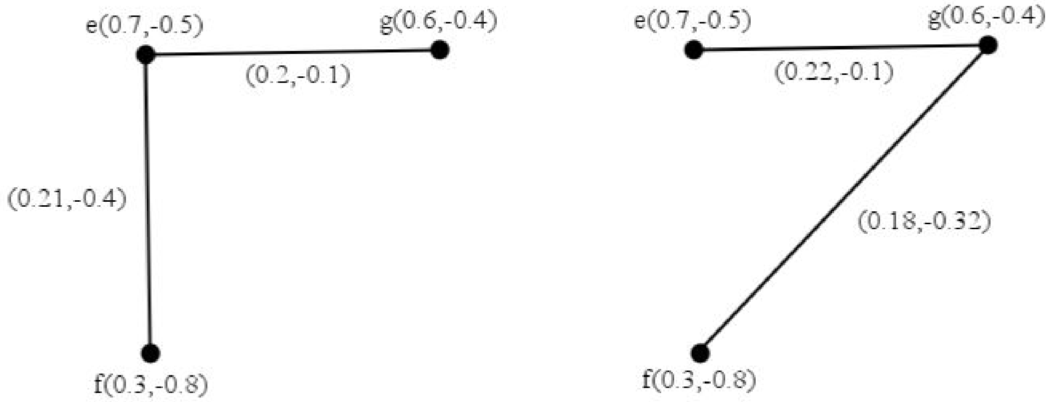

FIGURE 3. $\Im$ and its complement $\overline{\Im}$.

Here for all $r \in \aleph$, $\overline{\overline{\tau_S^P(r)}} = \overline{\tau_S^P(r)} = \tau_S^P(r)$, $\overline{\overline{\tau_S^N(r)}} = \overline{\tau_S^N(r)} = \tau_S^N(r)$, and for all $r, s \in \aleph$

$$\overline{\overline{\tau_T^P(rs)}} = \overline{\tau_S^P(r)\tau_S^P(s)} - \overline{\tau_T^P(rs)} = \tau_S^P(r)\tau_S^P(s) - (\tau_S^P(r)\tau_S^P(s) - \tau_T^P(rs)) = \tau_T^P(rs),$$
$$\overline{\overline{\tau_T^N(rs)}} = \overline{-|\tau_S^N(r)||\tau_S^N(s)|} - \overline{\tau_T^N(rs)} = -|\tau_S^N(r)||\tau_x^N(s)| - (-|\tau_S^N(r)||\tau_S^N(s)| - \tau_T^N(rs)) = \tau_T^N(rs).$$

Hence $\overline{\overline{\Im}} = \Im$.

Definition 3.11. *A homomorphism $\psi : \Im_1 \to \Im_2$ of two PBFGs $\Im_1$ and $\Im_2$ is a mapping $\psi : \aleph_1 \to \aleph_2$ satisfying the following conditions:*

(a): $\tau_{S_1}^P(r) \le \tau_{S_2}^P(\psi(r))$, $\tau_{S_1}^N(r) \ge \tau_{S_2}^N(\psi(r))$ for all $r \in \aleph_1$,

(b): $\tau_{T_1}^P(rs) \le \tau_{T_2}^P(\psi(r)\psi(s))$, $\tau_{T_1}^N(rs) \ge \tau_{T_2}^N(\psi(r)\psi(s))$ for all $rs \in \widetilde{\aleph_1^2}$.

Definition 3.12. *An isomorphism $\psi : \Im_1 \to \Im_2$ of two PBFGs $\Im_1$ and $\Im_2$ is a bijective mapping $\psi : \aleph_1 \to \aleph_2$ satisfying the following conditions:*

(c): $\tau_{S_1}^P(r) = \tau_{S_2}^P(\psi(r))$, $\tau_{S_1}^N(r) = \tau_{S_2}^N(\psi(r))$ for all $r \in \aleph_1$,

(d): $\tau_{T_1}^P(rs) = \tau_{T_2}^P(\psi(r)\psi(s))$, $\tau_{T_1}^N(rs) = \tau_{T_2}^N(\psi(r)\psi(s))$ for all $rs \in \widetilde{\aleph_1^2}$.

