SOLUTION OF INVERSE SOURCE PROBLEM IN THERMOACOUSTIC IMAGING

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IŞIK UNIVERSITY
JUNE, 2022

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Ph.D., Mathematics, Işık University, 2022

Submitted to the Graduate School of Science and Engineering
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy
in
Mathematics

IŞIK UNIVERSITY
JUNE, 2022

IŞIK UNIVERSITY
GRADUATE SCHOOL OF SCIENCE AND ENGINEERING

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ABSTRACT

This study aims to investigate and explore accurate analytical inverse solutions of thermoacoustic wave equation involved in microwave induced thermoacoustic imaging of breast. Using boundary conditions, we aimed to find more realistic solutions. For cross-sectional two-dimensional thermoacoustic imaging of breast, we explored solution of the wave equation using layered tissue model consisting of concentric annular layers on a cylindrical cross-section. To obtain the forward and inverse solutions of the thermoacoustic wave equation, we derived the Green's function involving Bessel and Hankel functions by employing the geometrical and acoustic parameters (densities and velocities) of layered media together with temporal initial condition, radiation conditions and continuity conditions on boundaries of layers. The image reconstruction based on this approach involves the layers parameters as the a priori information which can be estimated from the acquired thermoacoustic data. To test and compare our layered solution with conventional solution based on homogeneous medium assumption, we performed simulations using numerical test phantoms consisting of sources distributed in the layered structure. After then, we derived more general integral solution for thermoacoustic wave equation in frequency domain for an arbitrary convex domain in $\mathbb{R}^3$.

Keywords: Inverse source problem, Thermoacoustic wave equation, Green's functions, Integral equations, Nonhomogeneous medium

TERMOAKUSTİK GÖRÜNTÜLEMEDE TERS PROBLEM ÇÖZÜMÜ

ÖZET

Bu çalışma, tıbbi meme görüntülemede kullanılan mikrodalga uyarımlı termoakus- tik görüntüleme sisteminin dayandığı termoakustik dalga denkleminin doğru ana-litik ters çözümlerini araştırmayı ve keşfetmeyi amaçlamaktadır. Çalışmada, sınır koşullarını kullanarak daha gerçekçi çözümler bulmak hedeflendi. Meme dokusunun kesitsel iki boyutlu termoakustik görüntülemesi için, silindirik bir kesit üzerinde eşmerkezli dairesel katmanlardan oluşan katmanlı doku modeli kullanarak dalga denkleminin çözümü araştırıldı ve elde edildi. Termoakustik dalga denkleminin ileri ve ters çözümlerini elde etmek için, katmanlı ortamın geometrik ve akustik parametreleri (yoğunlukları ve hızları) ile zamansal başlangıç koşulu, radyasyon koşulları ve süreklilik sınır koşulları birlikte kullanılarak Bessel ve Hankel fonksiyonlarını içeren Green fonksiyonları bulundu. Bu yaklaşıma dayalı görüntü oluşturma, elde edilen termoakustik verilerden tahmin edilebilen katman parametrelerini ön bilgi olarak içerir. Katmanlı çözümümüzü homojen ortam varsayımına dayalı geleneksel çözümle test etmek ve karşılaştırmak için katmanlı yapıda dağılmış kaynaklardan oluşan sayısal test fantomları kullanarak simülasyonlar gerçekleştirildi. Daha sonra, $\mathbb{R}^3$'deki rastgele bir dışbükey bölge için termoakustik dalga denkleminin frekans uzayında daha genel bir integral çözüm elde edildi.

Anahtar kelimeler: Ters kaynak problemi, Termoakustik dalga denklemi, Green fonksiyonları, İntegral denklemler, Homojen olmayan ortam

ACKNOWLEDGEMENTS

I would like to thank to my advisor Prof. Banu Uzun for illuminating my path with her valuable knowledge and experiences. She has been very supportive and understanding on this difficult journey. She was a guide who thought every step in advance, picked me up every time I fell and always motivated me.

I want to express my deep gratitude to Prof. Mustafa Karaman, who unfortunately passed away, for the idea of this research and his contribution to this research by his extensive knowledge and enlightening vision.

I would also like to thank to my parents and my siblings for supporting me through all my life.

Lastly, special thanks to my husband and my children for their love, support and patience through my study.

This study was supported by The Scientific and Technological Research Council of Turkey (TÜBİTAK) Grant No: 213E038.

Demet ELMAS

To my children Yusuf and Elif . . .

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RF : Radio Frequency

MRI : Magnetic Resonance Imaging

TAT : Thermoacoustic Tomography

CHAPTER 1

1. INTRODUCTION

1.1 Thermoacoustic Tomography

Thermoacoustic tomography (TAT) is defined as cross-sectional or three dimensional imaging of biological tissues based on the thermoacoustic effect (Lihong, Xu, & Wang, 2006). In thermoacoustic imaging, non-ionizing radio frequency (RF) or microwave pulses are delivered into biological tissues. Some of the delivered energy is absorbed and converted into heat, leading to transient thermoelastic expansion, which in turn leads to ultrasonic emission. The generated ultrasonic waves are then detected by ultrasonic transducers located on the boundary of the object to form images of absorbtion properties of object. It is known that absorption is closely associated with physiological properties, such as hemoglobin concentration and oxygen saturation and that cancerous cells absorb several times more energy than the healthy ones. (Baranski & Czerski, 1976; Foster & Arkhipov, 1974; M. Xu & Wang, 2002; L. V. Wang, 2003). As a result, the magnitude of the thermoacoustic signal, which is proportional to the local energy deposition, reveals physiologically specific absorption contrast. This contrast provides determination of cancerous locations if the distribution of the absorption function is known.

1.2 Breast Cancer and Breast Imaging Modalities

Breast cancer accounts for 22.9 % of all cancers in women and approximately 13.7 % of cancer deaths in women is caused by breast cancer. Risk of getting breast cancer increases by age and the diagnosis ratio of breast cancer is almost 70 % (Boniol et al., 2007; Boyle & Ferlay, 2005). Breast cancer is the most common cancer type among women and is the second type of cancer with the highest mortality rate (Siegel, Miller, Fuchs, & Jemal, 2021). Similar to all other cancer types early diagnosis has a critical importance in dealing with breast cancer and to decrease the number of breast cancer related deaths. Today, mammography and ultrasound are widely used medical imaging devices for breast cancer diagnosis and follow-up. Mammography is a very cost-effective technique but the x-rays used for imaging have cancer-triggering harmful side effects. Also there are some difficulties in diagnosing pathologies in the glandular tissue with mammography. Computerized tomography which needs to be performed using higher doses of x-ray when compared to mammography is not used for diagnosis of breast cancer. Ultrasonic imaging is low-cost and has no harmful side-effects. Point resolution of this technique is high but contrast resolution is poor, this complicates early diagnosis of some cancers and distinguishing malignancy. In breast imaging, magnetic resonance imaging (MRI), with contrast, is useful for the diagnosis, but this imaging technique is relatively expensive and has no standart application protocols. RF and microwave breast tomography is a technique based on different electrical properties of tissues, having potentially high contrast resolution and no ionizing harmful side effect, but having poor point resolution (Lihong et al., 2006; Guy & Fytche, 2005). In recent years, imaging based on photoacoustic/theormoacoustic effect is an attractive research topic. Since TAT is a hybrid biomedical imaging modality, which employs electromagnetic energy in excitation and ultrasound waves in sensing, it combines high contrast due to electromagnetic absorption and high point resolution of ultrasound without harmful side effects.

1.3 Inverse Source Problem

Many imaging methods, for biological tissue, are based on the reconstruction of source distribution from data collected by transducers over a surface enclosing the region to be imaged. In microwave induced thermoacoustic tomography, the biological tissue is heated by microwaves for thermal expansion of a tissue. Then the tissue acts as an acoustic wave source. This process is represented mathematically by the following nonhomogeneous wave equation:

$$\nabla^2 p(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2 p(\mathbf{r},t)}{\partial t^2} = \frac{\beta}{C_p}\frac{\partial H(\mathbf{r},t)}{\partial t}, \tag{1.1}$$

where $p(r,t)$ is the acoustic pressure at position r and time t, c is the speed of sound, β is the isobaric volume expansion coefficient, C_p is the specific heat and $H(r,t)$ is the heating function. The left hand side of this equation describes the acoustic wave propagation and the right hand side represents the source term (Lihong et al., 2006; Karabutov & Gusev, 1993; Olsen, 1982; Council, 1996).

1.3.1 Derivation of Thermoacoustic Wave Equation

The acoustic wave equation governs the propagation of acoustic waves through a material medium. The equation describes the evolution of acoustic pressure p or particle velocity of a fluid $\overrightarrow{u} = u_x\overrightarrow{i} + u_y\overrightarrow{j} + u_z\overrightarrow{k}$ as a function of position r and time t.

The motion of the fluid can be described with its compression or expansion, by defining the relation between the particle velocity $\overrightarrow{u}$ and the instantaneous density ρ. Consider a small rectangular parallelepiped volume element $dV = dx\,dy\,dz$, which is fixed in space and through fluid travels:

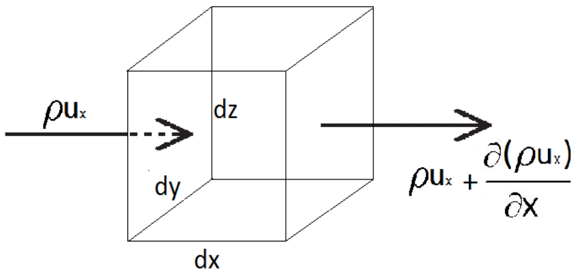

Figure 1.1 A spatially fixed volume element

The net influx of mass in the x direction into this spatially fixed volume is given by

$$\left(\rho u_x - \left(\rho u_x + \frac{\partial(\rho u_x)}{\partial x}dx\right)\right) dy\, dz = -\frac{\partial(\rho u_x)}{\partial x}dV. \tag{1.2}$$

The net influx of mass in the y and z directions can be expressed similarly and thus the total influx through the parallelepiped is given by the equation

$$-\left(\frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} + \frac{\partial(\rho u_z)}{\partial z}\right) dV = -\nabla.(\rho u_x)dV. \tag{1.3}$$

The net influx must be equal to the rate of temporal increase in the mass of the volume:

$$-\nabla.(\rho u_x)dV = \frac{\partial \rho}{\partial t}dV, \tag{1.4}$$

which gives the following equation of continuity:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho u_x) = 0. \tag{1.5}$$

Now, if we write $\rho = \rho_0(1+s)$ where s is condensation defined by $s = (\frac{\rho - \rho_0}{\rho_0})$ and ρ_0 is the equilibrium density, which should be sufficiently weak function of time, and assume that s is very small, (1.5) becomes

$$\rho_0 \frac{\partial s}{\partial t} + \nabla.(\rho_0 \overrightarrow{u}) = 0, \tag{1.6}$$

which is the linear continuity equation of mass.

Now, consider the fluid element $dV = dxdydz$ which moves with the fluid and contains a mass dm of a fluid. Let P be instantaneous pressure and P_0 be equilibrium pressure at given position so as the acoustic pressure $p = P - P_0$. By Newton's second law, the net force $d\overrightarrow{f} = \overrightarrow{a}dm$ on the element will give rise to acceleration.

Under the assumption of no thermal confinement and kinematic viscosity, the net force experienced by the element in the x direction is

$$df_x = \left(P - \left(P + \frac{\partial P}{\partial x}dx\right)\right)dydz = -\frac{\partial P}{\partial x}dV. \tag{1.7}$$

Again similar expressions can be written in both y and z directions and thus total force can be written as

$$df_x + df_y + df_z = -\nabla P dV. \tag{1.8}$$

On the other hand, on the infinitesimal element, gravitational force acts. Hence, the net force on the element is

$$d\overrightarrow{f} = -\nabla P dV + \overrightarrow{g}\rho dV. \tag{1.9}$$

The acceleration $\overrightarrow{a}$ of the fluid element is the time derivative of the velocity function $\overrightarrow{u}$ of it. Here, we must notice that $\overrightarrow{u}$ is a function of both space and time. So we can calculate the acceleration by using the chain rule:

$$\begin{aligned}
\overrightarrow{a} &= \frac{\partial u_x}{\partial t} + \frac{\partial u_x}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u_x}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u_x}{\partial z}\frac{\partial z}{\partial t} && (1.10)\\
&+ \frac{\partial u_y}{\partial t} + \frac{\partial u_y}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u_y}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u_y}{\partial z}\frac{\partial z}{\partial t} && (1.11)\\
&+ \frac{\partial u_z}{\partial t} + \frac{\partial u_z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u_z}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u_z}{\partial z}\frac{\partial z}{\partial t}. && (1.12)
\end{aligned}$$

Here, $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial t}$ and $\frac{\partial z}{\partial t}$ mean time derivative of displacement of a fluid in the direction x, y and z, respectively. The time derivative of displacement is equal to the velocity.

Hence, the acceleration is sum up to

$$\overrightarrow{a} = \frac{\partial \overrightarrow{u}}{\partial t} + u_x \frac{\partial \overrightarrow{u}}{\partial x} + u_y \frac{\partial \overrightarrow{u}}{\partial y} + u_z \frac{\partial \overrightarrow{u}}{\partial z}. \tag{1.13}$$

Then, $\overrightarrow{a}$ can be written as

$$\overrightarrow{a} = \frac{\partial \overrightarrow{u}}{\partial t} + (\overrightarrow{u}.\nabla)\overrightarrow{u}. \tag{1.14}$$

Now, $d\overrightarrow{f} = \overrightarrow{a}dm$ gives

$$-\nabla P + \overrightarrow{g}\rho = \rho\left(\frac{\partial \overrightarrow{u}}{\partial t} + (\overrightarrow{u}.\nabla)\overrightarrow{u}\right). \tag{1.15}$$

In the case of no acoustic excitation in the medium, $\overrightarrow{g}\rho_0 = \nabla P_0$, and hence $\nabla P = \nabla p + \overrightarrow{g}\rho_0$ so that substituting this in the above equation gives

$$-\frac{1}{\rho_0}\nabla p + \overrightarrow{g}s = (1+s)\left(\frac{\partial \overrightarrow{u}}{\partial t} + (\overrightarrow{u}.\nabla)\overrightarrow{u}\right). \tag{1.16}$$

If we make the assumptions that $|\overrightarrow{g}s| \ll \frac{|\nabla p|}{\rho_0}$, $|s| \ll 1$, and $|(\overrightarrow{u.\nabla})\overrightarrow{u}| \ll \left|\frac{\partial \overrightarrow{u}}{\partial dt}\right|$, then

$$\rho_0 \frac{\partial \overrightarrow{u}}{\partial t} = -\nabla p. \tag{1.17}$$

This is so called linear Euler's equation, valid for acoustic processes of small amplitude. Now, taking the divergence of (1.17) and the time derivative of (1.6), we get the following two equations, respectively:

$$\nabla.\left(\rho_0 \frac{\partial \overrightarrow{u}}{\partial t}\right) = -\nabla^2 p, \tag{1.18}$$

$$\rho_0 \frac{\partial^2 s}{\partial t^2} + \nabla.\left(\rho_0 \frac{\partial \overrightarrow{u}}{\partial t}\right) = 0. \tag{1.19}$$

Elimination of the divergence between these two equations gives

$$\nabla^2 p = \rho_0 \frac{\partial^2 s}{\partial t^2}. \tag{1.20}$$

Condensation s can be expressed as $s = \dfrac{p}{c^2\rho_0}$ where c is the thermodynamic speed of sound. So, (1.20) turns into the following acoustic wave equation

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}. \tag{1.21}$$

In thermoacoustic imaging, acoustic wave source inside the medium is caused by thermal expansion. After heating the medium, the pressure on the fluid is increased by acoustic pressure source, which in turn the total force change over the system. Hence, by Newton's second law, there would be an additional force in the equation (1.9) which leads to source term $\dfrac{\beta}{C_p}\dfrac{\partial H(\mathbf{r},t)}{\partial t}$ (Karabutov & Gusev, 1993; Kruger et al., 2000; Kuchment & Kunyansky, 2008) in the acoustic wave equation and it results in so called thermoacoustic wave equation:

$$\nabla^2 p(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2 p(\mathbf{r},t)}{\partial t^2} = \frac{\beta}{C_p}\frac{\partial H(\mathbf{r},t)}{\partial t}. \tag{1.22}$$

CHAPTER 2

2. LITERATURE SURVEY

Thermoacoustic imaging is an attractive research topic in recent years. Thermoacoustic effect generating acoustic sound by absorbed wave energy was discovered by Alexander Bell (Bell, 1880). Many studies developed mathematical methods for inverse solution of thermoaocustic wave equation representing thermoacoustic effect. Most of the research studies reported in the literature were based on homogeneous medium assumption (M. Xu & Wang, 2002; Y. Xu, Xu, & Wang, 2002b, 2002a; M. Xu & Wang, 2005; İdemen & Alkumru, 2012). These researchs includes analytic solutions using method of Green's functions, surface integrals and series expansions. But there are studies taking acoustic heterogeneties into account and using numerical methods, iterative approachs and operator theory (Agranovsky & Kuchment, 2007; Hristova, Kuchment, & Nyugen, 2008; Stefanov & Uhlman, 2009; Qian, Stefanov, Uhlman, & Zhao, 2011; Anastasio, Zhang, & Pan, 2005). The boundary conditions for thermoacoustic imaging have been investigated by Wang and Yang (L. V. Wang & Yang, 2007). In a more recent study, Schoonover and Anastasio (Schoonover & Anastasio, 2011) have presented an inverse solution based on piecewise homogeneous planar layers structure consisting source distribution only in one certain layer. Also, there are other studies combining conventional methods and acoustic speed distribution as apriori information so that reducing effect of inhomogeneity and improving image quality (Y. Xu & Wang, 2003; J. Wang et al., 2015; B. Wang, Zhao, Liu, Nie, & Liu, 2017; Liu, Lu, Zhu, & Jin, 2017).

