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CONVOLUTION CONDITIONS FOR q-CONVEXITY, q-STARLIKENESS AND q-SPIRALLIKENESS

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Abstract. In the present investigation, the authors derive necessary and sufficient conditions for q-starlike, q-convex functions of order β and q-spirallike, convex q-spirallike functions using convolution.

Keywords: q-derivative, q-starlike functions, q-convex functions, q-spirallike, convolution.

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1. Introduction

Convolution or Hadamard product is a powerful tool which forges links between the fundamental properties of operators and beautiful results from the classical theory of analytic functions. The Pólya-Schoenberg conjecture [4], generated a great deal of intrinsic interest in properties of convolutions. Silverman, Silvia and Telage's method described the construction of a function g_φ such that $Re\{g_\varphi(z)\} > 0$ if and only if $\frac{1}{z}(f * g_\varphi) \neq 0$. Generalizing Silverman, Silvia and Telage's results, we give characterization for q-analogue classes of convex, starlike and spirallike functions [8]. Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \; and \; |z| < 1\},$$

and satisfy the normalization conditions $f(0) = f'(0) - 1 = 0$, for every $z \in \mathcal{U}$. And $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all function which are univalent in $\mathcal{U}$. In view of Riemann mapping theorem, the unit disc $\mathcal{U}$ is usually treated as a standard domain because simply connected proper subset of the complex plane are conformally equivalent to $\mathcal{U}$. The functions whose ranges describe certain geometries like star, close-to-star, convex, close-to-convex, spiral, some in certain directions, some uniformly, some with respect to

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symmetric points and so on are known as geometric functions and these geometries can be described in succinct links between certain prescribed property of analytic functions and the geometries of their ranges. A subset $\mathcal{D}$ of the complex plane is called q-geometric if $qz \in \mathcal{D}$ for fixed $q \in \mathbb{R}$ whenever $z \in \mathcal{D}$. For a function f, real or complex valued on a q-geometric set $\mathcal{D}$, $|q| \neq 1$, the q-difference operator which was introduced by Jackson [2] is defined as

$$D_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad z \in \mathcal{D} - \{0\}. \tag{2}$$

In addition, the q-derivative at zero defined for $|q| > 1$, $D_q f(0) = D_{q^{-1}} f(0)$. In some literature the q-derivative at zero is defined as $f'(0)$ if it exists.
Equivalently (2), may be written as

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0$$

where

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1. \end{cases}$$

As a right inverse, Jackson [3] presented the q-integral of a function f as

$$\int_0^z f(t) d_q t = z(1-q) \sum_{n=0}^{\infty} q^n f(zq^n),$$

provided that the series converges. Under the hypothesis of the definition of q-difference operator, we have the following rules.
(i) $D_q(af(z) \pm bg(z)) = aD_q f(z) \pm bD_q g(z)$, where a and b any real (or complex) constants
(ii) $D_q(f(z)g(z)) = f(qz)D_q g(z) + D_q f(z)g(z) = f(z)D_q g(z) + D_q f(z)g(qz)$
(iii) $D_q\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)D_q f(z) - f(z)D_q g(z)}{g(qz)g(z)}$.

For $0 < q < 1$, we define the class $S_q^*(\beta)$ of q-starlike functions and the class $C_q(\beta)$ of q-convex functions of order $\beta (0 \leq \beta < 1)$ (see, [1], [7]), as below:

Definition 1.1. *A function $f(z) \in \mathcal{A}$ is said to be q-starlike of order β, $0 \leq \beta < 1$, if and only if*

$$Re\left\{\frac{zD_q f(z)}{f(z)}\right\} > \beta, \qquad for\ all\ z \in \mathcal{U}.$$

We denote by $S_q^*(\beta)$ the subclass of $\mathcal{A}$ consisting of all q-starlike functions of order β in the unit disk $\mathcal{U}$.