Proposition 3.7. *Let $\Im$ be a self-complementary PBFG, i.e., $\Im \cong \overline{\Im}$. Then*

(i): $\sum\limits_{r\neq s} \tau_T^P(rs) = \frac{1}{2}\sum\limits_{r\neq s} \tau_S^P(r)\tau_S^P(s)$,

(ii): $\sum\limits_{r\neq s} \tau_T^N(rs) = -\frac{1}{2}\sum\limits_{r\neq s} |\tau_S^N(r)||\tau_S^N(s)|$.

Proof. Let $\Im$ be a self-complementary PBFG. Then there exists an isomorphism $\psi : \Im \to \overline{\Im}$ such that

$$\begin{aligned}
\overline{\tau_S^P}(\psi(r)) &= \tau_S^P(r),\ \overline{\tau_S^N}(\psi(r)) = \tau_S^N(r) \text{ for all } r \in \aleph,\\
\overline{\tau_T^P}(\psi(r)\psi(s)) &= \tau_T^P(rs),\ \overline{\tau_T^N}(\psi(r)\psi(s)) = \tau_T^N(rs) \text{ for all } rs \in \widetilde{\aleph^2}.
\end{aligned}$$

(i): For all $rs \in \widetilde{\aleph^2}$. By Def. of $\overline{\Im}$, we have

$$\begin{aligned}
\overline{\tau_T^P}(\psi(r)\psi(s)) &= \overline{\tau_S^P}(\psi(r))\overline{\tau_S^P}(\psi(s)) - \tau_T^P(\psi(r)\psi(s))\\
\tau_T^P(rs) &= \tau_S^P(r)\tau_S^P(s) - \tau_T^P(\psi(r)\psi(s))\\
\sum_{r\neq s}\tau_T^P(rs) + \sum_{r\neq s}\tau_T^P(\psi(r)\psi(s)) &= \sum_{r\neq s}\tau_S^P(r)\tau_S^P(s)\\
\sum_{r\neq s}\tau_T^P(rs) &= \frac{1}{2}\sum_{r\neq s}\tau_S^P(r)\tau_S^P(s).
\end{aligned}$$

(ii): For all $rs \in \widetilde{\aleph^2}$. By Def. of $\overline{\Im}$, we have

$$\begin{aligned}
\overline{\tau_T^N}(\psi(r)\psi(s)) &= -|\overline{\tau_S^N}(\psi(r))||\overline{\tau_S^N}(\psi(s))| - \tau_T^N(\psi(r)\psi(s))\\
\tau_T^N(rs) &= -|\tau_S^N(r)||\tau_S^N(s)| - \tau_T^N(\psi(r)\psi(s))\\
\sum_{r\neq s}\tau_T^N(rs) + \sum_{r\neq s}\tau_T^N(\psi(r)\psi(s)) &= -\sum_{r\neq s}|\tau_S^N(r)||\tau_S^N(s)|\\
\sum_{r\neq s}\tau_T^N(rs) &= -\frac{1}{2}\sum_{r\neq s}|\tau_S^N(r)||\tau_S^N(s)|.
\end{aligned}$$

Hence proved. □

Proposition 3.8. *Let $\Im$ be a PBFG of $\mathbb{G}$. If $\tau_T^P(rs) = \frac{1}{2}\tau_S^P(r)\tau_S^P(s)$ and $\tau_T^N(rs) = -\frac{1}{2}|\tau_S^N(r)||\tau_S^N(s)|$ for all $r, s \in \aleph$, then $\Im$ is self-complementary.*