CHAPTER 3

3. APPROACH

3.1 Problem Statement

In this study, our purpose is to find an inverse solution of thermoacoustic wave equation in layered medium with taking boundary conditions into account.

The thermoacoustic wave propagation is governed by the nonhomogeneous wave equation

$$\nabla^2 p(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2 p(\mathbf{r},t)}{\partial t^2} = -p_0(\mathbf{r}).\delta'(t) \tag{3.1}$$

with boundary conditions

$$p(\mathbf{r},t)\big|_{\mathbf{r}\in S^-} = p(\mathbf{r},t)\big|_{\mathbf{r}\in S^+} \tag{3.2}$$

and

$$\left.\frac{1}{\rho(\mathbf{r})}\frac{\partial p(\mathbf{r},t)}{\partial n}\right|_{\mathbf{r}\in S^-} = \left.\frac{1}{\rho(\mathbf{r})}\frac{\partial p_{m+1}(\mathbf{r},t)}{\partial n}\right|_{\mathbf{r}\in S^+} \tag{3.3}$$

on each boundary S appearing in the space. Here, $p(\mathbf{r},t)$ and $\rho(\mathbf{r})$ are the acoustic wave and the density functions at position $\mathbf{r}$, respectively. $-p_0(\mathbf{r}).\delta'(t)$ is the source term corresponding the term $\frac{\beta}{C_p}\frac{\partial H(\mathbf{r},t)}{\partial t}$.

The outgoing wave function must satisfy the radiation condition

$$\left| \frac{\partial P}{\partial |\mathbf{r}|} - ikP \right| = 0 \quad \text{as} \quad \mathbf{r} \to \infty \tag{3.4}$$

and the incoming wave function must satisfy the radiation condition

$$\left| \frac{\partial P}{\partial |\mathbf{r}|} + ikP \right| = 0 \quad \text{as} \quad \mathbf{r} \to \infty. \tag{3.5}$$

Also, as a nature of the problem, nonhomogeneous thermoacoustic wave equation must satisfy the following initial conditions

$$p(\mathbf{r}, 0^+) = c^2(\mathbf{r}) p_0(\mathbf{r}) \tag{3.6}$$

$$p(\mathbf{r}, t) = 0 \quad \text{if} \quad t < 0 \tag{3.7}$$

$$\frac{\partial p(\mathbf{r}, 0)}{\partial t} = 0. \tag{3.8}$$

Inverse solution of thermoacoustic wave equation is determining source distrbution function (ratio of emision of microwave) inside medium from a known acoustic pressure by a transducer. Inverse source problem in thermoacoustic imaging has been studied for homogeneous medium by Xu and Wang (M. Xu & Wang, 2005) for specific measurement geometries: two parallel planes, an infinitely long circular cylinder and a sphere, and this solution was extended to the arbitrary measurement geometry by İdemen and Alkumru (İdemen & Alkumru, 2012). In these studies, in frequency domain, the source distribution inside the medium is determined by the following integral equation:

$$p_0(\mathbf{r}) = \frac{1}{\pi c^2} \int_{-\infty}^{\infty} \int_S P(\mathbf{r_s}, w) \frac{\partial G_h^{\text{in}}(\mathbf{r}, \mathbf{r_s}, w)}{\partial \mathbf{n_s}} \, dS \, dw, \tag{3.9}$$

where S is a measurement surface, $P(\mathbf{r_s}, w)$ is the acoustic pressure measured on the surface S and G_h is a free space Green's function. In our study, for the purpose of thermoacoustic imaging of breast, we worked with cylidrically N-layered medium. We extended the conventional solution and proved that the source distribution in each layer of this medium can be determined as

$$p_0(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\pi c^2(\mathbf{r})} \int_{-\infty}^{\infty} \int_{S_N} P(\mathbf{r_s}, w) \frac{1}{\rho(\mathbf{r_s})} \frac{\partial G^{\text{in}}(\mathbf{r}, \mathbf{r_s}, w)}{\partial \mathbf{n_s}}\, dS\, dw, \quad \mathbf{r} \in \text{Layer } i \tag{3.10}$$

for $1 \leq i \leq N$, where $P(\mathbf{r_s}, w)$ is the acoustic pressure measured on the surface S_N (located at outmost layer N^{th} layer), G is the corresponding Green's function of N-layered medium and $\rho(\mathbf{r})$ is a density function such that

$$\rho(\mathbf{r}) = \rho_i, \quad \mathbf{r} \in \text{Layer } i \tag{3.11}$$

for $1 \leq i \leq N$. After that we derived more general integral solution to the inverse problem of thermoacoustic wave equation in arbitrary convex medium as

$$\begin{aligned} p_0(\mathbf{r}') = {} & \frac{1}{2\pi c^2(\mathbf{r})} \\ & \times \int_{-\infty}^{\infty} \int_{S} \left(\frac{1}{\rho(\mathbf{r})} \nabla P(\mathbf{r}, w) G^{in}(\mathbf{r}, \mathbf{r}', w) - P(\mathbf{r}, w) \frac{1}{\rho(\mathbf{r})} \nabla G^{in}(\mathbf{r}, \mathbf{r}', w) \right) .\mathbf{n}\, dS_{\mathbf{r}}\, dw, \end{aligned} \tag{3.12}$$

where $P(\mathbf{r}, w)$ is a pressure function known on the measurement surface S and $G^{in}(\mathbf{r}, \mathbf{r}', w)$ is the Green's function corresponding to nonhomogeneous media.

CHAPTER 4

4. FORWARD SOLUTION AND INITIAL CONDITION

The Fourier transform is a useful tool to solve differential equations. We denote the Fourier transform of a function $f(t)$ with upper case of the letter f, that is

$$F(w) = \int_{-\infty}^{\infty} f(t)e^{-iwt}dt.$$

By taking Fourier transform of thermoacoustic wave equation

$$\nabla^2 p(\mathbf{r}, t) - \frac{1}{c^2}\frac{\partial^2 p(\mathbf{r}, t)}{\partial t^2} = -p_0(\mathbf{r}).\delta'(t), \tag{4.1}$$

we obtain following nonhomogeneous Helmholtz equation

$$\nabla^2 P(\mathbf{r}, w) + k^2 P(\mathbf{r}, w) = iwp_0(r) \tag{4.2}$$

where $k = w/c$ is the wave number, $P(\mathbf{r}, w)$ is the temporal Fourier Transform of $p(\mathbf{r}, t)$.

In our derivations, we considered that $w > 0$ and $P(\mathbf{r}, w)$ is corresponding to outgoing wave. After then, for the completeness in frequency domain, we defined $P(\mathbf{r}, -w) = P(\mathbf{r}, w)^*$ for $w < 0$ as complex conjugate of pressure function for positive frequency. The outgoing and incoming waves were represented by superscripts 'out' and 'in' for pressure function and we used the fact that $P^{in}(\mathbf{r}, w) = (P^{out}(\mathbf{r}, w))^*$.

4.1 Integral Representation of Forward Solution of Nonhomogeneous Wave Equation

In this study, we made use of Green's function to solve nonhomogeneous wave equation. Green's function is the unit impulse response of a medium. When a point source is located at $\mathbf{r} = \mathbf{r}'$, Green's function is the solution of the equation

$$\nabla^2 P(\mathbf{r}, w) + k^2 P(\mathbf{r}, w) = \delta(\mathbf{r} - \mathbf{r}') \tag{4.3}$$

where $\delta(.)$ is the Dirac delta function. The wave equation $P(\mathbf{r}, w)$ satisfies

$$\nabla^2 P(\mathbf{r}, w) + k^2 P(\mathbf{r}, w) = iwp_0(\mathbf{r}) \tag{4.4}$$

and Green's function G satisfies

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') + k^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \tag{4.5}$$

If we multiply the first equation by G and the second one by $P(\mathbf{r}, w)$ and subtract side by side, we get

$$G\nabla^2 P - P\nabla^2 G = iwp_0(\mathbf{r})G - P(\mathbf{r}, w)\delta(\mathbf{r} - \mathbf{r}'). \tag{4.6}$$

If we integrate of the above equation in a very large disk of radius R, which involves the source support V, we write

$$\int_{|\mathbf{r}|\leq R} G\nabla^2 P - P\nabla^2 G \, d\mathbf{r} = iw \int_V p_0(\mathbf{r})G \, d\mathbf{r} - P(\mathbf{r}', w). \tag{4.7}$$

Now, we apply Green's Theorem to the first integral:

$$\int_{|\mathbf{r}|=R} G\frac{\partial P}{\partial r} - P\frac{\partial G}{\partial r} \, dS = iw \int_V p_0(\mathbf{r})G \, d\mathbf{r} - P(\mathbf{r}', w). \tag{4.8}$$

If we add and subtract the term $ikGP$ in the left integral and take the limit as R goes to infinity, we easily see that the left integral goes to zero by radiation conditions (5.11).

Hence, the solution of forward problem is obtained as follows:

$$P(\mathbf{r}', w) = iw \int_V p_0(\mathbf{r}) G(\mathbf{r}, \mathbf{r}')\, d\mathbf{r}. \tag{4.9}$$

4.2 Result of Initial Condition

We know that the forward solution of the wave equation is given by

$$P(\mathbf{r}, w) = iw \int_{\mathbb{R}^3} p_0(\mathbf{r}')\; G(\mathbf{r}', \mathbf{r}, w)\, dv_{\mathbf{r}'}.$$

When we take the inverse Fourier transform of both sides of the above equation, we obtain

$$p(\mathbf{r}, t) = \frac{i}{2\pi} \int_{\mathbb{R}^2} p_0(\mathbf{r}') \left(\int_{-\infty}^{\infty} wG(\mathbf{r}', \mathbf{r}, w)\, e^{-i\omega t}\, dw \right) dv_{\mathbf{r}'}.$$

Now, by discontinuity at $t = 0$, we get the equality

$$\frac{1}{2} \left(p(\mathbf{r}, 0^-) + p(\mathbf{r}, 0^+) \right) = \frac{i}{2\pi} \int_{\mathbb{R}^3} p_0(\mathbf{r}') \left(\int_{-\infty}^{\infty} wG(\mathbf{r}', \mathbf{r}, w)\, dw \right) dv_{\mathbf{r}'}.$$

On the other hand, the left hand side of the above equation must be equal to source function $p_0(\mathbf{r})$ by the initial conditions of the thermoacoustic wave equation (6.4) and (6.5), therefore

$$p_0(\mathbf{r}) = \frac{i}{\pi c^2} \int_{\mathbb{R}^3} p_0(\mathbf{r}') \left(\int_{-\infty}^{\infty} wG(\mathbf{r}', \mathbf{r}, w)\, dw \right) dv_{\mathbf{r}'} \tag{4.10}$$

must be satisfied. $p_0(\mathbf{r})$ in the equation (4.10) is an arbitrary source function, hence this equation gives

$$\frac{i}{\pi c^2(\mathbf{r})} \int_{-\infty}^{\infty} wG(\mathbf{r}', \mathbf{r}, w)\, dw = \delta(\mathbf{r}' - \mathbf{r}) \tag{4.11}$$

for any $\mathbf{r}'$ and $\mathbf{r}$.

We use the result (4.11) in the proof of inverse solution.

CHAPTER 5

5. INVERSE SOLUTION IN TWO DIMENSIONAL CIRCULARLY LAYERED MEDIUM

At the beginning of our study, we took nonhomogeneous region as circularly two layered medium in two dimensional space to understand the problem in a simpler layered media. In this aspect, we characterized two layered media, stated inverse problem and derived a solution of thermoacoustic wave equation on this layered medium as represented in this chapter:

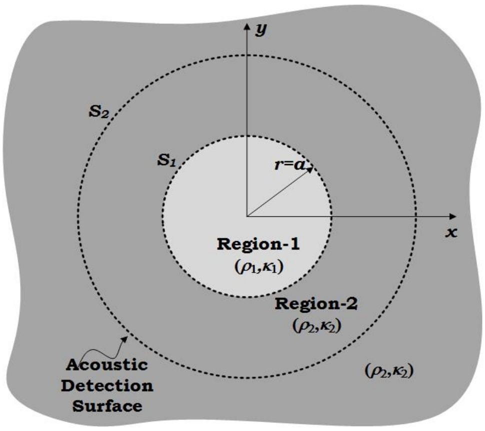

Figure 5.1 Configuration of two layered medium

Consider two regions in $\mathbb{R}^2$ with different acoustic properties as depicted in Figure 5.1. The interface of regions is the circle with center $(0, 0)$ and radius $r = a$, denoted by S_1. We call the inside and the outside of the circle S_1 as the Region 1 and Region 2, respectively. Suppose there is a circular transducer, called S_2 in Region 2 enclosing Region 1 as in the Figure 5.1. We call the area covered by S_1 as V_1 and the area between

S_1 and S_2 as V_2. We want to determine the source distribution in the region covered by the transducer.

The acoustic waves are measured by the transducer for a sufficiently long time interval so that the waves emitted from every source location reach the transducer. When the two regions are different, there will be reflections and transmissions at the boundary S_1. The thermoacoustic wave propagation is governed by the nonhomogeneous wave equation

$$\nabla^2 p(\mathbf{r}, t) - \frac{1}{c^2}\frac{\partial^2 p(\mathbf{r}, t)}{\partial t^2} = -p_0(\mathbf{r}).\delta'(t) \tag{5.1}$$

with boundary conditions

$$p_1(\mathbf{r}, t) = p_2(\mathbf{r}, t)\big|_{\mathbf{r}\in S_1}$$

and

$$\frac{1}{\rho_1}\frac{\partial p_1(\mathbf{r}, t)}{\partial n} = \frac{1}{\rho_2}\frac{\partial p_2(\mathbf{r}, t)}{\partial n}\Bigg|_{\mathbf{r}\in S_1}$$

on the boundary S_1. Here, p_1 and p_2 are the acoustic waves and ρ_1 and ρ_2 are the densities for Region 1 and Region 2, respectively. In an inverse source problem, $p_0(\mathbf{r})$ is to be reconstructed given that acoustic field is measured by the transducer and known on the surface S_2. We know that (5.1) corresponds to following nonhomogeneous Helmholtz equation in frequency domain:

$$\nabla^2 P(\mathbf{r}, w) + k^2 P(\mathbf{r}, w) = iwp_0(\mathbf{r}) \tag{5.2}$$

where $k = w/c$ is the wave number, $P(\mathbf{r}, w)$ is the temporal Fourier Transform of $p(\mathbf{r}, t)$.

5.1 Green's Function of Medium

Green's function is the solution of homogeneous wave equation except the point $\mathbf{r}'$ where the point source located:

$$\nabla^2 G(\mathbf{r}, w) + k^2 G(\mathbf{r}, w) = \delta(\mathbf{r} - \mathbf{r}') \tag{5.3}$$

where $\delta(.)$ is the Dirac delta function.

Because of the circular symmetry of the configuration of regions and transducer, we prefer to work in polar coordinates. The Helmholtz wave equation given in (3.4) is expressed in polar coordinates (r, ϕ) as,

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r}\frac{\partial G}{\partial r} + \frac{1}{r^2}\frac{\partial^2 G}{\partial \phi^2} + k^2 G = \frac{1}{r}\delta(r - r')\delta(\phi - \phi'). \tag{5.4}$$

To solve above nonhomogeneous equation in layered medium we need to solve homogeneous one in homogeneous medium firstly:

Consider the two regions are identical that is the medium is homogeneous. Now, suppose that there is a point source at (r', ϕ'). Green's function is the solution of homogeneous wave equation except the point where the source located. We apply separation of variables method to solve the following homogeneous equation

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r}\frac{\partial G}{\partial r} + \frac{1}{r^2}\frac{\partial^2 G}{\partial \phi^2} + k^2 G = 0. \tag{5.5}$$

Let $I(r, \phi)$ be a solution of (5.5) and suppose

$$I(r, \phi) = \mathfrak{R}(r)\Phi(\phi).$$

Substituting I in (5.5) yields

$$\mathfrak{R}''\Phi + \frac{1}{r}\mathfrak{R}'\phi + \frac{1}{r^2}\mathfrak{R}\Phi'' + k^2\mathfrak{R}\Phi = 0$$

which implies that

$$\frac{\mathfrak{R}''}{\mathfrak{R}}r^2 + r\frac{\mathfrak{R}'}{\mathfrak{R}} + k^2 r^2 = -\frac{\Phi'}{\Phi} \tag{5.6}$$

where primes refer to ordinary differentiation with respect to the independent variable, r or ϕ. Since each side of (5.6) depends on different variable, they must be equal to same constant, say λ:

$$\frac{\mathfrak{R}''}{\mathfrak{R}}r^2 + r\frac{\mathfrak{R}'}{\mathfrak{R}} + k^2r^2 = -\frac{\Phi'}{\Phi} = \lambda. \tag{5.7}$$

Hence, we obtain two ordinary differential equation from (5.7):

$$\Phi'' + \lambda\Phi = 0 \tag{5.8}$$

and

$$r^2R'' + rR' + (k^2r^2 - \lambda)R = 0. \tag{5.9}$$

By periodicity of the function Φ, the fundamental solutions of first equation (5.8) are $e^{in\phi}$ and $e^{-in\phi}$, where n is an integer and $\lambda = n^2$.