Definition 1.2. *A function $f(z) \in \mathcal{A}$ is said to be q-convex of order β, $0 \leq \beta < 1$, if and only if*

$$Re\left\{\frac{D_q\left(zD_q f(z)\right)}{D_q f(z)}\right\} > \beta, \qquad for\ all\ z \in \mathcal{U}.$$

We denote by $C_q(\beta)$ the subclass of $\mathcal{A}$ consisting of all q-convex functions of order β in the unit disk $\mathcal{U}$.

Definition 1.3. *The convolution or Hadamard product, of two analytic functions*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ (|z| < R_1) \ \text{and}\ g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \ (|z| < R_2),$$

is defined as the power series

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \qquad |z| < R_1 R_2.$$

It can be easily seen that

$$zD_q f * g = f * zD_q g. \tag{3}$$

2. Main results

Theorem 2.1. *The function $f \in C_q(\beta)$ in $|z| < R \leq 1$ if and only if*

$$\frac{1}{z}\left[f * \frac{z + \left(\frac{[2]_q(x+2\beta-1)}{2-2\beta} + 1\right) qz^2 + \frac{(1+q-[2]_q)(x+2\beta-1)}{2-2\beta} qz^3}{(1-z)(1-qz)(1-q^2z)}\right] \neq 0.$$

Proof. The function $f \in C_q(\beta)$ if and only if

$$Re\left\{\frac{D_q\left(zD_q f(z)\right)}{D_q f(z)}\right\} > \beta, \qquad for\ all\ z \in \mathcal{U}. \tag{4}$$

Since $\frac{D_q(zD_qf)}{D_qf} = 1$ at $z = 0$, so (7) is equivalent to

$$\frac{\frac{D_q(zD_qf)}{D_qf} - \beta}{1-\beta} \neq \frac{x-1}{x+1}, \ (|z| < R,\ |x| = 1,\ x \neq -1)$$

which implies

$$(1+x)D_q\left(zD_q f\right) + (1 - 2\beta - x)D_q f \neq 0. \tag{5}$$

Setting $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we have

$$D_q f = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}$$

$$D_q\left(zD_q f\right) = 1 + \sum_{n=2}^{\infty} [n]_q^2 a_n z^{n-1} = D_q f * \frac{1}{(1-z)(1-qz)}.$$

The left hand side of (5) is equivalent to

$$\begin{aligned}
(1+x)&\left[D_q f * \sum_{n=1}^{\infty}[n]_q z^{n-1}\right] + D_q f * \sum_{n=1}^{\infty}(1-2\beta-x)z^{n-1} \\
&= D_q f * \sum_{n=1}^{\infty}\left[(1-2\beta-x) + (1+x)[n]_q\right] z^{n-1} \\
&= D_q f * \left(\frac{1-2\beta-x}{1-z} + \frac{1+x}{(1-z)(1-qz)}\right) \\
&= D_q f * \left(\frac{(2-2\beta) + (x+2\beta-1)qz}{(1-z)(1-qz)}\right).
\end{aligned}$$

Thus

$$\frac{1}{z}\left[zD_q f * \frac{z + \frac{x+2\beta-1}{2-2\beta} qz^2}{(1-z)(1-qz)}\right] \neq 0. \tag{6}$$

By using (3), we can write (6) as

$$\frac{1}{z}\left[f * \frac{z+\left(\frac{[2]_q(x+2\beta-1)}{2-2\beta}+1\right)qz^2+\frac{(1+q-[2]_q)(x+2\beta-1)}{2-2\beta}qz^3}{(1-z)(1-qz)(1-q^2z)}\right] \neq 0,$$

which completes the proof. □

As $q \to 1^-$, we have following result proved by Silverman and et al. in [8].