Proof. Let $\Im$ be a PBFG satisfying $\tau_T^P(rs) = \frac{1}{2}\tau_S^P(r)\tau_S^P(s)$ and $\tau_T^N(rs) = -\frac{1}{2}|\tau_S^N(r)||\tau_S^N(s)|$ for all $r, s \in \aleph$. Then the identity mapping $I : \aleph \to \aleph$ is an isomorphism from $\Im$ to $\overline{\Im}$. Clearly, I satisfies the condition (c) of Definition 3.12. Since $\tau_T^P(rs) = \frac{1}{2}\tau_S^P(r)\tau_S^P(s)$ and $\tau_T^N(rs) = -\frac{1}{2}|\tau_S^N(r)||\tau_S^N(s)|$ for all $r, s \in \aleph$, using the Def. of complement, we have

$$\begin{aligned}
\overline{\tau_T^P}(I(r)I(s)) &= \overline{\tau_T^P}(rs) = \tau_S^P(r)\tau_S^P(s) - \tau_T^P(rs) = \tau_S^P(r)\tau_S^P(s) - \frac{1}{2}\tau_S^P(r)\tau_S^P(s) = \tau_T^P(rs),\\
\overline{\tau_T^N}(I(r)I(s)) &= \overline{\tau_T^N}(rs) = -|\tau_S^N(r)||\tau_S^N(s)| - \tau_T^N(rs) = -|\tau_S^N(r)||\tau_S^N(s)| - \left(-\frac{1}{2}|\tau_S^N(r)||\tau_S^N(s)|\right) = \tau_T^N(rs).
\end{aligned}$$

Thus the condition (d) of Definition 3.12 is also satisfied by I. Therefore $\Im$ is self-complementary. □

Proposition 3.9. *Let $\Im_1$ and $\Im_2$ be two PBFGs. Then $\Im_1 \cong \Im_2$ if and only if $\overline{\Im_1} \cong \overline{\Im_2}$.*

Proof. Suppose that $\Im_1$ and $\Im_2$ are two isomorphic PBFGs, then there exists a bijective mapping $\psi : \aleph_1 \to \aleph_2$ satisfying

$$\begin{aligned}
\tau_{S_1}^P(r) &= \tau_{S_2}^P(\psi(r)),\ \tau_{S_1}^N(r) = \tau_{S_2}^N(\psi(r)) \text{ for all } r \in \aleph_1,\\
\tau_{T_1}^P(rs) &= \tau_{T_2}^P(\psi(r)\psi(s)),\ \tau_{T_1}^N(rs) = \tau_{T_2}^N(\psi(r)\psi(s)) \text{ for all } rs \in \widetilde{\aleph_1^2}.
\end{aligned}$$

Using the Def. of complement, for all $rs \in \widetilde{\aleph_1^2}$ we have

$$\begin{aligned}\overline{\tau_{T_1}^P}(rs) &= \tau_{S_1}^P(r)\tau_{S_1}^P(s) - \tau_{T_1}^P(rs) = \tau_{S_2}^P(\psi(r))\tau_{S_2}^P(\psi(s)) - \tau_{T_2}^P(\psi(r)\psi(s)) = \overline{\tau_{T_2}^P}(\psi(r)\psi(s)),\\ \overline{\tau_{T_1}^N}(rs) &= -|\tau_{S_1}^N(r)||\tau_{S_1}^N(s)| - \tau_{T_1}^N(rs) = -|\tau_{S_2}^N(\psi(r))||\tau_{S_2}^N(\psi(s))| - \tau_{T_2}^N(\psi(r)\psi(s)) = \overline{\tau_{T_2}^N}(\psi(r)\psi(s)).\end{aligned}$$

Therefore $\overline{\Im_1} \cong \overline{\Im_2}$. Analogously, we can prove the converse part. □