Second equation (5.9) is known as Bessel differential equation. For the sake of completeness, we rederived the solution of Bessel equation and state the derivation in Appendix. Two independent solutions of Bessel equation are the first kind of Bessel function $J_n(kr)$ and the second kind of Bessel function $Y_n(kr)$. Alternatively, linear combination of these two functions, Hankel functions of first kind $H_n^1(kr) = J_n(kr) + iY_n(kr)$ and of second kind $H_n^2(kr) = J_n(kr) - iY_n(kr)$ can be used as fundamental solutions. As a result, the homogeneous solution of (5.5) has the form

$$I(r,\phi) = \sum_{n=-\infty}^{\infty} \left(A_nJ_n(k_1r) + B_nY_n(k_1r)\right)e^{in\phi}.$$

Now, we can define the Green's function for homogeneous medium as follows:

$$G(r,\phi;r',\phi') = \begin{cases} \sum\limits_{n=-\infty}^{\infty} \left(A_n J_n(kr) + B_n Y_n(kr)\right) e^{in\phi}, & \text{if } r < r' \\ \sum\limits_{n=-\infty}^{\infty} \left(C_n H_n^1(kr) + D_n H_n^2(kr)\right) e^{in\phi}, & \text{if } r' < r. \end{cases} \tag{5.10}$$

Since Green's functions must be defined for all r in the domain, and Y_n has singularity at $r = 0$, B_n must be identically zero. G must also satisfy Sommerfeld's radiation condition, that is

$$\lim_{r\to\infty}\left[r^{\frac{d-1}{2}}(\frac{\partial}{\partial r} - ik)G(r)\right] = 0. \tag{5.11}$$

Since $H_n^1(kr)$ diverges when r goes to infinity, C_n must be zero. So,

$$G(r,\phi;r',\phi') = \begin{cases} \sum\limits_{n=-\infty}^{\infty} A_n J_n(kr) e^{in\phi}, & \text{if } r < r' \\ \sum\limits_{n=-\infty}^{\infty} D_n H_n^2(kr) e^{in\phi}, & \text{if } r' < r. \end{cases} \tag{5.12}$$

By continuity of Green's function at $r = r'$,

$$\sum_{n=-\infty}^{\infty} A_n J_n(kr') e^{in\phi} = \sum_{n=-\infty}^{\infty} D_n H_n^2(kr') e^{in\phi}. \tag{5.13}$$

Here, $\left\{e^{in\phi}\right\}$ are orthogonal functions which satisfy

$$\int_0^{2\pi} e^{in\phi} e^{im\phi} d\phi = \begin{cases} 0, & \text{if } m = n \\ 2\pi, & \text{if } m \neq n \end{cases} \tag{5.14}$$

on the interval $[0, 2\pi]$. If we take inner product of both sides of above equation (5.13) by $e^{-im\phi}$, we get

$$\int\limits_0^{2\pi} \left(\sum_{n=-\infty}^{\infty} A_n J_n(kr') e^{in\phi}\right) e^{im\phi} d\phi = \int\limits_0^{2\pi} \left(\sum_{n=-\infty}^{\infty} D_n H_n^2(kr') e^{in\phi}\right) e^{im\phi} d\phi$$

which gives

$$\sum_{n=-\infty}^{\infty} \int_0^{2\pi} A_n J_n(kr')e^{in\phi}e^{im\phi}d\phi = \sum_{n=-\infty}^{\infty} \int_0^{2\pi} D_n H_n^2(kr')e^{in\phi}e^{im\phi}d\phi$$

implying

$$2\pi A_m J_m(kr) = 2\pi D_m H_m^2(kr) \tag{5.15}$$

by orthogonality of exponential functions. Since m is arbitrary in (5.15), we conclude that

$$A_n J_n(kr)e^{in\phi} = D_n H_n^2(kr)e^{in\phi} \quad \text{for all} \quad n. \tag{5.16}$$

Now, let us call

$$a_n = \begin{cases} A_n J_n(kr), & \text{if } r < r' \\ \\ D_n H_n^2(kr), & \text{if } r > r' \end{cases} \tag{5.17}$$

so we can write G as in the series form

$$G = \sum_{n=-\infty}^{\infty} a_n e^{in\phi}.$$

Substituting the G in the polar Helmholtz equation (5.4) results in

$$\begin{aligned} \frac{\partial^2}{\partial r^2} \sum_{n=-\infty}^{\infty} a_n(r)e^{in\phi} + \frac{1}{r}\frac{\partial}{\partial r} \sum_{n=-\infty}^{\infty} a_n(r)e^{in\phi} & \\ + \frac{1}{r^2} \sum_{n=-\infty}^{\infty} -n^2 a_n(r)e^{in\phi} + k^2 \sum_{n=-\infty}^{\infty} a_n(r)e^{in\phi} & \\ = \frac{1}{r}\delta(r-r')\delta(\phi-\phi'). & \end{aligned} \tag{5.18}$$

Again taking inner product by $e^{-im\phi}$, the above partial differential equation turns into ordinary differential equation:

$$\frac{d^2a_n}{dr^2} + \frac{1}{r}\frac{da_n}{dr} - (k^2 - \frac{n^2}{r^2})a_n = \frac{1}{r}\delta(r-r')\frac{e^{-im\phi'}}{2\pi}. \tag{5.19}$$

Now, by multiplying the equation (5.19) by r and integrating both sides with respect to r from $r' - \varepsilon$ to $r' + \varepsilon$, we obtain

$$\int_{r'-\varepsilon}^{r'+\varepsilon} \left(r\frac{d^2a_n(r)}{(dr)^2} + \frac{da_n(r)}{dr} + \left(k^2 - \frac{n^2}{r^2}\right) a_n(r) \right) dr = \int_{r'-\varepsilon}^{r'+\varepsilon} \left(\delta(r-r')\frac{e^{-im\phi'}}{2\pi} \right) dr$$

which implies that

$$\int_{r'-\varepsilon}^{r'+\varepsilon} r\frac{d^2a_n(r)}{(dr)^2}\,dr + \int_{r'-\varepsilon}^{r'+\varepsilon} \frac{da_n(r)}{dr}\,dr + \int_{r'-\varepsilon}^{r'+\varepsilon} \left(k^2 - \frac{n^2}{r^2}\right) a_n(r)\,dr = \frac{e^{-in\phi}}{2\pi}. \tag{5.20}$$

We apply integration by parts to the first integral in (5.20) and obtain

$$r\frac{da_n(r)}{dr}\bigg|_{r'-\varepsilon}^{r'+\varepsilon} - \int_{r'-\varepsilon}^{r'+\varepsilon} \frac{da_n(r)}{dr}\,dr + \int_{r'-\varepsilon}^{r'+\varepsilon} \frac{da_n(r)}{dr}\,dr + \int_{r'-\varepsilon}^{r'+\varepsilon} \left(k^2 - \frac{n^2}{r^2}\right) a_n(r)\,dr = \frac{e^{-in\phi}}{2\pi}$$

which gives

$$(r'+\varepsilon)\ \frac{da_n(r)}{dr}\bigg|_{r'+\varepsilon} - (r'-\varepsilon)\ \frac{da_n(r)}{dr}\bigg|_{r'-\varepsilon} + \int_{r'-\varepsilon}^{r'+\varepsilon} \left(k^2 - \frac{n^2}{r^2}\right) a_n(r)\,dr = \frac{e^{-in\phi}}{2\pi}. \tag{5.21}$$

Now, when we take the limit of both sides of equation (5.21) as ε goes to zero, the integral on the left goes to zero due to the continuity of a_n at r'. Hence, it yields that

$$r' \lim_{r\to r'^+} \frac{da_n(r)}{dr} - r' \lim_{r\to r'^-} \frac{da_n(r)}{dr} = \frac{e^{-in\phi}}{2\pi}. \tag{5.22}$$

This condition is called jump discontinuity condition of Green's functions. By definition of a_n (5.17), jump discontinuity gives

$$\frac{d\left(D_n H_n^2(kr)\right)}{dr}\bigg|_{r=r'} - \frac{d\left(A_n J_n(kr)\right)}{dr}\bigg|_{r=r'} = \frac{e^{-in\phi}}{2\pi r'}$$

which implies

$$D_n H_n^{2'}(kr') - A_n J_n'(kr') = \frac{e^{-in\phi}}{2\pi kr'} \quad \text{for all} \quad n. \tag{5.23}$$

Now, if we multiply the equation (5.16) by $H_n^{2'}(kr')$ and the equation (5.23) by $H_n^2(kr')$ and add the equations side by side, we obtain

$$A_n \left(J_n(kr') H_n^2(kr') - J_n'(kr') H_n^{2'}(kr') \right) = \frac{e^{-in\phi}}{2\pi kr'} H_n^2(kr'). \tag{5.24}$$

The Wronskian of J_n and H_n^2 can be computed as follows [1]

$$W(J_n(r), H_n^2(r)) = -\frac{2i}{\pi r}. \tag{5.25}$$

Hence, we obtain the coefficient A_n from (5.24) as

$$A_n = \frac{1}{4i} H_n^2(kr') e^{-in\phi}. \tag{5.26}$$

Subsituting A_n in (5.16), we get the coefficient D_n as

$$D_n = -\frac{1}{4i} J_n(kr') e^{-in\phi}. \tag{5.27}$$

Hence, the Green's function representing the unit impulse response of the homogeneous medium is

$$G(r, \phi; r', \phi') = \begin{cases} \frac{1}{4i} \sum\limits_{n=-\infty}^{\infty} H_n^2(kr') J_n(kr) e^{in(\phi - \phi')}, & \text{if } r < r' \\ -\frac{1}{4i} \sum\limits_{n=-\infty}^{\infty} J_n(kr') H_n^2(kr) e^{in(\phi - \phi')}, & \text{if } r > r'. \end{cases} \tag{5.28}$$

5.1.1 Green's Function of Two Layered Medium

Suppose that two regions have different densities, respectively ρ_1, ρ_2 and there is a point source at (r', ϕ') inside Region 1. There will be reflections and transmissions on the interface circle. Hence, Green's function must satisfy the boundary conditions. We can define Green's function for layered medium as follows:

$$G(r,\phi;r',\phi') = \begin{cases} \sum\limits_{n=-\infty}^{\infty} \left(A_n J_n(k_1 r) + B_n Y_n(k_1 r)\right) e^{in\phi}, & \text{if} \quad r < r' \\ \sum\limits_{n=-\infty}^{\infty} \left(C_n J_n(k_1 r) + D_n Y_n(k_1 r)\right) e^{in\phi}, & \text{if} \quad r' < r < a \\ \sum\limits_{n=-\infty}^{\infty} \left(E_n H_n^1(k_2 r) + F_n H_n^2(k_2 r)\right) e^{in\phi}, & \text{if} \quad r > a. \end{cases} \tag{5.29}$$

Since Green's functions must be defined for all r in the domain, and Y_n has singularity at $r = 0$, B_n must be identically zero.

G must also satisfy Sommerfeld's radiation condition (5.11). Hence, $E_n = 0$ due to the divergence of H_n^1 at infinity. Thus,

$$G(r,\phi;r',\phi') = \begin{cases} \sum\limits_{n=-\infty}^{\infty} A_n J_n(k_1 r) e^{in\phi}, & \text{if } r < r' \\ \sum\limits_{n=-\infty}^{\infty} \left(C_n J_n(k_1 r) + D_n Y_n(k_1 r)\right) e^{in\phi}, & \text{if} \quad r' < r < a \\ \sum\limits_{n=-\infty}^{\infty} F_n H_n^2(k_2 r) e^{in\phi}, & \text{if} \quad r > a. \end{cases} \tag{5.30}$$

The Green's function G must be continuous at $r = r'$, this gives

$$\sum_{n=-\infty}^{\infty} A_n J_n(k_1 r') = \sum_{n=-\infty}^{\infty} C_n J_n(k_1 r') + D_n Y_n(k_1 r'). \tag{5.31}$$

If we take the inner product of (5.31), we obtain the following equation by similar calculations in homogeneous case:

$$A_n J_n(k_1 r') = C_n J_n(k_1 r') + D_n Y_n(k_1 r'). \tag{5.32}$$

Secondly, the jump discontinuity condition (5.22) at $r = r'$ yields that

$$\left.\frac{d\left(C_n J_n(k_1 r') + D_n Y_n(k_1 r')\right)}{dr}\right|_{r=r'} - \left.\frac{d\left(A_n J_n(k_1 r)\right)}{dr}\right|_{r=r'} = \frac{e^{-in\phi}}{2\pi r'} \quad \text{for all} \quad n$$

which implies

$$C_n J_n'(k_1 r') + D_n Y_n'(k_1 r') - A_n J_n(k_1 r') = \frac{e^{-in\phi}}{2\pi k r'} \quad \text{for all} \quad n. \tag{5.33}$$

Now, we apply the boundary conditions to Green's function G. The continuity of acoustic pressure at the boundary $r = a$, that is

$$\lim_{r\to a^-} G = \lim_{r\to a^+} G,$$

provides

$$\sum_{n=-\infty}^{\infty} C_n J_n(k_1 a) + D_n Y_n(k_1 a) = \sum_{n=-\infty}^{\infty} F_n H_n^2(k_2 a).$$

If we take inner product of (5.1.1) by $e^{im\varphi}$, it results in

$$C_n J_n(k_1 a) + D_n Y_n(k_1 a) = F_n H_n^2(k_2 a) \quad \text{for all} \quad n \tag{5.34}$$

by orthogonality of exponential functions, again. The normal derivative of the function G at the boundary circle D corresponds to the derivative of G with respect to r since the circle D is centered at the origin. Hence, the second boundary condition

$$\frac{1}{\rho_1} \lim_{r\to a^-} \frac{\partial G}{\partial n} = \frac{1}{\rho_2} \lim_{r\to a^+} \frac{\partial G}{\partial n}$$

gives

$$\frac{1}{\rho_1} \sum_{n=-\infty}^{\infty} (C_n k_1 J_n'(k_1 a) + D_n k_1 Y_n'(k_1 a)) = \frac{1}{\rho_2} \sum_{n=-\infty}^{\infty} F_n k_2 H_n^{2'}(k_2 a)$$

which in turn leads to

$$\frac{k_1}{\rho_1}(C_n J_n'(k_1 a) + D_n Y_n'(k_1 a)) = \frac{k_2}{\rho_2} F_n H_n^{2'}(k_2 a) \quad \text{for all} \quad n \tag{5.35}$$

by applying inner product operation, again. Now, we have four equations (5.32), (5.33), (5.34) and (5.35) to obtain the coefficients of the Green's function of nonhomogeneous medium. If we multiply (5.32) by $J_n'(k_1 r')$ and (5.33) by $J_n(k_1 r')$ and adding side by side, we get

$$D_n(J_n(k_1r')Y_n'(k_1r') - J_n'(k_1r')Y_n'(k_1r')) = \frac{e^{-in\varphi'}}{2\pi k_1 r'}J_n(k_1r'). \tag{5.36}$$

The Wronskian of J_n and Y_n can be derived as

$$W(J_n(r), Y_n(r)) = J_n(r)Y_n'(r) - J_n'(r)Y_n(r) = \frac{2}{\pi r} \tag{5.37}$$

which yields

$$D_n = \frac{e^{-in\varphi'}}{4}J_n(k_1r'). \tag{5.38}$$

If we multiply (5.32) by $Y_n'(k_1r')$ and (5.33) by $Y_n(k_1r')$ and add side by side and using the Wronskian (5.37), we get

$$A_n - C_n = \frac{e^{-in\varphi'}}{4}Y_n(k_1r'). \tag{5.39}$$

Now, we multiply the equation (5.33) by $(1/\rho_1)J_n'(k_1a)$ and the equation (5.34) by $J_n'(k_1a)$ and subtract side by side. Then subsituting D_n in the derived equation and using the Wronskian (5.37), we obtain

$$F_n = \frac{2}{\pi\rho_1 a}\left(\frac{J_n(k_1r')}{\left(\frac{k_2}{\rho_2}J_n(k_1a)H_n^{2\prime}(k_2a) - \frac{k_1}{\rho_1}J_n'(k_1a)H_n^2(k_2a)\right)}\right)\frac{e^{-in\phi}}{4}.$$

When we substitute D_n in (5.33) and (5.34) and multiply the equations by $(k_2/\rho_2)H_n^{2\prime}(k_2a)$ and $H_n^2(k_2a)$, respectively, and subtract side by side, we obtain C_n as

$$C_n = \left(\frac{\left(\frac{k_2}{\rho_2}Y_n(k1_a)H_n^{2\prime}(k_2a) - \frac{k_1}{\rho_1}Y_n'(k1_a)H_n^2(k_2a)\right)J_nk_1r'}{\left(\frac{k_2}{\rho_2}J_n(k1_a)H_n^{2\prime}(k_2a) - \frac{k_1}{\rho_1}J_n'(k1_a)H_n^2(k_2a)\right)}\right)\frac{e^{-in\phi}}{4}.$$