Corollary 2.1. *The function $f \in C(\beta)$ in $|z| < R \leq 1$ if and only if*

$$\frac{1}{z}\left[f * \frac{z+\frac{x+\beta}{1-\beta}z^2}{(1-z)^3}\right] \neq 0.$$

As $q \to 1^-$ and $\beta = 0$, we have following result proved by Ruschewyh in [6].

Corollary 2.2. *The function $f \in C$ in $|z| < R \leq 1$ if and only if*

$$\frac{1}{z}\left[f * z\frac{1-xz}{(1-z)^3}\right] \neq 0.$$

Theorem 2.2. *The function $f \in S_q^*(\beta)$ in $|z| < R \leq 1$ if and only if*

$$\frac{1}{z}\left[f * \frac{z+\frac{x+2\beta-1}{2-2\beta}qz^2}{(1-z)(1-qz)}\right] \neq 0, \ (|z| < R, \ |x| = 1).$$

Proof. Since f is q-starlike of order β if and only if $g(z) = \int_0^z \frac{f(\zeta)}{\zeta} d_q\zeta$ is q-convex of order β, we have

$$\frac{1}{z}\left[g * \frac{z+\left(\frac{[2]_q(x+2\beta-1)}{2-2\beta}+1\right)qz^2+\frac{(1+q-[2]_q)(x+2\beta-1)}{2-2\beta}qz^3}{(1-z)(1-qz)(1-q^2z)}\right] = \frac{1}{z}\left[f * \frac{z+\frac{x+2\beta-1}{2-2\beta}qz^2}{(1-z)(1-qz)}\right].$$

Thus the result follows from Theorem 2.1. □

As $q \to 1^-$, we have following result proved by Silverman and et al. in [8].

Corollary 2.3. *The function $f \in S^*(\beta)$ in $|z| < R \leq 1$ if and only if*

$$\frac{1}{z}\left[g * \frac{z+\frac{x+\beta}{1-\beta}z^2}{(1-z)^3}\right] = \frac{1}{z}\left[f * \frac{z+\frac{x+2\beta-1}{2-2\beta}z^2}{(1-z)^2}\right].$$

As $q \to 1^-$, and $\beta = \frac{1}{2}$ we have following result proved by Ruschewyh in [6].

Corollary 2.4. *The function $f \in S^*(\frac{1}{2})$ in $|z| < R \leq 1$ if and only if*

$$\frac{1}{z}\left[g * \frac{z+(2x+1)z^2}{(1-z)^3}\right] = \frac{1}{z}\left[f * \frac{z+xz^2}{(1-z)^2}\right].$$

Now by using the concept of q-derivative we define the classes of q-spirallike and convex q-spirallike functions as the following

Definition 2.1. *A function $f \in \mathcal{A}$ is said to be q-spirallike if and only if*

$$Re\left\{e^{i\lambda}\frac{zD_qf(z)}{f(z)}\right\} > 0, \ |z| < R \leq 1, \ \lambda \text{ real with } |\lambda| < \frac{\pi}{2}.$$

Definition 2.2. *A function $f \in \mathcal{A}$ is said to be convex q-spirallike if and only if*

$$Re\left\{e^{i\lambda}\left(\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)}\right)\right\} > 0,\ |z| < R \leq 1,\ \lambda \text{ real with } |\lambda| < \frac{\pi}{2}.$$

Theorem 2.3. *For $|z| < R \leq 1$, λ real with $|\lambda| < \frac{\pi}{2}$ and $|x| = 1$, we have*

$$Re\left\{e^{i\lambda}\left(\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)}\right)\right\} > 0$$

if and only if

$$\frac{1}{z}\left[f * \frac{z + \left(\frac{[2]_q(x-e^{-2i\lambda})}{1+e^{-2i\lambda}} + 1\right)qz^2 + \frac{(1-q)(x-e^{-2i\lambda})}{1+e^{-2i\lambda}}qz^3}{(1-z)(1-qz)(1-q^2z)}\right] \neq 0.$$