Definition 3.13. *A generalized PBFG with a finite set $\aleph$ as the underlying set is a pair $\Im = (S, T)$, where $S = (\tau_S^P, \tau_S^N)$ is a BFS in $\aleph$ and $T = (\tau_T^P, \tau_T^N)$ is a BFS in $\widetilde{\aleph^2}$ such that*

$$\tau_T^P(rs) \leq \tau_S^P(r)\tau_S^P(s) \text{ and } \tau_T^N(rs) \geq -|\tau_S^N(r)||\tau_S^N(s)| \text{ for all } rs \in \widetilde{\aleph^2}.$$

4. Application of PBFGs in multi-agent decision making

A wide variety of human decision making, especially multi-agent decision making, is based on bipolar or double-sided judgmental thinking on a negative side and a positive side as it captures the bipolar or double-sided nature of human perception and cognition. Also in decision making problems, there is a number of uncertainties and in some situations, there exist some relations among agents in a multi-agent decision making problem. So, it is an interesting area of applications in PBFG theory.

For a multi-agent decision making problem, let $\mathcal{P} = \{p_1, p_2, \ldots, p_m\}$ be a set of planes (alternatives) and $A = \{r_1, r_2, \ldots, r_n\}$ be a set of uncertain agents (criteria), which can be described by a BFS $\{(r, \tau_S^-(r), \tau_S^+(r)) \mid r \in \aleph\}$, whose weight information is completely unknown. Also each agent identified with a vertex and links between agents with relations (edges) in PBFG. The implementation of any plan will force some or all agents to take actions, during which benefits will be produced. To drive the maximal benefit, how to choose the best plan, is a multi-agent decision making problem using PBFG.

In a PBFG $\Im = (S, T)$, for a plan, assume that if an agent $r_i \in A$ takes an action, we choose $z_i = 1$, otherwise $z_i = 0$. Then the benefit of each agent r_i can be calculated by using

$$b_i = (\tau_S^-(r_i), \tau_S^+(r_i))z_i + \overline{z}_{\mathcal{N}_i} \; i = 1, 2, \ldots, n. \tag{3}$$

where $\mathcal{N}_i$ is the set of the agent r_i's neighbors and

$$\overline{z}_{\mathcal{N}_i} = \sum_{j \in \mathcal{N}_i} (\tau_T^-(r_i r_j), \tau_T^+(r_i r_j))\xi(r_i r_j)z_j,$$

$\xi(r_i r_j) \in [0, 1]$ is the influence coefficient between relevant agents.

The weights of all agents can be calculated using product bipolar fuzzy graphic structure

$$\varkappa_i = \frac{\mathrm{d}(r_i)}{\sum\limits_{j=1}^{n} \mathrm{d}(r_j)}, i = 1, 2, \ldots, n. \tag{4}$$

And finally to select the best plan, the overall benefit of the plan can be calculated by using an aggregation operator (weighted averaging operator)

$$b = \sum_{i=1}^{n} \varkappa_i b_i, \; i = 1, 2, \ldots, n. \tag{5}$$

where $\varkappa = (\varkappa_1, \varkappa_2, \ldots, \varkappa_n)^T$ is a weight vector of $(b_1, b_2, \ldots, b_n)^T$.

Example 4.1. *Consider a social network of 7 agents. The links between agents reflect social interactions and connected agents are neighbors. The links between agents cannot always be positive. If two agents in a social network have some conflicts in that case the* *links between agents can be negative. We present a PBFG for such a case. We consider a PBFG of a social network as shown in Fig. 4.*