Finally, when we substitute C_n in (5.39), we obtain

$$A_n = \left(\frac{\left(\frac{k_2}{\rho_2}Y_n(k1_a)H_n^{2\prime}(k_2a) - \frac{k_1}{\rho_1}Y_n'(k1_a)H_n^2(k_2a)\right)J_nk_1r'}{\left(\frac{k_2}{\rho_2}J_n(k1_a)H_n^{2\prime}(k_2a) - \frac{k_1}{\rho_1}J_n'(k1_a)H_n^2(k_2a)\right)} + Y_n(kr')\right)\frac{e^{-in\phi}}{4}.$$

Thus, the Green's function representing the unit impulse response of nonhomogeneous medium is

$$G(r,\phi;r',\phi') = \begin{cases} \sum\limits_{n=-\infty}^{\infty} A_n J_n(k_1 r) e^{in\phi}, & \text{if } r < r' \\ \sum\limits_{n=-\infty}^{\infty} \left(C_n J_n(k_1 r) + D_n Y_n(k_1 r)\right) e^{in\phi}, & \text{if } \quad r' < r < a \\ \sum\limits_{n=-\infty}^{\infty} F_n H_n^2(k_2 r) e^{in\phi}, & \text{if } \quad r > a \end{cases} \tag{5.40}$$

in which

$$\begin{aligned} A_n &= \left(\frac{\left(\frac{k_2}{\rho_2} Y_n(k_1 a) H_n^{2\prime}(k_2 a) - \frac{k_1}{\rho_1} Y_n'(k_1 a) H_n^2(k_2 a)\right) J_n k_1 r'}{\left(\frac{k_2}{\rho_2} J_n(k_1 a) H_n^{2\prime}(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) H_n^2(k_2 a)\right)} + Y_n(kr') \right) \frac{e^{-in\phi}}{4}, \\ C_n &= \left(\frac{\left(\frac{k_2}{\rho_2} Y_n(k1_a) H_n^{2\prime}(k_2 a) - \frac{k_1}{\rho_1} Y_n'(k1_a) H_n^2(k_2 a)\right) J_n k_1 r'}{\left(\frac{k_2}{\rho_2} J_n(k1_a) H_n^{2\prime}(k_2 a) - \frac{k_1}{\rho_1} J_n'(k1_a) H_n^2(k_2 a)\right)} \right) \frac{e^{-in\phi}}{4}, \\ D_n &= \frac{J_n(k_1 r')}{4} e^{-in\phi}, \\ F_n &= \frac{2}{\pi \rho_1 a} \left(\frac{J_n(k_1 r')}{\left(\frac{k_2}{\rho_2} J_n(k_1 a) H_n^{2\prime}(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) H_n^2(k_2 a)\right)} \right) \frac{e^{-in\phi}}{4}. \end{aligned}$$

The Green's function denoted by G_1, representing the unit impulse response of the layered medium when the source (r',ϕ') located in Region 1, is given by

$$G_1(r',\phi';r,\phi) = \begin{cases} \sum\limits_{n=-\infty}^{\infty} A_n J_n(k_1 r) e^{in\phi}, & \text{if } r < r' \\ \sum\limits_{n=-\infty}^{\infty} \left(C_n J_n(k_1 r) + D_n Y_n(k_1 r)\right) e^{in\phi}, & \text{if } \quad r' < r < a \\ \sum\limits_{n=-\infty}^{\infty} F_n H_n^2(k_2 r) e^{in\phi}, & \text{if } \quad r > a \end{cases} \tag{5.41}$$

in which

$$A_n = \left(\frac{\left(\frac{k_2}{\rho_2} Y_n(k_1 a) H_n^{2\prime}(k_2 a) - \frac{k_1}{\rho_1} Y_n'(k_1 a) H_n^2(k_2 a)\right) J_n(k_1 r')}{\left(\frac{k_2}{\rho_2} J_n(k_1 a) H_n^{2\prime}(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) H_n^2(k_2 a)\right)} + Y_n(k_1 r') \right) \frac{e^{-in\phi'}}{4},$$

$$C_n = \left(\frac{\left(\frac{k_2}{\rho_2} Y_n(k_1 a) H_n^{2\prime}(k_2 a) - \frac{k_1}{\rho_1} Y_n'(k_1 a) H_n^2(k_2 a)\right) J_n(k_1 r')}{\left(\frac{k_2}{\rho_2} J_n(k_1 a) H_n^{2\prime}(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) H_n^2(k_2 a)\right)} \right) \frac{e^{-in\phi'}}{4},$$

$$D_n = J_n(k_1 r') \frac{e^{-in\phi'}}{4},$$

$$F_n = \frac{2}{\pi \rho_1 a} \left(\frac{J_n(k_1 r')}{\left(\frac{k_2}{\rho_2} J_n(k_1 a) H_n^{2\prime}(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) H_n^2(k_2 a)\right)} \right) \frac{e^{-in\phi'}}{4}.$$

The Green's function denoted by G_2 representing the unit impulse response of the layered medium when the source (r', ϕ') located in Region 2, is given by

$$G_2(r', \phi'; r, \phi) = \begin{cases} \sum\limits_{n=-\infty}^{\infty} A_n J_n(k_1 r) e^{in\phi}, & \text{if } r < a \\ \sum\limits_{n=-\infty}^{\infty} \left(C_n J_n(k_1 r) + D_n Y_n(k_1 r)\right) e^{in\phi}, & \text{if } \quad a < r < r' \\ \sum\limits_{n=-\infty}^{\infty} F_n H_n^2(k_2 r) e^{in\phi}, & \text{if } \quad r > r' \end{cases} \tag{5.42}$$

in which

$$A_n = \frac{2}{\pi \rho_2 a} \left(\frac{H_n^2(k_2 r')}{\left(\frac{k_2}{\rho_2} J_n(k_1 a) H_n^{2\prime}(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) H_n^2(k_2 a)\right)} \right) \frac{e^{-in\phi'}}{4},$$

$$C_n = \left(-\frac{\left(\frac{k_2}{\rho_2} J_n(k_1 a) {Y_n}'(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) Y_n(k_2 a)\right) H_n^2(k_2 r')}{\left(\frac{k_2}{\rho_2} J_n(k_1 a) H_n^{2\prime}(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) H_n^2(k_2 a)\right)} \right) \frac{e^{-in\phi'}}{4},$$

$$D_n = \left(\frac{\left(\frac{k_2}{\rho_2} J_n(k_1 a) {J_n}'(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) J_n(k_2 a)\right) H_n^2(k_2 r')}{\left(\frac{k_2}{\rho_2} J_n(k_1 a) H_n^{2\prime}(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) H_n^2(k_2 a)\right)} \right) \frac{e^{-in\phi'}}{4},$$

$$F_n = \left(-\frac{\left(\frac{k_2}{\rho_2} J_n(k_1 a) {Y_n}'(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) Y_n(k_2 a)\right) J_n(k_2 r')}{\left(\frac{k_2}{\rho_2} J_n(k_1 a) H_n^{2\prime}(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) H_n^2(k_2 a)\right)} \right.$$

$$\left. + \frac{\left(\frac{k_2}{\rho_2} J_n(k_1 a) {J_n}'(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) J_n(k_2 a)\right) Y_n(k_2 r')}{\left(\frac{k_2}{\rho_2} J_n(k_1 a) H_n^{2\prime}(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) H_n^2(k_2 a)\right)} \right) \frac{e^{-in\phi'}}{4}.$$

In each case, we obtain the coefficient A_n, C_n, D_n and F_n using the boundary conditions on S_1 through the following equations, respectively:

$$\begin{aligned}
&C_n J_n(k_1 r') + D_n Y_n(k_1 r') - A_n J_n(k_1 r') = 0 \\
&C_n J_n'(k_1 r') + D_n Y_n'(k_1 r') - A_n J_n(k_1 r') = \frac{e^{-in\phi}}{2\pi k r'} \\
&F_n H_n^2(k_2 a) - C_n J_n(k_1 a) + D_n Y_n(k_1 a) = 0 \\
&\frac{k_1}{\rho_1}(C_n J_n'(k_1 a) + D_n Y_n'(k_1 a)) = \frac{k_2}{\rho_2} F_n H_n^{2\prime}(k_2 a) \quad \text{for all} \quad n
\end{aligned}$$

$$
\begin{aligned}
&F_n H_n^2(k_2 r') - C_n J_n(k_2 r') - D_n Y_n(k_2 r') = 0\\
&F_n H_n^{2'}(k_2 r') - (C_n J_n'(k_2 r') + D_n Y_n'(k_2 r')) = \frac{e^{-in\phi}}{2\pi k_2 r'}\\
&C_n J_n(k_2 a) + D_n Y_n(k_2 a) - A_n J_n(k_1 a) = 0\\
&\frac{k_1}{\rho_1} A_n J_n'(k_1 a) = \frac{k_2}{\rho_2}(C_n J_n'(k_2 a) + D_n Y_n'(k_2 a)) \quad \text{for all} \quad n.
\end{aligned}
$$

These equations can be expressed in the following matrix forms:

$$
\begin{bmatrix}
-J_n(k_1 r') & J_n(k_1 r') & Y_n(k_1 r') & 0\\
-J_n'(k_1 r') & J_n'(k_1 r') & Y_n'(k_1 r') & 0\\
0 & J_n(k_1 a) & Y_n(k_1 a) & -H_n^2(k_2 a)\\
0 & \frac{k_1}{\rho_1} J_n'(k_1 a) & \frac{k_1}{\rho_1} Y_n'(k_1 a) & \frac{k_2}{\rho_2} F_n H_n^{2'}(k_2 a)
\end{bmatrix}
\begin{bmatrix} A_n\\ C_n\\ D_n\\ F_n \end{bmatrix}
=
\begin{bmatrix} 0\\ \frac{e^{-in\phi}}{2\pi k_1 a}\\ 0\\ 0 \end{bmatrix}
$$

and

$$
\begin{bmatrix}
-J_n(k_1 a) & J_n(k_1 a) & Y_n(k_1 a) & 0\\
-J_n'(k_1 a) & J_n'(k_1 a) & Y_n'(k_1 a) & 0\\
0 & J_n(k_1 r') & Y_n(k_1 r') & -H_n^2(k_2 r')\\
0 & \frac{k_1}{\rho_1} J_n'(k_1 r') & \frac{k_1}{\rho_1} Y_n'(k_1 r') & \frac{k_2}{\rho_2} F_n H_n^{2'}(k_2 r')
\end{bmatrix}
\begin{bmatrix} A_n\\ C_n\\ D_n\\ F_n \end{bmatrix}
=
\begin{bmatrix} 0\\ 0\\ 0\\ \frac{e^{-in\phi}}{2\pi k_1 a} \end{bmatrix}.
$$

The determinants of these coefficent matrices are identical, depends on n and equal to

$$
\beta_n = \left(\frac{k_2}{\rho_2} J_n(k_1 a) H_n^{2'}(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) H_n^2(k_2 a) \right). \tag{5.43}
$$

After that, we call the parts of G_1 as G_{11} and G_{12} with respect to location of observation point:

$$G_{11} = \begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} \Bigg(\frac{\frac{k_2}{\rho_2} Y_n(k_1 a) H_n^{2\prime}(k_2 a) - \frac{k_1}{\rho_1} Y_n'(k_1 a) H_n^2(k_2 a)}{\beta_n} J_n(k_1 r') J_n(k_1 r) \\ \qquad + Y_n(k_1 r') J_n(k_1 r) \Bigg) e^{in(\phi-\phi')}, \quad \text{when} \quad r < r' \\ \\ \displaystyle\sum_{n=-\infty}^{\infty} \Bigg(\frac{\frac{k_2}{\rho_2} Y_n(k_1 a) H_n^{2\prime}(k_2 a) - \frac{k_1}{\rho_1} Y_n'(k_1 a) H_n^2(k_2 a)}{\beta_n} J_n(k_1 r') J_n(k_1 r) \\ \qquad + J_n(k_1 r') Y_n(k_1 r) \Bigg) e^{in(\phi-\phi')}, \quad \text{when} \quad r' < r \end{cases}$$

$$G_{12} = \sum_{n=-\infty}^{\infty} \frac{2}{\pi \rho_1 a} \frac{1}{\beta_n} J_n(k_1 r') H_n^2(k_2 r) e^{in(\phi-\phi')}.$$

Similarly, we call the parts of G_2 as G_{12} and G_{22} with respect to location of observation point:

$$G_{21} = \sum_{n=-\infty}^{\infty} \frac{2}{\pi \rho_2 a} \frac{1}{\beta_n} H_n^2(k_2 r') J_n(k_1 r) e^{in(\phi-\phi')}$$

$$G_{22} = \begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} \Bigg(-\frac{\frac{k_2}{\rho_2} J_n(k_1 a) Y_n{}'(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) Y_n(k_2 a)}{\beta_n} H_n^2(k_2 r') J_n(k_2 r) \\ \qquad + \frac{\frac{k_2}{\rho_2} J_n(k_1 a) J_n{}'(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) J_n(k_2 a)}{\beta_n} H_n^2(k_2 r') Y_n(k_2 r) \Bigg) e^{in(\phi-\phi')}, \\ \text{when} \quad r < r' \\ \\ \displaystyle\sum_{n=-\infty}^{\infty} \Bigg(-\frac{\frac{k_2}{\rho_2} J_n(k_1 a) Y_n{}'(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) Y_n(k_2 a)}{\beta_n} J_n(k_2 r') H_n^2(k_2 r) \\ \qquad + \frac{\frac{k_2}{\rho_2} J_n(k_1 a) J_n{}'(k_2 a) - \frac{k_1}{\rho_1} J_n'(k_1 a) J_n(k_2 a)}{\beta_n} Y_n(k_2 r') H_n^2(k_2 r) \Bigg) e^{in(\phi-\phi')}, \\ \text{when} \quad r' < r. \end{cases}$$

We notice that each $G_1(\mathbf{r}',\mathbf{r})$ and $G_2(\mathbf{r}',\mathbf{r})$ can be represented in the form

$$\sum_{n=-\infty}^{\infty} g_n(r',r)e^{in(\phi-\phi')}.$$

Sometimes, for simplicity in notation, we write

$$G_1(\mathbf{r}',\mathbf{r}) = \sum_{n=-\infty}^{\infty} G_1(r',r)e^{in(\phi-\phi')}$$

and

$$G_2(\mathbf{r}',\mathbf{r}) = \sum_{n=-\infty}^{\infty} G_2(r',r)e^{in(\phi-\phi')}$$

keeping n dependence of $G_1(r',r)$ and $G_2(r',r)$ in mind.

5.2 Inverse Solution

Now, we examine our claim

$$p_0(\mathbf{r}) = \begin{cases} \dfrac{\rho(\mathbf{r})}{\pi c^2(\mathbf{r})} \displaystyle\int_{-\infty}^{\infty} iw \int_{S_2} P(\mathbf{r_s},w)\frac{1}{\rho(\mathbf{r_s})}\frac{\partial G_{12}^{\text{in}}(\mathbf{r},\mathbf{r_s})}{\partial \mathbf{n_s}}\, dS\, dw, & \mathbf{r}\in R_1 \\ \\ \dfrac{\rho(\mathbf{r})}{\pi c^2(\mathbf{r})} \displaystyle\int_{-\infty}^{\infty} iw \int_{S_2} P(\mathbf{r_s},w)\frac{1}{\rho(\mathbf{r}_s)}\frac{\partial G_{22}^{\text{in}}(\mathbf{r},\mathbf{r_s})}{\partial \mathbf{n_s}}\, dS\, dw, & \mathbf{r}\in R_2. \end{cases} \tag{5.44}$$

Let us write our claim as independent of index set and call the integral expression as $q(r)$:

$$q(\mathbf{r}) = \int_{-\infty}^{\infty} iw \int_{S_2} P(\mathbf{r_s},w)\frac{1}{\rho(\mathbf{r_s})}\frac{\partial G^{\text{in}}(\mathbf{r},\mathbf{r_s})}{\partial \mathbf{n_s}}\, dS\, dw. \tag{5.45}$$

The acoustic pressure measured on the surface S_2 is given by forward solution of the wave equation (4.9):

$$P(\mathbf{r_s},w) = iw\int_{V'} p_0(\mathbf{r}')G^{\text{out}}(\mathbf{r}',\mathbf{r_s})dV_{\mathbf{r}'}$$

where $V' = V_1 \cup V_2$. We substitute the forward solution in $q(r)$:

$$\begin{aligned}
q(\mathbf{r}) &= \int_{-\infty}^{\infty} iw \int_{S_2} P(\mathbf{r_s}, w) \frac{1}{\rho(\mathbf{r_s})} \frac{\partial G^{\text{in}}(\mathbf{r}, \mathbf{r_s})}{\partial \mathbf{n}_s} \, dS \, dw \\
&= \int_{-\infty}^{\infty} iw \int_{S_2} P(\mathbf{r_s}, w) \frac{1}{\rho(\mathbf{r_s})} \nabla G^{\text{in}}(\mathbf{r}, \mathbf{r_s}).\mathbf{n}_s \, dS \, dw \\
&= \int_{-\infty}^{\infty} iw \int_{S_2} \left(\int_{V'} p_0(\mathbf{r}') G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) dV_{\mathbf{r}'} \right) \nabla \frac{1}{\rho(\mathbf{r_s})} G^{\text{in}}(\mathbf{r}, \mathbf{r_s}).\mathbf{n}_s \, dS \, dw \\
&= \int_{V'} p_0(\mathbf{r}') \left(\int_{-\infty}^{\infty} iw \int_{S_2} \left(G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) \frac{1}{\rho(\mathbf{r_s})} \nabla G^{\text{in}}(\mathbf{r}, \mathbf{r_s}) \right) .\mathbf{n}_s \, dS \, dw \right) dV_{\mathbf{r}'}.
\end{aligned}$$