Proof. We have, $Re\left\{e^{i\lambda}\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right)\right\} > 0$ if and only if

$$\frac{e^{i\lambda}\frac{D_q(zD_qf)}{D_qf} - i\sin\lambda}{\cos\lambda} \neq \frac{x-1}{x+1},\ (|z| < R,\ |x| = 1,\ x \neq -1)$$

which implies

$$(1+x)D_q\left(zD_qf\right) + (e^{-2i\lambda} - x)D_qf \neq 0. \tag{7}$$

Setting $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we have

$$D_q\left(zD_qf\right) = 1 + \sum_{n=2}^{\infty}[n]_q^2 a_n z^{n-1} = D_qf * \frac{1}{(1-z)(1-qz)}.$$

The left hand side of (7) is equivalent to

$$\begin{aligned}(1+x)&\left[D_qf * \sum_{n=1}^{\infty}[n]_q z^{n-1}\right] + D_qf * \sum_{n=1}^{\infty}(e^{-2i\lambda} - x)z^{n-1}\\
&= D_qf * \sum_{n=1}^{\infty}\left[(e^{-2i\lambda} - x) + (1+x)[n]_q\right]z^{n-1}\\
&= D_qf * \left(\frac{e^{-2i\lambda} - x}{1-z} + \frac{1+x}{(1-z)(1-qz)}\right)\\
&= D_qf * \left(\frac{(1+e^{-2i\lambda}) + (x - e^{-2i\lambda})qz}{(1-z)(1-qz)}\right).\end{aligned}$$

Thus

$$\frac{1}{z}\left[zD_qf * \frac{z + \frac{x-e^{-2i\lambda}}{1+e^{-2i\lambda}}qz^2}{(1-z)(1-qz)}\right] \neq 0. \tag{8}$$

By using (3), we can write (8) as

$$\frac{1}{z}\left[f * \frac{z + \left(\frac{[2]_q(x-e^{-2i\lambda})}{1+e^{-2i\lambda}} + 1\right)qz^2 + \frac{(1-q)(x-e^{-2i\lambda})}{1+e^{-2i\lambda}}qz^3}{(1-z)(1-qz)(1-q^2z)}\right] \neq 0,$$

which completes the proof. □

As $q \to 1^-$, we have following result proved by Silverman and et al. in [8].

Corollary 2.5. *For $|z| < R \leq 1$, λ real with $|\lambda| < \frac{\pi}{2}$ and $|x| = 1$, we have*

$$Re\left\{e^{i\lambda}\left(\frac{(zf'(z))'}{f'(z)}\right)\right\} > 0$$

if and only if

$$\frac{1}{z}\left[f * \frac{z + \frac{2x+1-e^{-2i\lambda}}{1+e^{-2i\lambda}}z^2}{(1-z)^3}\right] \neq 0.$$

Theorem 2.4. *For $|z| < R \leq 1$, λ real with $|\lambda| < \frac{\pi}{2}$ and $|x| = 1$, we have*

$$Re\left\{e^{i\lambda}\frac{zD_qf(z)}{f(z)}\right\} > 0$$

if and only if

$$\frac{1}{z}\left[f * \frac{z + \frac{x-e^{-2i\lambda}}{1+e^{-2i\lambda}}qz^2}{(1-z)(1-qz)}\right] \neq 0.$$

Proof. The result follows from Theorem 2.3 in the same manner that Theorem 2.2 followed from Theorem 2.1. □

As $q \to 1^-$, we have following result proved by Silverman and et al. in [8].

Corollary 2.6. *For $|z| < R \leq 1$, λ real with $|\lambda| < \frac{\pi}{2}$ and $|x| = 1$, we have*

$$Re\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\} > 0$$

if and only if

$$\frac{1}{z}\left[f * \frac{z + \frac{x-e^{-2i\lambda}}{1+e^{-2i\lambda}}z^2}{(1-z)^2)}\right] \neq 0.$$

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