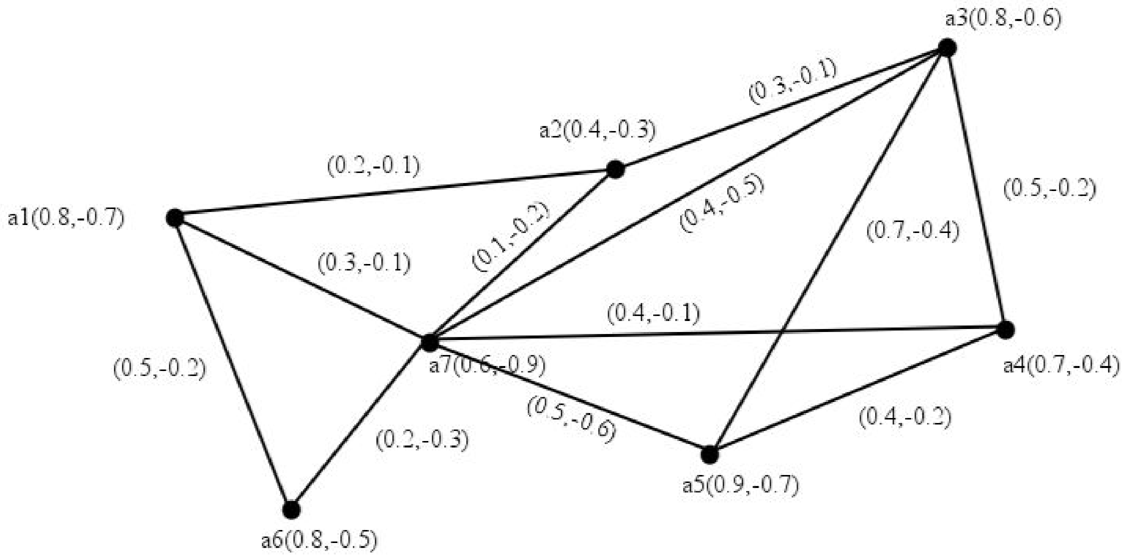

FIGURE 4. PBFG with 7 vertices (agents).

TABLE 1. Bipolar fuzzy neighbourhood of agents.

Agents	$\mathcal{N}$*(Agents)*
a_1	$\{(a_2, 0.2, -0.1), (a_6, 0.5, -0.2), (a_7, 0.3, -0.1)\}$
a_2	$\{(a_1, 0.2, -0.1), (a_3, 0.3, -0.1), (a_7, 0.1, -0.2)\}$
a_3	$\{(a_2, 0.3, -0.1), (a_4, 0.5, -0.2), (a_5, 0.7, -0.4), (a_7, 0.4, -0.5)\}$
a_4	$\{(a_3, 0.5, -0.2), (a_5, 0.4, -0.2), (a_7, 0.4, -0.1)\}$
a_5	$\{(a_3, 0.7, -0.4), (a_4, 0.4, -0.2), (a_7, 0.5, -0.6)\}$
a_6	$\{(a_1, 0.5, -0.2), (a_7, 0.2, -0.3)\}$
a_7	$\{(a_1, 0.3, -0.1), (a_2, 0.1, -0.2), (a_3, 0.4, -0.5)$
	$(a_4, 0.4, -0.1), (a_5, 0.5, -0.6), (a_6, 0.2, -0.3)\}$

The degree of each agent is given by

$$\mathrm{d}(a_1) = (1.0, -0.4), \mathrm{d}(a_2) = (0.6, -0.4), \mathrm{d}(a_3) = (1.9, -1.2), \mathrm{d}(a_4) = (1.3, -0.5),$$
$$\mathrm{d}(a_5) = (1.6, -1.2), \mathrm{d}(a_6) = (0.7, -0.5), \mathrm{d}(a_7) = (1.9, -1.8).$$

Calculate their scores using the score function $\mathcal{S}_i = \tau_i^N + \tau_i^P$ [10]*:*

$$\mathcal{S}(\mathrm{d}(a_1)) = 0.6, \mathcal{S}(\mathrm{d}(a_2)) = 0.2, \mathcal{S}(\mathrm{d}(a_3)) = 0.7, \mathcal{S}(\mathrm{d}(a_4)) = 0.8, \mathcal{S}(\mathrm{d}(a_5)) = 0.4, \mathcal{S}(\mathrm{d}(a_6)) = 0.2, \mathcal{S}(\mathrm{d}(a_7)) = 0.1.$$