Let us call the term in outer integrals as follows:

$$P(\mathbf{r}, \mathbf{r}') = \int_{-\infty}^{\infty} iw \int_{S_2} G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) \frac{1}{\rho(\mathbf{r_s})} \nabla G^{\text{in}}(\mathbf{r}, \mathbf{r_s}).\mathbf{n}_s \, dS \, dw. \tag{5.46}$$

We know that Green's function is continuous on whole space. Also, the normal derivative of Green's function with a scaling factor density function is also continuous. Hence, the expression in the above integral is continuous which makes possible to apply the divergence theorem as follows:

$$\begin{aligned}
P(\mathbf{r}, \mathbf{r}') &= \int_{-\infty}^{\infty} iw \int_{S_2} G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) \frac{1}{\rho(\mathbf{r_s})} \nabla_s G^{\text{in}}(\mathbf{r}, \mathbf{r_s}).\mathbf{n}_s \, dS \, dw \\
&= \int_{-\infty}^{\infty} iw \int_{V_1 \cup V_2} \frac{1}{\rho(r_s)} \nabla_s \left(G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) \nabla_s G^{\text{in}}(\mathbf{r}, \mathbf{r_s}) \right) dV_s \, dw \\
&= \int_{-\infty}^{\infty} iw \int_{V_1 \cup V_2} \frac{1}{\rho(r_s)} \Big(\nabla_s G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) \nabla_s G^{\text{in}}(\mathbf{r}, \mathbf{r_s}) \\
&\qquad + G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) \nabla_s^2 G^{\text{in}}(\mathbf{r}, \mathbf{r_s}) \Big) dV_s \, dw.
\end{aligned} \tag{5.47}$$

The solutions G^{in} and G^{out} satisfy the Helmholtz equation:

$$\nabla_s^2 G^{in}(\mathbf{r}, \mathbf{r_s}, w) + k_s^2 G^{in}(\mathbf{r}, \mathbf{r_s}, w) = \delta(\mathbf{r} - \mathbf{r_s}), \tag{5.48}$$

$$\nabla_s^2 G^{out}(\mathbf{r}', \mathbf{r_s}, w) + k_s^2 G^{out}(\mathbf{r}', \mathbf{r_s}, w) = \delta(\mathbf{r}' - \mathbf{r_s}) \tag{5.49}$$

in which $k_s = w/c_s$ and c_s is the acoustic speed in the region where r_s in. If we multiply (5.48) by $G^{out}(\mathbf{r}', \mathbf{r_s}, w)$ and (5.49) by $G^{in}(\mathbf{r}, \mathbf{r_s}, w)$ and subtract each other, we get

$$G^{out}\nabla_s^2 G^{in} = G^{in}\nabla_s^2 G^{out} + \delta(\mathbf{r}-\mathbf{r_s})G^{out} - \delta(\mathbf{r}'-\mathbf{r_s})G^{in}. \tag{5.50}$$

By adding the term $G^{out}\nabla_s^2 G^{in} + 2\nabla_s G^{out}\nabla_s G^{in}$ to both sides of (5.50), we obtain

$$\begin{aligned}
&2\left(G^{out}\nabla_s^2 G^{in} + \nabla_s G^{out}\nabla_s G^{in}\right)\\
=&G^{out}\nabla_s^2 G^{in} + 2\nabla_s G^{out}\nabla_s G^{in} + G^{in}\nabla_s^2 G^{out} + \delta(\mathbf{r}-\mathbf{r_s})G^{out} - \delta(\mathbf{r}'-\mathbf{r_s})G^{in}\\
=&\nabla_s.\left(G^{out}\nabla_s G^{in}\right) + \nabla_s.\left(\nabla_s G^{in} G^{out}\right) + \delta(\mathbf{r}-\mathbf{r_s})G^{out} - \delta(\mathbf{r}'-\mathbf{r_s})G^{in}\\
=&\nabla_s.\left(G^{out}\nabla_s G^{in} + \nabla_s G^{in} G^{out}\right) + \delta(\mathbf{r}-\mathbf{r_s})G^{out} - \delta(\mathbf{r}'-\mathbf{r_s})G^{in}\\
=&\nabla_s.\nabla_s(G^{in}G^{out}) + \delta(\mathbf{r}-\mathbf{r_s})G^{out} - \delta(\mathbf{r}'-\mathbf{r_s})G^{in}.
\end{aligned} \tag{5.51}$$

When we substitute the last equality (5.51) instead of the integrand seen in the integral (5.47), $P(\mathbf{r},\mathbf{r}')$ can be written as

$$P(\mathbf{r},\mathbf{r}') = \frac{1}{2}\int_{-\infty}^{\infty} iw \int_{V_1\cup V_2} \frac{1}{\rho(\mathbf{r_s})}\nabla_s.\left(\nabla_s(G^{\text{out}}(\mathbf{r}',\mathbf{r_s})G^{\text{in}}(\mathbf{r},\mathbf{r_s}))\right)\, dV_s\, dw \tag{5.52}$$

$$+\frac{1}{2}\int_{-\infty}^{\infty} iw \int_{V_1\cup V_2} \frac{1}{\rho(\mathbf{r_s})}\delta(\mathbf{r}-\mathbf{r_s})G^{\text{out}}(\mathbf{r}',\mathbf{r_s}) - \frac{1}{\rho(\mathbf{r_s})}\delta(\mathbf{r}'-\mathbf{r_s})G^{\text{in}}(\mathbf{r},\mathbf{r_s})\, dV_s\, dw. \tag{5.53}$$

We first deal with second term (5.53) of above expression. The Dirac delta function has the following property:

$$\int_V f(x)\delta(x-a) = \begin{cases} f(a), & \text{if } a \in V \\ 0, & \text{if } a \notin V \end{cases}. \tag{5.54}$$

Therefore, we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty} iw \int_{V_1 \cup V_2} \frac{1}{\rho(\mathbf{r_s})}\delta(\mathbf{r}-\mathbf{r_s})G^{\text{out}}(\mathbf{r}',\mathbf{r_s}) - \frac{1}{\rho(r_s)}\delta(\mathbf{r}'-\mathbf{r_s})G^{\text{in}}(\mathbf{r},\mathbf{r_s})\,dV_s\,dw \\
&= \int_{-\infty}^{\infty} iw \left(\frac{1}{\rho(\mathbf{r})}G^{\text{out}}(\mathbf{r}',\mathbf{r}) - \frac{1}{\rho(r')}G^{\text{in}}(\mathbf{r},\mathbf{r}')\right)\,dw \\
&= \frac{1}{\rho(\mathbf{r})}\int_{-\infty}^{\infty} iwG^{\text{out}}(\mathbf{r}',\mathbf{r})\,dw + \frac{1}{\rho(\mathbf{r}')}\left(\int_{-\infty}^{\infty} iwG^{\text{out}}(\mathbf{r},\mathbf{r}')\,dw\right)^* \\
&= \frac{1}{\rho(\mathbf{r})}\pi c^2(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') + \frac{1}{\rho(\mathbf{r}')}\pi c^2(\mathbf{r}')\delta(\mathbf{r}'-\mathbf{r}) \\
&= \pi\frac{c^2(\mathbf{r}')\rho(\mathbf{r}) + c^2(\mathbf{r})\rho(\mathbf{r}')}{\rho(\mathbf{r})\rho(\mathbf{r}')}\delta(\mathbf{r}-\mathbf{r}')
\end{aligned}
\tag{5.55}
$$

by using the result (4.11) obtained from the initial conditions. Now, to explore the first term of $P(\mathbf{r},\mathbf{r}')$, we substitute the Green's functions in the expression for all location combinations of $\mathbf{r}$, $\mathbf{r}'$ and $\mathbf{r_s}$ in $V_1 \cup V_2$. Through calculations, we realize that the conditions $r_s > r$ and $r_s > r'$ make it easier to deal with the given integral. To satisfy these conditions, we again turn back to surface integral for the first part of $P(\mathbf{r},\mathbf{r}')$.

$$
\begin{aligned}
&\int_{-\infty}^{\infty} iw \int_{V_1 \cup V_2} \frac{1}{\rho(r_s)}\nabla_s.\left(\nabla_s(G^{\text{out}}(\mathbf{r}',\mathbf{r_s})G^{\text{in}}(\mathbf{r},\mathbf{r_s}))\right)\,dV_s\,dw \\
&\int_{-\infty}^{\infty} iw \int_{S_2} \frac{1}{\rho(r_s)}\nabla_s(G^{\text{out}}(\mathbf{r}',\mathbf{r_s})G^{\text{in}}(\mathbf{r},\mathbf{r_s})).\mathbf{n_s}\,dS\,dw \\
&= \frac{1}{\rho_2}\int_{-\infty}^{\infty} iw \int_{S_2} \frac{\partial}{\partial r_s}\left(G^{\text{out}}(\mathbf{r}',\mathbf{r_s})G^{\text{in}}(\mathbf{r},\mathbf{r_s})\right)\,dS\,dw \\
&= \frac{1}{\rho_2}\int_{-\infty}^{\infty} iw \int_0^{2\pi} \frac{\partial}{\partial r_s}\left(G^{\text{out}}(\mathbf{r}',\mathbf{r_s})G^{\text{in}}(\mathbf{r},\mathbf{r_s})\right) r_s\,d\phi_s\,dw \\
&= \frac{r_s}{\rho_2}\frac{\partial}{\partial r_s}\left(\int_{-\infty}^{\infty} iw \int_0^{2\pi}\left(\sum_{n=-\infty}^{\infty} G^{\text{out}}(r',r_s)e^{-in(\phi'-\phi_s)}\right)\right. \\
&\qquad\qquad \left.\times\left(\sum_{m=-\infty}^{\infty} G^{\text{in}}(r,r_s)e^{im(\phi-\phi_s)}\right)\,d\phi_s\,dw\right) \\
&= \frac{r_s}{\rho_2}\frac{\partial}{\partial r_s}\left(\int_{-\infty}^{\infty} iw \int_0^{2\pi}\left(\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} G^{\text{out}}(r',r_s)G^{\text{in}}(r,r_s)\right.\right. \\
&\qquad\qquad \left.\left.\times\, e^{im\phi'-in\phi}e^{i\phi_s(m-n)}\right)\,d\phi_s\,dw\right),
\end{aligned}
$$

since the normal derivative on circle is equal to the derivative with respect to the variable r in polar coordinates. By the orthogonality of the exponential functions $\{e^{in\phi}\}$, we obtain

$$
\begin{aligned}
&\frac{r_s}{\rho_2}\frac{\partial}{\partial r_s}\left(\int_{-\infty}^{\infty} iw \int_0^{2\pi}\left(\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} G^{\text{out}}(r', r_s)G^{\text{in}}(r, r_s)\right.\right.\\
&\qquad\qquad\qquad\qquad\qquad\left.\left.\times\, e^{im\phi'-in\phi}e^{i\phi_s(m-n)}\right) d\phi_s\, dw\right)\\
&= \frac{r_s}{\rho_2}\frac{\partial}{\partial r_s}\left(2\pi\int_{-\infty}^{\infty} iw\left(\sum_{n=-\infty}^{\infty} G^{\text{out}}(r', r_s)G^{\text{in}}(r, r_s)e^{in(\phi'-\phi)}\right) dw\right)\\
&= 2\pi\frac{r_s}{\rho_2}\frac{\partial}{\partial r_s}\left(\sum_{n=-\infty}^{\infty} e^{in(\phi'-\phi)}\int_{-\infty}^{\infty} iwG^{\text{out}}(r', r_s)G^{\text{in}}(r, r_s)\ dw\right). \qquad (5.56)
\end{aligned}
$$

Earlier, we examined some properties of the integral in (5.56). We know that Green's function depends on frequency variable w. For simplicity, we eliminated this variable in representation of Green's function. Now, we use w dependence of Green's function. Let

$$H(w) = G^{\text{out}}(r', r_s)G^{\text{in}}(r, r_s).$$

Then,

$$
\begin{aligned}
\int_{-\infty}^{\infty} iwG^{\text{out}}(r', r_s)G^{\text{in}}(r, r_s)\ dw &= \int_{-\infty}^{\infty} iwH(w)\, dw\\
&= \int_{-\infty}^{0} iwH(w)\, dw + \int_0^{\infty} iwH(w)\, dw\\
&= \int_{\infty}^{0} i - w'H(-w') -\ dw' + \int_0^{\infty} iwH(w)\, dw\\
&= \int_0^{\infty} -iwH(-w)\, dw + \int_0^{\infty} iwH(w)\, dw\\
&= \int_0^{\infty} -iwH(w)^*\, dw + \int_0^{\infty} iwH(w)\, dw\\
&= \int_0^{\infty} iw\left(H(w) - H(w)^*\right)\ dw\\
&= \int_0^{\infty} iw2\boldsymbol{Im}(H(w)) \qquad (5.57)
\end{aligned}
$$

by substituting $w = -w'$ in the integral $\int_{-\infty}^{0} iwH(w)\, dw$ and using the definition of negative frequency for wave functions. This result shows that the real part of the integrand has no contribution to the integral. Now, we substitute the radial part of Green's function in (5.56). On S_2, the second variable $\mathbf{r_s}$ in G^{out} and G^{in} is an element of Region 2. But $\mathbf{r}$ and $\mathbf{r}'$ are free to be in any region. Thus, we have four cases for the combination of product terms depending on locations of points $\mathbf{r}$ and $\mathbf{r}'$: $G^{\text{out}}_{12}(\mathbf{r}', \mathbf{r_s})G^{\text{in}}_{12}(\mathbf{r}, \mathbf{r_s})$, $G^{\text{out}}_{22}(\mathbf{r}', \mathbf{r_s})G^{\text{in}}_{12}(\mathbf{r}, \mathbf{r_s})$, $G^{\text{out}}_{12}(\mathbf{r}', \mathbf{r_s})G^{\text{in}}_{22}(\mathbf{r}, \mathbf{r_s})$, and $G^{\text{out}}_{22}(\mathbf{r}', \mathbf{r_s})G^{\text{in}}_{22}(\mathbf{r}, \mathbf{r_s})$. We show that the products are purely real functions:

$$
\begin{aligned}
&G^{\text{out}}_{12}(r', r_s)G^{\text{in}}_{12}(r, r_s)\\
&= \frac{2}{\pi\rho_1 a}\frac{J_n(k_1 r')H^2_n(k_2 r_s)}{\left(\frac{k_2}{\rho_2}J_n(k_1 a)H^{2\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^2_n(k_2 a)\right)}\\
&\times \frac{2}{\pi\rho_1 a}\frac{J_n(k_1 r)H^1_n(k_2 r_s)}{\left(\frac{k_2}{\rho_2}J_n(k_1 a)H^{1\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^1_n(k_2 a)\right)}\\
&= \frac{2}{\pi\rho_1 a}\frac{J_n(k_1 r')J_n(k_1 r)\parallel H^1_n(k_1 r_s)\parallel}{\parallel \frac{k_2}{\rho_2}J_n(k_1 a)H^{1\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^1_n(k_2 a)\parallel},
\end{aligned}
$$

$$
\begin{aligned}
&G^{\text{out}}_{22}(r', r_s)G^{\text{in}}_{12}(r, r_s)\\
&= \left(-\frac{\frac{k_2}{\rho_2}J_n(k_1 a){Y_n}'(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)Y_n(k_2 a)J_n(k_2 r')}{\frac{k_2}{\rho_2}J_n(k_1 a)H^{2\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^2_n(k_2 a)}\right.\\
&+ \left.\frac{\frac{k_2}{\rho_2}J_n(k_1 a){J_n}'(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)J_n(k_2 a)Y_n(k_2 r')}{\frac{k_2}{\rho_2}J_n(k_1 a)H^{2\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^2_n(k_2 a)}\right) H^2_n(k_2 r_s)\\
&\times \frac{2}{\pi\rho_1 a}\frac{J_n(k_1 r)H^1_n(k_2 r_s)}{\left(\frac{k_2}{\rho_2}J_n(k_1 a)H^{1\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^1_n(k_2 a)\right)}\\
&= \left(-\frac{\frac{k_2}{\rho_2}J_n(k_1 a){Y_n}'(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)Y_n(k_2 a)J_n(k_2 r')}{\parallel \frac{k_2}{\rho_2}J_n(k_1 a)H^{2\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^2_n(k_2 a)\parallel}\right.\\
&+ \left.\frac{\frac{k_2}{\rho_2}J_n(k_1 a){J_n}'(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)J_n(k_2 a)Y_n(k_2 r')}{\parallel \frac{k_2}{\rho_2}J_n(k_1 a)H^{2\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^2_n(k_2 a)\parallel}\right) J_n(k_1 r)\parallel H^2_n(k_2 r_s)\parallel,
\end{aligned}
$$

$$
\begin{aligned}
&G^{\text{out}}_{12}(r', r_s)G^{\text{in}}_{22}(r, r_s) \\
&= \frac{2}{\pi\rho_1 a}\frac{J_n(k_1 r')H^2_n(k_2 r_s)}{\left(\frac{k_2}{\rho_2}J_n(k_1 a)H^{2\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^2_n(k_2 a)\right)} \\
&\times \left(-\frac{\frac{k_2}{\rho_2}J_n(k_1 a){Y_n}'(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)Y_n(k_2 a)J_n(k_2 r)}{\frac{k_2}{\rho_2}J_n(k_1 a)H^{1\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^1_n(k_2 a)}\right. \\
&+ \left.\frac{\frac{k_2}{\rho_2}J_n(k_1 a){J_n}'(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)J_n(k_2 a)Y_n(k_2 r)}{\frac{k_2}{\rho_2}J_n(k_1 a)H^{1\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^1_n(k_2 a)}\right) H^1_n(k_2 r_s) \\
&= \left(-\frac{\frac{k_2}{\rho_2}J_n(k_1 a){Y_n}'(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)Y_n(k_2 a)J_n(k_2 r)}{\parallel \frac{k_2}{\rho_2}J_n(k_1 a)H^{2\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^2_n(k_2 a) \parallel}\right. \\
&+ \left.\frac{\frac{k_2}{\rho_2}J_n(k_1 a){J_n}'(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)J_n(k_2 a)Y_n(k_2 r)}{\parallel \frac{k_2}{\rho_2}J_n(k_1 a)H^{2\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^2_n(k_2 a) \parallel}\right) J_n(k_1 r') \parallel H^2_n(k_2 r_s) \parallel,
\end{aligned}
$$

and

$$
\begin{aligned}
&G^{\text{out}}_{22}(r', r_s)G^{\text{in}}_{22}(r, r_s) \\
&= \left(-\frac{\frac{k_2}{\rho_2}J_n(k_1 a){Y_n}'(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)Y_n(k_2 a)J_n(k_2 r')}{\frac{k_2}{\rho_2}J_n(k_1 a)H^{2\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^2_n(k_2 a)}\right. \\
&+ \left.\frac{\frac{k_2}{\rho_2}J_n(k_1 a){J_n}'(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)J_n(k_2 a)Y_n(k_2 r')}{\frac{k_2}{\rho_2}J_n(k_1 a)H^{2\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^2_n(k_2 a)}\right) H^2_n(k_2 r_s) \\
&\times \left(-\frac{\frac{k_2}{\rho_2}J_n(k_1 a){Y_n}'(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)Y_n(k_2 a)J_n(k_2 r)}{\frac{k_2}{\rho_2}J_n(k_1 a)H^{1\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^1_n(k_2 a)}\right. \\
&+ \left.\frac{\frac{k_2}{\rho_2}J_n(k_1 a){J_n}'(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)J_n(k_2 a)Y_n(k_2 r)}{\frac{k_2}{\rho_2}J_n(k_1 a)H^{1\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^1_n(k_2 a)}\right) H^1_n(k_2 r_s) \\
&= \left(-\frac{\left(\frac{k_2}{\rho_2}J_n(k_1 a){Y_n}'(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)Y_n(k_2 a)\right)^2}{\parallel \frac{k_2}{\rho_2}J_n(k_1 a)H^{2\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^2_n(k_2 a) \parallel}J_n(k_2 r')J_n(k_2 r)\right. \\
&+ \left.\frac{\left(\frac{k_2}{\rho_2}J_n(k_1 a){J_n}'(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)J_n(k_2 a)\right)^2}{\parallel \frac{k_2}{\rho_2}J_n(k_1 a)H^{2\prime}_n(k_2 a) - \frac{k_1}{\rho_1}J'_n(k_1 a)H^2_n(k_2 a) \parallel}Y_n(k_2 r')Y_n(k_2 r)\right) \parallel H^2_n(k_2 r_s) \parallel .
\end{aligned}
$$

Bessel functions J_n's, Y_n's and the modulus of any complex valued functions are real valued functions which implies that all the above products are real. Therefore, the integral (5.56) is equal to zero.

Consequently, we obtained the function $P(\mathbf{r}, \mathbf{r}')$ as follows

$$
\begin{aligned}
P(\mathbf{r}, \mathbf{r}') =& \int_{-\infty}^{\infty} iw \int_{S_2} G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) \nabla_s G^{\text{in}}(\mathbf{r}, \mathbf{r_s}).\mathbf{n_s} \, dS \, dw \\
=& \frac{1}{2} \int_{-\infty}^{\infty} iw \int_{S_2} \nabla_s \left(G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) G^{\text{in}}(\mathbf{r}, \mathbf{r_s}) \right) .\mathbf{n_s} \, dS \, dw \\
&+ \frac{1}{2} \int_{-\infty}^{\infty} iw \int_{V_1 \cup V_2} \delta(\mathbf{r} - \mathbf{r_s}) G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) - \delta(\mathbf{r}' - \mathbf{r_s}) G^{\text{in}}(\mathbf{r}, \mathbf{r_s}) \, dV_s \, dw \\
=& \frac{\pi}{2} \frac{c^2(\mathbf{r}')\rho(\mathbf{r}) + c^2(\mathbf{r})\rho(\mathbf{r}')}{\rho(\mathbf{r})\rho(\mathbf{r}')} \delta(\mathbf{r} - \mathbf{r}').
\end{aligned}
$$

Hence,

$$
\begin{aligned}
q(\mathbf{r}) =& \int_{V'} p_0(\mathbf{r}') P(\mathbf{r}, \mathbf{r}') \, dV' \\
=& \int_{V'} p_0(\mathbf{r}') \frac{\pi}{2} \frac{c^2(\mathbf{r}')\rho(\mathbf{r}) + c^2(\mathbf{r})\rho(\mathbf{r}')}{\rho(\mathbf{r})\rho(\mathbf{r}')} \delta(\mathbf{r} - \mathbf{r}') \, dV' \\
=& \frac{\pi c^2(\mathbf{r})}{\rho(\mathbf{r})} p_0(\mathbf{r})
\end{aligned}
$$

where $\rho(r)$ is a density function. Therefore,

$$
p_0(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\pi c^2(\mathbf{r})} \int_{-\infty}^{\infty} iw \int_{S_2} P(r_s, w) \frac{1}{\rho(\mathbf{r_s})} \frac{\partial G^{\text{in}}(\mathbf{r}, \mathbf{r_s})}{\partial \mathbf{n_s}} \, dS \, dw. \tag{5.58}
$$

that is

$$
p_0(\mathbf{r}) = \begin{cases} \dfrac{\rho(\mathbf{r_1})}{\pi c^2(\mathbf{r})} \displaystyle\int_{-\infty}^{-\infty} iw \int_{S_2} P(\mathbf{r_s}, w) \frac{1}{\rho(\mathbf{r_s})} \frac{\partial G_{12}^{\text{in}}(\mathbf{r}, \mathbf{r_s})}{\partial \mathbf{n_s}} \, dS \, dw, & \mathbf{r} \in R_1 \\ \\ \dfrac{\rho(\mathbf{r_2})}{\pi c^2(\mathbf{r})} \displaystyle\int_{-\infty}^{-\infty} iw \int_{S_2} P(\mathbf{r_s}, w) \frac{1}{\rho(\mathbf{r_s})} \frac{\partial G_{22}^{\text{in}}(\mathbf{r}, \mathbf{r_s})}{\partial \mathbf{n_s}} \, dS \, dw, & \mathbf{r} \in R_2. \end{cases} \tag{5.59}
$$

CHAPTER 6

6. INVERSE SOLUTION IN THREE DIMENSIONAL CYLINDRICALLY N-LAYERED MEDIUM

In this chapter, after examining and solving thermoacoustic equation in circularly two layered medium in two dimension, we state and solve the thermoacoustic wave equation in there dimensional space for cylindrically N-layered medium:

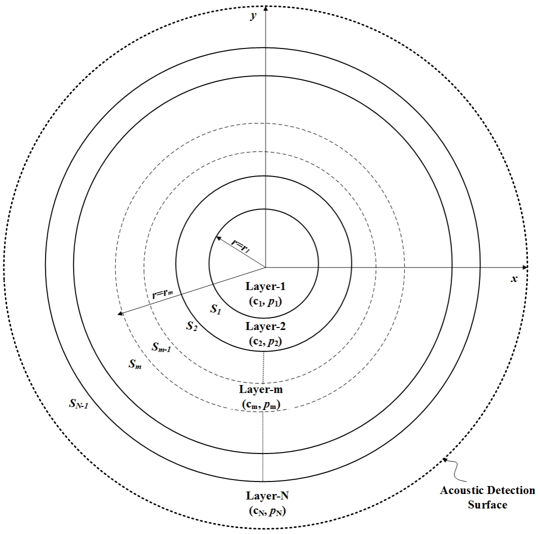

Figure 6.1 Z-Cross Section of N-Layered Medium

Consider a region having N-concentric annular cylindrical layers with different acoustic properties in space $\mathbb{R}^3$ whose z-cross-section is as depicted in Figure 6.1. The interface of consecutive m^{th} and $(m+1)^{th}$ layers is a cylinder with center $(0,0)$ and radius $r = r_m$, denoted by S_m. We call the volume between S_{m-1} and S_m as Region m. Suppose there is a cylindrical transducer, called S_N in Region N enclosing the other regions as in the Figure 6.1. We call the volume covered by transducer S_N as V. We want to determine the source distribution of the region covered by transducer.

The acoustic waves are measured by the transducer for a sufficiently long time interval so that the waves emitted from every source location reach to the transducer. When the regions are different, there will be reflections and transmissions at the boundaries S_m for $1 \leq m \leq N$. Thermoacoustic wave propagation in this layered medium is governed by the nonhomogeneous wave equation

$$\nabla^2 p(\mathbf{r}, t) - \frac{1}{c^2}\frac{\partial^2 p(\mathbf{r}, t)}{\partial t^2} = -p_0(\mathbf{r}).\delta'(t) \tag{6.1}$$

with 2(N-1) boundary conditions

$$p_m(\mathbf{r}, t) = p_{m+1}(\mathbf{r}, t)\big|_{\mathbf{r}\in S_m} \tag{6.2}$$

and

$$\frac{1}{\rho_m}\frac{\partial p_m(\mathbf{r}, t)}{\partial \mathbf{n}} = \frac{1}{\rho_{m+1}}\frac{\partial p_{m+1}(\mathbf{r}, t)}{\partial \mathbf{n}}\Bigg|_{\mathbf{r}\in S_m} \tag{6.3}$$

on each boundary S_m for $1 \leq m \leq N$. Here, p_m and p_{m+1} are the acoustic waves and ρ_m and ρ_{m+1} are the densities for Region m and Region $(m+1)$, respectively. Also, as a nature of the problem, nonhomogeneous thermoacoustic wave equation must satisfy the following initial condition as we stated earlier

$$p(\mathbf{r}, 0^+) = c^2(\mathbf{r})p_0(\mathbf{r}) \quad \text{and} \quad \frac{\partial p(\mathbf{r}, 0^+)}{\partial t} = 0 \tag{6.4}$$

$$p(\mathbf{r}, t) = 0 \quad \text{if} \quad t < 0. \tag{6.5}$$

In an inverse source problem, $p_0(\mathbf{r})$ is to be reconstructed given that acoustic field is measured by the transducer and is known on the surface S_N. We know that the equation (6.1) in frequency domain correspondes to the nonhomogenous Helmholtz equation

$$\nabla^2 P(\mathbf{r}, w) + k^2 P(\mathbf{r}, w) = -iwp_0(\mathbf{r}), \tag{6.6}$$

where $P(\mathbf{r}, w)$ is the temporal Fourier Transform of $p(\mathbf{r}, t)$. In derivations as we did in two dimensions for two layered medium, we again consider that $w > 0$ and $P(\mathbf{r}, w)$ was corresponding to outgoing wave. After that, for the completeness in frequency domain, we define $P(\mathbf{r}, -w) = P(\mathbf{r}, w)^*$ for $w < 0$ as complex conjugate of pressure function for positive frequency. The outgoing and incoming waves were represented by superscripts 'out' and 'in' for pressure function and we used the fact that $P^{in}(\mathbf{r}, w) = (P^{out}(\mathbf{r}, w))^*$. We again make use of Green's functions:

6.1 Green's Function of Medium

The Green's function is the solution of homogeneous wave equation except the point $\mathbf{r}'$ where the point source located:

$$\nabla^2 G^{out}(\mathbf{r}', \mathbf{r}, w) + k^2 G^{out}(\mathbf{r}', \mathbf{r}, w) = -\delta(\mathbf{r} - \mathbf{r}') \tag{6.7}$$

where $\delta(.)$ is the Dirac delta function. It is convenient to study using cylindrical coordinates for the N-layered cylindrical configuration. Before transforming the Helmholtz equation (6.7) to cylindrical system, we take the spatial Fourier transform in z-direction to derive forward solution. We represent the spatial transform with a tilde symbol above of a function name, that is

$$\tilde{f}(k_z) = \int_{-\infty}^{\infty} f(z) e^{-ik_z z} dz, \tag{6.8}$$

and from equation (6.7), we obtain two dimensional Helmholtz equation

$$\nabla^2 \tilde{G}(\mathbf{r}', \mathbf{r}, k_z, w) + (k^2 - k_z^2)\tilde{G}(\mathbf{r}', \mathbf{r}, k_z, w) = \delta(\mathbf{r} - \mathbf{r}') e^{-ik_z z'} \tag{6.9}$$

where $k = w/c$ is the wave number, k_z is the spatial frequency. The wave equation given in (6.9) is expressed in cylindrical coordinates (r, ϕ, z) as

$$\frac{\partial^2 \tilde{G}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{G}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{G}}{\partial \phi^2} + (k^2 - k_z^2)\tilde{G} = \frac{1}{r}\delta(r - r')\delta(\phi - \phi')e^{-ik_z z'}. \tag{6.10}$$

We previously showed that the homogeneous solution of above equation has a series form consisting of Bessel's functions and exponential functions. When the term $\sqrt{(k^2 - k_z^2)}$ is a real number, two independent solutions are the first kind of Bessel function $J_n(\sqrt{(k^2 - k_z^2)}r)$ and the second kind of Bessel function $Y_n(\sqrt{(k^2 - k_z^2)}r)$. Alternatively, linear combination of these two functions, Hankel functions of first kind $H_n^1(\sqrt{(k^2 - k_z^2)}r) = J_n(\sqrt{(k^2 - k_z^2)}r) + iY_n(\sqrt{(k^2 - k_z^2)}r)$ and of second kind $H_n^2(kr) = J_n(kr) - iY_n(kr)$ can be used as fundamental solutions. On the other hand, when $\sqrt{(k^2 - k_z^2)}$ is not a real number, the two independent solutions are called first and second kind modified Bessel functions and denoted by $I_n(kr)$ and $K_n(kr)$, respectively.

As a result, the homogeneous solution of (5.5) has the form

$$I(r, \phi) = \begin{cases} \sum_{n=-\infty}^{\infty} \left(A_n J_n(\sqrt{(k^2 - k_z^2)}r) + B_n Y_n(\sqrt{(k^2 - k_z^2)}r) \right) e^{in\phi}, \\ \qquad\qquad \text{if} \quad \| k \| \geq \| k_z \| \\ \sum_{n=-\infty}^{\infty} \left(C_n I_n(\sqrt{(k_z^2 - k^2)}r) + D_n M_n(\sqrt{(k_z^2 - k^2)}r) \right) e^{in\phi}, \\ \qquad\qquad \text{if} \quad \| k \| \leq \| k_z \| . \end{cases} \tag{6.11}$$

The Bessel functions $J_n(r)$, $Y_n(r)$, $I_n(r)$ and $M_n(r)$ are real valued functions for positive real arguments. Hence all the terms except unknown coefficients in (6.11) are all real. When $\| k \| \geq \| k_z \|$, we apply Sommerfeld radiation condition and when $\| k \| \leq \| k_z \|$, we choose evanescent waves for outer most layer, so that the waves will not grow to infinity. In light of these, the derivations made for the argument $\sqrt{(k^2 - k_z^2)}r$ are the same as the derivations made for the argument $\sqrt{(k_z^2 - k^2)}r$ in inverse solution proof. Therefore, we suppose $\| k \| \geq \| k_z \|$ and progress under this assumption. For simplicity in expressions, we represent $\sqrt{(k^2 - k_z^2)}$ as $\mathbf{k}$, keeping in mind k_z dependence of $\mathbf{k}$.

When the point source $\mathbf{r}'$ locates in Layer m, we denote Green's function as G_m ($1 \leq m \leq N$). Each Green's function G_m represents the unit impulse response of the layered medium and is partially defined with respect to observation point **r**:

$$G_m(\mathbf{r}', \mathbf{r}, w) = \begin{cases} \sum\limits_{n=-\infty}^{\infty} e^{in\phi} \int\limits_{-\infty}^{\infty} e^{ik_z(z-z')} \left(A_{nj} J_n(\mathbf{k}_j r) + B_{nj} Y_n(\mathbf{k}_j r)\right) \, dk_z, \\ \text{if} \quad r < r' \text{and r in layer j} \\ \\ \sum\limits_{n=-\infty}^{\infty} e^{in\phi} \int\limits_{-\infty}^{\infty} e^{ik_z(z-z')} \left(C_{nj} J_n(\mathbf{k}_j r) + D_{nj} Y_n(\mathbf{k}_j r)\right) \, dk_z, \\ \text{if} \quad r' < r \;\; \text{and r in layer j.} \end{cases} \tag{6.12}$$

We call the parts of G_m with respect to location of observation points as G_{mj} for $1 \leq j \leq N$. In derivations of inverse problem, the observation points are on the transducer, so we need to calculate only the last parts G_{mN} of Green's function G_m wherever the source location m is.