Then by using Eq. (4), weights of each agent can be calculated as:

$$\varkappa_1 = 0.20,\ \varkappa_2 = 0.07,\ \varkappa_3 = 0.23,\ \varkappa_4 = 0.27,\ \varkappa_5 = 0.13,\ \varkappa_6 = 0.07,\ \varkappa_7 = 0.03.$$

If there is a plan $\mathcal{P}$*, in which just agent* a_7 *takes an action, then* $z_7 = 1$ *and* $z_i = 0$ $(i = 1, 2, \ldots, 6.)$ *For example, action may be interpreted as acquiring information by agent* a_7*.*

Also take $\xi(a_ia_j) = 0.5$ *for* $i, j = 1, 2, \ldots, 7$ *and* $i \neq j$*, then by Eq. (3), the benefits of all agents are:*

$$
\begin{aligned}
b_1^{(\mathcal{P})} &= (\tau_S^N(a_1), \tau_S^P(a_1))z_1 + \overline{z}_{\mathcal{N}_1} = (\tau_T^P(a_1a_7), \tau_T^N(a_1a_7))\xi(a_1a_7)z_7 = (0.15, -0.05)\\
b_2^{(\mathcal{P})} &= (\tau_S^P(a_2), \tau_S^N(a_2))z_2 + \overline{z}_{\mathcal{N}_2} = (\tau_T^P(a_2a_7), \tau_T^N(a_2a_7))\xi(a_2a_7)z_7 = (0.05, -0.1)\\
b_3^{(\mathcal{P})} &= (\tau_S^P(a_3), \tau_S^N(a_3))z_3 + \overline{z}_{\mathcal{N}_3} = (\tau_T^P(a_3a_7), \tau_T^N(a_3a_7))\xi(a_3a_7)z_7 = (0.2, -0.25)\\
b_4^{(\mathcal{P})} &= (\tau_S^P(a_4), \tau_S^N(a_4))z_4 + \overline{z}_{\mathcal{N}_4} = (\tau_T^P(a_4a_7), \tau_T^N(a_4a_7))\xi(a_4a_7)z_7 = (0.2, -0.05)\\
b_5^{(\mathcal{P})} &= (\tau_S^P(a_5), \tau_S^N(a_5))z_5 + \overline{z}_{\mathcal{N}_5} = (\tau_T^P(a_5a_7), \tau_T^N(a_5a_7))\xi(a_5a_7)z_7 = (0.25, -0.3)\\
b_6^{(\mathcal{P})} &= (\tau_S^P(a_6), \tau_S^N(a_6))z_6 + \overline{z}_{\mathcal{N}_6} = (\tau_T^P(a_6a_7), \tau_T^N(a_6a_7))\xi(a_6a_7)z_7 = (0.10, -0.15)\\
b_7^{(\mathcal{P})} &= (\tau_S^P(a_7), \tau_S^N(a_7))z_7 + \overline{z}_{\mathcal{N}_7} = (0.6, -0.9).
\end{aligned}
$$

By using Eq. (5) overall benefit of plane $\mathcal{P}$ *is* $b^{(\mathcal{P})} = \sum\limits_{i=1}^{n} \varkappa_i b_i^{(\mathcal{P})} = (0.1974, -0.2011).$

5. Conclusions

In comparison to classical and fuzzy models, bipolar fuzzy models provide more precision, flexibility, and compatibility to the system. In this paper, we have discussed the basic properties of operations on PBFGs. The direct product of two PBFGs is a PBFG, as we have demonstrated. However, two PBFGs' Cartesian product, strong product, and lexicographic product are not PBFGs in general. Where as, the Cartesian product, strong product and lexicographic product are PBFGs if $\Im$ is a product bipolar fuzzy edge graph. We have defined the notion of complement of PBFGs along with its properties. Finally, we have given an application of PBFGs in multi-agent decision making. It is required and valuable to extend the concept of operations on PBFG to product single-valued neutrosophic graphs in future study.

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