The coefficients in each Green's function G_m are obtained by $(2N + 2)$ equalities coming from the boundary conditions, Green's function's conditions and radiation conditions:

The given boundary conditions (6.2) and (6.3) state that acoustic pressure function is continuous and its normal derivative is continuous with a scaling factor on the layer boundaries $r = r_i$:

$$\begin{aligned} & A_{n(i+1)} J_n(\mathbf{k}_{i+1} r_i) + B_{n(i+1)} Y_n(\mathbf{k}_{i+1} r_i) - A_{n(i)} J_n(\mathbf{k}_i r_i) - B_{n(i)} Y_n(\mathbf{k}_i r_i) = 0, \\ & \frac{\mathbf{k}_{i+1}}{\rho_{i+1}} \left(A_{n(i+1)} J_n'(\mathbf{k}_{i+1} r_i) + B_{n(i+1)} Y_n'(\mathbf{k}_{i+1} r_i)\right) \\ & - \frac{\mathbf{k}_i}{\rho_i} \left(A_{n(i)} J_n'(\bar{k}_i r_i) + B_{n(i)} Y_n'(\mathbf{k}_i r_i)\right) = 0 \end{aligned} \tag{6.13}$$

for $1 \leq i \leq m - 1$,

$$
\begin{aligned}
&C_{n(i+1)}J_n(\mathbf{k}_{i+1}r_i) + D_{n(i+1)}Y_n(\mathbf{k}_{i+1}r_i) - C_{n(i)}J_n(\mathbf{k}_i r_i) - D_{n(i)}Y_n(\mathbf{k}_i r_i) = 0, \\
&\frac{\mathbf{k}_{i+1}}{\rho_{i+1}}\left(C_{n(i+1)}J_n'(\mathbf{k}_{i+1}r_i) + D_{n(i+1)}Y_n'(\mathbf{k}_{i+1}r_i)\right) \\
&-\frac{\mathbf{k}_i}{\rho_i}\left(C_{n(i)}J_n'(\mathbf{k}_i r_i) + D_{n(i)}Y_n'(\mathbf{k}_i r_i)\right) = 0
\end{aligned}
\tag{6.14}
$$

for $m \leq i \leq N-1$.

On the other hand, Green's function is continuous and its normal derivative has jump discontinuity on a cylinder $r = r'$ where the point source locates (Stakgold, 1979). Hence, these conditions give us

$$
\begin{aligned}
&C_{n(m)}J_n(\mathbf{k}_m r') + D_{n(m)}Y_n(\mathbf{k}_m r') - A_{n(m)}J_n(\mathbf{k}_m r') - B_{n(m)}Y_n(\mathbf{k}_m r') = 0, \\
&\mathbf{k}_m r'\left((C_{n(m)} - A_{n(m)})J_n'(\mathbf{k}_m r') + (D_{n(m)} - B_{n(m)})Y_n'(\mathbf{k}_m r')\right) = e^{-ik_z z'}\frac{e^{-in\phi'}}{2\pi}.
\end{aligned}
\tag{6.15}
$$

Additionally, second kind of Bessel function Y_n is undefined when $\mathbf{r} = 0$. Therefore in Layer 1, Green's function cannot include Y_n implying

$$
B_{n1} = 0. \tag{6.16}
$$

Lastly, the pressure function must satisfy Sommerfeld radiation condition

$$
\lim_{|\mathbf{r}|\to\infty}\left(\frac{\partial}{\partial|\mathbf{r}|} - ik\right)P(\mathbf{r}, w) = 0 \tag{6.17}
$$

which leads to

$$
D_{nN} - iC_{nN} = 0. \tag{6.18}
$$

This system of equations is represented in a matrix form as follows: If we call the terms seen in the boundary conditions (6.13) and (6.14)

$$-J_n(\mathbf{k}_m r_m) = a_{m1}, \qquad -Y_n(\mathbf{k}_m r_m) = a_{m2},$$

$$J_n(\mathbf{k}_{m+1} r_m) = a_{m3}, \qquad Y_n(\mathbf{k}_{m+1} r_m) = a_{m4,}$$

$$-\tfrac{\mathbf{k}_m}{\rho_m} J_n'(\mathbf{k}_m r_m) = b_{m1}, \qquad -\tfrac{\mathbf{k}_m}{\rho_m} Y_n'(\mathbf{k}_m r_m) = b_{m2},$$

$$\tfrac{\mathbf{k}_{m+1}}{\rho_{m+1}} J_n'(\mathbf{k}_{m+1} r_m) = b_{m3}, \quad \tfrac{\mathbf{k}_{m+1}}{\rho_{m+1}} Y_n'(\mathbf{k}_{m+1} r_m) = b_{m4}$$

and in Green's function's conditions (6.15)

$$J_n(\mathbf{k}_m r') = k_1, \qquad Y_n(\mathbf{k}_m r') = k_2,$$

$$\mathbf{k}_m r' J_n'(\mathbf{k}_m r') = l_1, \quad \mathbf{k}_m r' Y_n'(\mathbf{k}_m r') = l_2$$

as given above, and if we let

$$L_{m1} = \begin{bmatrix} a_{m1} & a_{m2} \\ b_{m1} & b_{m2} \end{bmatrix} \quad \text{and} \quad L_{m2} = \begin{bmatrix} a_{m3} & a_{m4} \\ b_{m3} & b_{m4} \end{bmatrix},$$

we can write the system of equations in the matrix form as follows:

$$\mathbf{B_n} \cdot \begin{bmatrix} A_{n1} \\ B_{n1} \\ \vdots \\ B_{n(m)} \\ C_{n(m)} \\ D_{n(m)} \\ \vdots \\ D_{nN} \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \vdots \\ \\ 0 \\ e^{-ik_z z'} \frac{e^{-in\phi'}}{2\pi} \\ 0 \\ \vdots \end{bmatrix} \tag{6.19}$$

in which $\mathbf{B_n}$ is the coefficient matrix of the system of equations is found as follows:

$$\mathbf{B_n} = \left[\begin{array}{cccccccc}
0\ 1 & 0\ 0 & & & & & & \\
\left[L_{11}\right] & \left[L_{12}\right] & & & & & & \\
 & \left[L_{21}\right] & \left[L_{22}\right] & & & & & \\
\mathbf{0} & & \ddots & & & & & \\
 & & & -k_1\ -k_2 & k_1\ k_2 & & & \\
 & & & -l_1\ -l_2 & l_1\ l_2 & & \mathbf{0} & \\
 & & & & \left[L_{m1}\right] & \left[L_{m2}\right] & & \\
 & & & & & \ddots & & \\
 & & & & & & \left[L_{(N-1)1}\right] & \left[L_{(N-1)2}\right] \\
 & & & & & & 0\quad 0 & i\quad -1
\end{array}\right]. \tag{6.20}$$

We apply elementary row-column operations to coefficient matrix $\mathbf{B_n}$ to obtain its determinant β_n. If we add $(2m+1)$th column to $(2m-1)$th column and $(2m+2)$th column to $(2m)$th column, we obtain

$$\mathbf{B_n} = \left[\begin{array}{cccccccc}
0\ 1 & 0\ 0 & & & & & & \\
\left[L_{11}\right] & \left[L_{12}\right] & & & & & & \\
 & \left[L_{21}\right] & \left[L_{22}\right] & & & & & \\
\mathbf{0} & & \ddots & & & & & \\
 & & & 0\ 0 & k_1\ k_2 & & & \\
 & & & 0\ 0 & l_1\ l_2 & & \mathbf{0} & \\
 & & & \left[L_{m1}\right] & \left[L_{m1}\right] & \left[L_{m2}\right] & & \\
 & & & & & \ddots & & \\
 & & & & & & \left[L_{(N-1)1}\right] & \left[L_{(N-1)2}\right] \\
 & & & & & & 0\quad 0 & i\quad -1
\end{array}\right]. \tag{6.21}$$

We firstly choose the row containing k_1, k_2 to calculate determinant B_n as:

$$
\begin{aligned}
&(-1)^{2m-1+2m+1}k_1
\begin{vmatrix}
0\ 1 & 0\ 0 & & & & & & \\
\begin{bmatrix}L_{11}\end{bmatrix} & \begin{bmatrix}L_{12}\end{bmatrix} & & & & & & \\
 & \begin{bmatrix}L_{21}\end{bmatrix} & \begin{bmatrix}L_{22}\end{bmatrix} & & & & & \\
\mathbf{0} & & \ddots & & & & & \\
 & & & 0\ 0 & l_2 & & & \mathbf{0} \\
 & & & \begin{bmatrix}L_{m1}\end{bmatrix} & \begin{matrix}a_{m2}\\ b_{m2}\end{matrix} & \begin{bmatrix}L_{m2}\end{bmatrix} & & \\
 & & & & & \ddots & & \\
 & & & & & & \begin{bmatrix}L_{(N-1)1}\end{bmatrix} & \begin{bmatrix}L_{(N-1)2}\end{bmatrix} \\
 & & & & & & 0 \quad 0 & i \quad -1
\end{vmatrix} \\
&+(-1)^{2m-1+2m+2}k_2
\begin{vmatrix}
0\ 1 & 0\ 0 & & & & & & \\
\begin{bmatrix}L_{11}\end{bmatrix} & \begin{bmatrix}L_{12}\end{bmatrix} & & & & & & \\
 & \begin{bmatrix}L_{21}\end{bmatrix} & \begin{bmatrix}L_{22}\end{bmatrix} & & & & & \\
\mathbf{0} & & \ddots & & & & & \\
 & & & 0\ 0 & l_1 & & & \mathbf{0} \\
 & & & \begin{bmatrix}L_{m1}\end{bmatrix} & \begin{matrix}a_{m1}\\ b_{m1}\end{matrix} & \begin{bmatrix}L_{m2}\end{bmatrix} & & \\
 & & & & & \ddots & & \\
 & & & & & & \begin{bmatrix}L_{(N-1)1}\end{bmatrix} & \begin{bmatrix}L_{(N-1)2}\end{bmatrix} \\
 & & & & & & 0 \quad 0 & i \quad -1
\end{vmatrix}.
\end{aligned}
\tag{6.22}
$$

Then we use the rows containing l_1 and l_2 and obtain

$$
(-1)^{2m-1+2m-1} k_1 l_2
\begin{vmatrix}
0\ \ 1 & 0\ \ 0 & & & & & \\
\left[L_{11}\right] & \left[L_{12}\right] & & & & & \\
 & \left[L_{21}\right] & \left[L_{22}\right] & & & & \\
\mathbf{0} & & \ddots & & & & \\
 & & & \left[L_{m1}\right] & \left[L_{m2}\right] & & \\
 & & & & \ddots & & \mathbf{0} \\
 & & & & & \left[L_{(N-1)1}\right] & \left[L_{(N-1)2}\right] \\
 & & & & & 0 \quad 0 & i \quad -1
\end{vmatrix}
$$

$$
-(-1)^{2m-1+2m-1} k_2 l_1
\begin{vmatrix}
0\ \ 1 & 0\ \ 0 & & & & & \\
\left[L_{11}\right] & \left[L_{12}\right] & & & & & \\
 & \left[L_{21}\right] & \left[L_{22}\right] & & & & \\
\mathbf{0} & & \ddots & & & & \\
 & & & \left[L_{m1}\right] & \left[L_{m2}\right] & & \\
 & & & & \ddots & & \mathbf{0} \\
 & & & & & \left[L_{(N-1)1}\right] & \left[L_{(N-1)2}\right] \\
 & & & & & 0 \quad 0 & i \quad -1
\end{vmatrix}. \tag{6.23}
$$

for derivation, respectively, hence we write the determinant as below:

$$
\beta_n = (k_1 l_2 - k_2 l_1).
\begin{vmatrix}
0\ \ 1 & 0\ \ 0 & & & & & \\
L_{11} & L_{12} & & & & & \\
 & L_{21} & L_{22} & & & & \\
\mathbf{0} & & \ddots & & & & \\
 & & & L_{m1} & L_{m2} & & \mathbf{0} \\
 & & & & \ddots & & \\
 & & & & & L_{(N-1)1} & L_{(N-1)2} \\
 & & & & & 0 \quad 0 & i \quad -1
\end{vmatrix} \tag{6.24}
$$

Here, we notice that the terms seen in the determinant brackets are coming from boundary conditions, only the product term $(k_1 l_2 - k_2 l_1)$ includes information about source location (the region m). Now, we examine this term:

$$
\begin{aligned}
k_1 l_2 - k_2 l_1 =& J_n(\mathbf{k}_m r')\mathbf{k}_m r' Y_n'(\mathbf{k}_m r') - Y_n(\mathbf{k}_m r')\mathbf{k}_m r' J_n'(\mathbf{k}_m r') \\
=& \mathbf{k}_m r' \left(J_n(\mathbf{k}_m r') Y_n'(\mathbf{k}_m r') - Y_n(\mathbf{k}_m r') J_n'(\mathbf{k}_m r') \right) \\
=& \mathbf{k}_m r' \frac{2}{\pi \mathbf{k}_m r'} \\
=& \frac{2}{\pi}
\end{aligned}
$$

using the Wronskian properties of Bessel functions. Therefore the examined term is independent of the source location. Hence, the determinant β_n is the same for all n (index set for Bessel functions' order) and for all m (location of source point $\mathbf{r}'$).

In the derivation of inverse solution, we need the last part of Green's function G_m. So, now, we obtain the coefficients of G_{mN}, that is C_{nN} and D_{nN}: Firstly, we know that $C_{nN} = -iD_{nN}$ by radiation condition (6.18). To calculate the coefficient D_{nN} we apply Kramer's rule:

$$
D_{nN} = \frac{\alpha_n}{\beta_n} \tag{6.25}
$$

in which

$$
\alpha_n = \begin{vmatrix}
0\ \ 1 & 0\ \ 0 & & & & & 0 \\
\left[L_{11}\right] & \left[L_{12}\right] & & & & & \vdots \\
 & \left[L_{21}\right] & \left[L_{22}\right] & & & & \\
\mathbf{0} & & \ddots & & & & \\
 & & -k_1\ -k_2 & k_1\ k_2 & & & \\
 & & -l_1\ -l_2 & l_1\ l_2 & & & \Gamma \\
 & & & \left[L_{m1}\right] & \left[L_{m2}\right] & & 0 \\
 & & & & \ddots & & \vdots \\
 & & & & & \left[L_{(N-1)1}\right] & \left[L_{(N-1)2}\right] \\
 & & & & & 0\ \ 0 & 1\ \ 0
\end{vmatrix} \tag{6.26}
$$

with $\Gamma = e^{-ik_z z'} \dfrac{e^{-in\phi'}}{2\pi}$. We choose the row containing the term Γ to calculate the determinant α_n and obtain

$$
\alpha_n = (-1)^{2m+2+2N+2} e^{-ik_z z'} \tfrac{e^{-in\phi'}}{2\pi} \cdot
\left|
\begin{array}{ccccccc}
0\ 1 & 0\ 0 & & & & & \\
\begin{bmatrix} L_{11} \end{bmatrix} & \begin{bmatrix} L_{12} \end{bmatrix} & & & & & \\
& \begin{bmatrix} L_{21} \end{bmatrix} & \begin{bmatrix} L_{22} \end{bmatrix} & & & & \\
\mathbf{0} & & \ddots & & & & \mathbf{0} \\
& & & -k_1 - k_2 \ \ k_1\ k_2 & & & \\
& & & & \begin{bmatrix} L_{m1} \end{bmatrix} & \begin{bmatrix} L_{m2} \end{bmatrix} & \\
& & & & & \ddots & \\
& & & & & \begin{bmatrix} L_{(N-1)1} \end{bmatrix} & \begin{matrix} a_{(N-1)3} \\ b_{(N-1)3} \end{matrix} \\
& & & & & 0 \quad 0 & 1
\end{array}
\right| . \tag{6.27}
$$

We call the determinant factor as R_n^m. The most important property of R_n^m used in inverse problem derivation is that all the terms contained in R_n^m are real, hence the determinant R_n^m is itself is real. At the end, the coefficent D_{nN} is written in the form

$$
D_{nN} = e^{-ik_z z'} \frac{e^{-in\phi'}}{2\pi} \frac{R_n^m}{\beta_n}. \tag{6.28}
$$

Hence,

$$
C_{nN} = -ie^{-ik_z z'} \frac{e^{-in\phi'}}{2\pi} \frac{R_n^m}{\beta_n} \tag{6.29}
$$

which implies that

$$
\begin{aligned}
& G_{mN}(\mathbf{r}', \mathbf{r}, k_z) \\
&= \int_{-\infty}^{\infty} e^{ik_z z} \sum_{n=-\infty}^{\infty} \left(-ie^{-ik_z z'} \frac{e^{-in\phi'}}{2\pi} \frac{R_n^m}{\beta_n}(r', r) J_n(\mathbf{k}_N r) \right. && (6.30) \\
& \quad \left. + e^{-ik_z z'} \frac{e^{-in\phi'}}{2\pi} \frac{R_n^m}{\beta_n}(r', r) Y_n(\mathbf{k}_N r) \right) e^{in\phi'} \, dk_z && (6.31) \\
&= \int_{-\infty}^{\infty} e^{ik_z z} e^{-ik_z z'} \sum_{n=-\infty}^{\infty} \frac{R_n^m}{\beta_n}(r', r) \left(-iJ_n(\mathbf{k}_N r) + Y_n(\mathbf{k}_N r) \right) \frac{e^{-in\phi'}}{2\pi} e^{in\phi'} \, dk_z \\
&= \int_{-\infty}^{\infty} e^{ik_z (z-z')} \sum_{n=-\infty}^{\infty} \frac{R_n^m}{\beta_n}(r', r) \left(-iH_n^1(\mathbf{k}_N r) \right) \frac{e^{in(\phi-\phi')}}{2\pi} \, dk_z. && (6.32)
\end{aligned}
$$

6.2 Inverse Solution

In two dimensional space for circularly two layered medium, we proved that inverse solution of thermoacoustic wave equation, thermoacoustic source distribution, is given by

$$
p_0(\mathbf{r}) = \begin{cases} \dfrac{\rho(\mathbf{r})}{c^2(\mathbf{r})\pi} \displaystyle\int_{-\infty}^{\infty} \int_{S_2} P(\mathbf{r_s}, w) \frac{1}{\rho(\mathbf{r_s})} \frac{\partial G_1^{\text{in}}(\mathbf{r}, \mathbf{r_s})}{\partial \mathbf{n_s}} \, dS \, dw, & \mathbf{r} \in R_1 \\[2ex] \dfrac{\rho(\mathbf{r})}{c^2(\mathbf{r})\pi} \displaystyle\int_{-\infty}^{\infty} \int_{S_2} P(\mathbf{r_s}, w) \frac{1}{\rho(r_s)} \frac{\partial G_2^{\text{in}}(\mathbf{r}, \mathbf{r_s})}{\partial \mathbf{n_s}} \, dS \, dw, & \mathbf{r} \in R_2 \end{cases} \tag{6.33}
$$

where $P(\mathbf{r_s}, w)$ is the acoustic pressure measured on the surface S_2, G_1, G_2 are the corresponding Green's function of the medium and $\rho(r)$ is a density function

$$
\rho(\mathbf{r}) = \begin{cases} \rho_1, & \mathbf{r} \in R_1 \\ \rho_2, & \mathbf{r} \in R_2. \end{cases}
$$

Now, we prove our solution (6.33) can be extended for three dimensional N-layered configuration as

$$
p_0(\mathbf{r}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{S_N} P(\mathbf{r_s}, w) \frac{1}{\rho(\mathbf{r_s})} \frac{\partial G_{iN}^{\text{in}}(\mathbf{r}, \mathbf{r_s})}{\partial \mathbf{n_s}} \, dS \, dw, \quad \mathbf{r} \in R_i \tag{6.34}
$$

where $P(\mathbf{r_s}, w)$ is the acoustic pressure measured on the surface S_N, G_{iN} is the corresponding Green's function for $1 \leq i \leq N$ and $\rho(\mathbf{r})$ is a density function such that

$$\rho(\mathbf{r}) = \rho_i, \quad \mathbf{r} \in R_i \tag{6.35}$$

for $1 \leq i \leq N$. Let us write (6.34) as independent of index set and call the integral expression as $q(\mathbf{r})$:

$$q(\mathbf{r}) = \int_{-\infty}^{\infty} \int_{S_N} P(\mathbf{r_s}, w) \frac{1}{\rho(\mathbf{r_s})} \frac{\partial G^{\text{in}}(\mathbf{r}, \mathbf{r_s})}{\partial \mathbf{n_s}} \, dS \, dw. \tag{6.36}$$

The acoustic pressure measured on the surface S_N is given by forward solution of the wave equation (4.9):

$$P(\mathbf{r_s}, w) = iw \int_{V'} p_0(\mathbf{r}') G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) dV_{\mathbf{r}'}.$$

We substitute the forward solution in $q(r)$:

$$\begin{aligned}
q(\mathbf{r}) &= \int_{-\infty}^{\infty} \int_{S_N} P(\mathbf{r_s}, w) \frac{1}{\rho(\mathbf{r_s})} \frac{\partial G^{\text{in}}(\mathbf{r}, \mathbf{r_s})}{\partial \mathbf{n_s}} \, dS \, dw \\
&= \int_{-\infty}^{\infty} \int_{S_N} P(\mathbf{r_s}, w) \frac{1}{\rho(\mathbf{r_s})} \nabla G^{\text{in}}(\mathbf{r}, \mathbf{r_s}).\mathbf{n_s} \, dS \, dw \\
&= \int_{-\infty}^{\infty} iw \int_{S_N} \left(\int_{V'} p_0(\mathbf{r}') G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) dV_{\mathbf{r}'} \right) \nabla \frac{1}{\rho(\mathbf{r_s})} G^{\text{in}}(\mathbf{r}, \mathbf{r_s}).\mathbf{n_s} \, dS \, dw \\
&= \int_{V'} p_0(\mathbf{r}') \left(\int_{-\infty}^{\infty} iw \int_{S_N} \left(G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) \frac{1}{\rho(r_s)} \nabla G^{\text{in}}(\mathbf{r}, \mathbf{r_s}) \right) .\mathbf{n_s} \, dS \, dw \right) dV_{\mathbf{r}'}.
\end{aligned}$$

Let us call the term in outer integrals as follows:

$$P(\mathbf{r}, \mathbf{r}') = \int_{-\infty}^{\infty} iw \int_{S_N} G^{\text{out}}(\mathbf{r}', \mathbf{r_s}, w) \frac{1}{\rho(\mathbf{r_s})} \nabla_s G^{\text{in}}(\mathbf{r}, \mathbf{r_s}, w).\mathbf{n_s} \, dS \, dw. \tag{6.37}$$

We know that the Green's function is continuous on whole space. Also the normal derivative of Green's function with a scaling factor (the density function) is continuous, too. Hence, the expression in the above integral is continuous which makes possible to apply the Divergence theorem as follows:

$$
\begin{aligned}
P(\mathbf{r},\mathbf{r}') &= \int_{-\infty}^{\infty} iw \int_{S_N} G^{\text{out}}(\mathbf{r}',\mathbf{r_s},w)\frac{1}{\rho(\mathbf{r_s})}\nabla_s G^{\text{in}}(\mathbf{r},\mathbf{r}_s,w).\mathbf{n_s}\,dS\,dw \\
&= \int_{-\infty}^{\infty} iw \int_{V} \nabla_s\left(G^{\text{out}}(\mathbf{r}',\mathbf{r_s},w)\nabla_s\frac{1}{\rho(\mathbf{r_s})}G^{\text{in}}(\mathbf{r},\mathbf{r_s},w)\right)dV_s\,dw. \qquad (6.38)
\end{aligned}
$$

Since each layer is homogeneous in itself, the density function $\rho(\mathbf{r_s})$ is constant on each volume V_i for $1 \leq i \leq N$. Therefore

$$
\begin{aligned}
P(\mathbf{r},\mathbf{r}') = \int_{-\infty}^{\infty} iw \int_{V} \frac{1}{\rho(\mathbf{r_s})}\Big(&\nabla_s G^{\text{out}}(\mathbf{r}',\mathbf{r_s},w)\nabla_s G^{\text{in}}(\mathbf{r},\mathbf{r_s},w) \qquad (6.39)\\
&+G^{\text{out}}(\mathbf{r}',\mathbf{r_s},w)\nabla_s^2 G^{\text{in}}(\mathbf{r},\mathbf{r_s},w)\Big)\,dV_s\,dw. \qquad (6.40)
\end{aligned}
$$

The solutions G^{in} and G^{out} satisfy the Helmholtz equation:

$$\nabla_s^2 G^{in}(\mathbf{r},\mathbf{r_s},w) + k_s^2 G^{in}(\mathbf{r},\mathbf{r_s},w) = -\delta(\mathbf{r}-\mathbf{r_s}), \qquad (6.41)$$

$$\nabla_s^2 G^{out}(\mathbf{r}',\mathbf{r_s},w) + k_s^2 G^{out}(\mathbf{r}',\mathbf{r_s},w) = -\delta(\mathbf{r}'-\mathbf{r_s}) \qquad (6.42)$$

in which $k_s = w/c_s$ and c_s is the acoustic speed in the region where r_s in. If we multiply (6.41) by $G^{out}(\mathbf{r}',\mathbf{r_s},w)$ and (6.42) by $G^{in}(\mathbf{r},\mathbf{r_s},w)$ and subtract each other, we get

$$G^{out}\nabla_s^2 G^{in} = G^{in}\nabla_s^2 G^{out} - \delta(\mathbf{r}-\mathbf{r_s})G^{out} + \delta(\mathbf{r}'-\mathbf{r_s})G^{in}. \qquad (6.43)$$

By adding the term $G^{out}\nabla_s^2 G^{in} + 2\nabla_s G^{out}\nabla_s G^{in}$ to both sides of (6.43), we obtain

$$
\begin{aligned}
&2\left(G^{out}\nabla_s^2 G^{in} + \nabla_s G^{out}\nabla_s G^{in}\right)\\
=&G^{out}\nabla_s^2 G^{in} + 2\nabla_s G^{out}\nabla_s G^{in} + G^{in}\nabla_s^2 G^{out} + \delta(\mathbf{r}-\mathbf{r_s})G^{out} - \delta(\mathbf{r}'-\mathbf{r_s})G^{in}\\
=&\nabla_s.\left(G^{out}\nabla_s G^{in}\right) + \nabla_s.\left(\nabla_s G^{in} G^{out}\right) + \delta(\mathbf{r}-\mathbf{r_s})G^{out} - \delta(\mathbf{r}'-\mathbf{r_s})G^{in}\\
=&\nabla_s.\left(G^{out}\nabla_s G^{in} + \nabla_s G^{in} G^{out}\right) + \delta(\mathbf{r}-\mathbf{r_s})G^{out} - \delta(\mathbf{r}'-\mathbf{r_s})G^{in}\\
=&\nabla_s.\nabla_s(G^{in}G^{out}) + \delta(\mathbf{r}-\mathbf{r_s})G^{out} - \delta(\mathbf{r}'-\mathbf{r_s})G^{in}. \qquad (6.44)
\end{aligned}
$$

When we substitute the last equality (6.44) instead of the integrand seen in the integral (6.40), $P(\mathbf{r},\mathbf{r}')$ can be written as

$$P(\mathbf{r},\mathbf{r}') = \frac{1}{2}\int_{-\infty}^{\infty} iw \int_V \frac{1}{\rho(\mathbf{r_s})}\nabla_s.\nabla_s \left(G^{\text{out}}(\mathbf{r}',\mathbf{r_s},w)G^{\text{in}}(\mathbf{r},\mathbf{r_s},w)\right)\, dV_s\, dw \tag{6.45}$$

$$+\frac{1}{2}\int_{-\infty}^{\infty} iw \int_V \frac{1}{\rho(\mathbf{r_s})}\delta(\mathbf{r}-\mathbf{r_s})G^{\text{out}}(\mathbf{r}',\mathbf{r_s},w) - \frac{1}{\rho(\mathbf{r_s})}\delta(\mathbf{r}'-\mathbf{r_s})G^{\text{in}}(\mathbf{r},\mathbf{r_s},w)\, dV_s\, dw. \tag{6.46}$$

We first deal with second term (6.46) of above expression. The Dirac delta function has the following property:

$$\int_V f(x)\delta(x-a) = \begin{cases} f(a), \text{ if } a \in V \\ 0, \quad\;\; \text{if } a \notin V \end{cases}. \tag{6.47}$$

Therefore, we obtain

$$\int_{-\infty}^{\infty} iw \int_V \frac{1}{\rho(\mathbf{r_s})}\delta(\mathbf{r}-\mathbf{r_s})G^{\text{out}}(\mathbf{r}',\mathbf{r_s}) - \frac{1}{\rho(\mathbf{r_s})}\delta(\mathbf{r}'-\mathbf{r_s})G^{\text{in}}(\mathbf{r},\mathbf{r_s})\, dV_s\, dw \tag{6.48}$$

$$= \int_{-\infty}^{\infty} iw \left(\frac{1}{\rho(\mathbf{r})}G^{\text{out}}(\mathbf{r}',\mathbf{r}) - \frac{1}{\rho(\mathbf{r}')}G^{\text{in}}(\mathbf{r},\mathbf{r}')\right)\, dw \tag{6.49}$$

$$= \frac{1}{\rho(\mathbf{r})}\int_{-\infty}^{\infty} iwG^{\text{out}}(\mathbf{r}',\mathbf{r})\, dw + \frac{1}{\rho(\mathbf{r}')}\left(\int_{-\infty}^{\infty} iwG^{\text{out}}(\mathbf{r},\mathbf{r}')\, dw\right)^* \tag{6.50}$$

$$= \frac{1}{\rho(\mathbf{r})}\pi c^2(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') + \frac{1}{\rho(\mathbf{r}')}\pi c^2(\mathbf{r}')\delta(\mathbf{r}'-\mathbf{r}) \tag{6.51}$$

$$= \pi\frac{c^2(\mathbf{r}')\rho(\mathbf{r}) + c^2(\mathbf{r})\rho(\mathbf{r}')}{\rho(\mathbf{r})\rho(\mathbf{r}')}\delta(\mathbf{r}-\mathbf{r}') \tag{6.52}$$

by using the result (4.11) obtained by the initial condition. Now, to explore the first term of $P(\mathbf{r},\mathbf{r}')$, we substitute the Green's functions in the expression for all location combinations of $\mathbf{r}$, $\mathbf{r}'$ and $\mathbf{r_s}$ in $V = \bigcup_i^n V_i$. Through calculations, we realize that the conditions $r_s > r$ and $r_s > r'$ make it easier to deal with the given integral. To satisfy these conditions, we again turn back to surface integral for the first part of $P(\mathbf{r},\mathbf{r}')$. We use again the argument that is the density function is constant on each layer and derive

$$
\begin{aligned}
&\int_{-\infty}^{\infty} iw \int_V \frac{1}{\rho(\mathbf{r_s})} \nabla_s . \left(\nabla_s (G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) G^{\text{in}}(\mathbf{r}, \mathbf{r_s})) \right) \, dV_s \, dw \\
&\int_{-\infty}^{\infty} iw \int_{S_N} \frac{1}{\rho(\mathbf{r_s})} \nabla_s (G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) G^{\text{in}}(\mathbf{r}, \mathbf{r_s})) . \mathbf{n_s} \, dS \, dw \\
&= \frac{1}{\rho_N} \int_{-\infty}^{\infty} iw \int_{S_N} \frac{\partial}{\partial r_s} \left(G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) G^{\text{in}}(\mathbf{r}, \mathbf{r_s}) \right) \, dS \, dw \\
&= \frac{1}{\rho_N} \int_{-\infty}^{\infty} iw \int_{-\infty}^{\infty} \int_0^{2\pi} \frac{\partial}{\partial r_s} \left(G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) G^{\text{in}}(\mathbf{r}, \mathbf{r_s}) \right) r_s \, d\phi_s \, dz_s \, dw \\
&= \frac{r_s}{\rho_N} \frac{\partial}{\partial r_s} \left[\int_{-\infty}^{\infty} iw \int_{-\infty}^{\infty} \int_0^{2\pi} \left(\int_{-\infty}^{\infty} e^{-ik_z(z'-z_s)} \sum_{n=-\infty}^{\infty} G^{\text{out}}(r', r_s, k_z) e^{-in(\phi'-\phi_s)} \, dk_z \right) \right. \\
&\times \left. \left(\int_{-\infty}^{\infty} e^{ik_z^*(z-z_s)} \sum_{m=-\infty}^{\infty} G^{\text{in}}(r, r_s, k_z^*) e^{im(\phi-\phi_s)} \, dk_z^* \right) d\phi_s \, dz_s \, dw \right] \\
&= \frac{r_s}{\rho_N} \frac{\partial}{\partial r_s} \left[\int_{-\infty}^{\infty} iw \int_{-\infty}^{\infty} \int_0^{2\pi} \left(\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ik_z(z'-z_s)} G^{\text{out}}(r', r_s, k_z) \, dk_z e^{-in(\phi'-\phi_s)} \right) \right. \\
&\times \left. \left(\sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ik_z^*(z-z_s)} G^{\text{in}}(r, r_s, k_z^*) \, dk_z^* e^{im(\phi-\phi_s)} \right) d\phi_s \, dz_s \, dw \right] \\
&= \frac{r_s}{\rho_N} \frac{\partial}{\partial r_s} \left[\int_{-\infty}^{\infty} iw \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left(\int_0^{2\pi} e^{im\phi' - in\phi} e^{i\phi_s(m-n)} \, d\phi_s \right) \right. \\
&\times \left. \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ik_z(z'-z_s)} G^{\text{out}}(r', r_s, k_z) e^{ik_z^*(z-z_s)} G^{\text{in}}(r, r_s, k_z^*) \, dw \right],
\end{aligned}
$$

since the normal derivative on a cylinder is equal to the derivative with respect to the variable r_s in cylindrical coordinates and the partial derivative operator is independent of integral variable w. By the orthogonality of exponential functions $\{e^{in\phi}\}$ on the interval $[0, 2\pi]$, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} iw \int_{S_2} \nabla_s G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) G^{\text{in}}(\mathbf{r}, \mathbf{r_s})\, dw \\
&= 2\pi \frac{r_s}{\rho_N} \frac{\partial}{\partial r_s} \sum_{n=-\infty}^{\infty} e^{in(\phi'-\phi)} \int_{-\infty}^{\infty} iw \\
&\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ik_z(z'-z_s)} G^{\text{out}}(r', r_s, k_z) e^{ik_z^*(z-z_s)} G^{\text{in}}(r, r_s, k_z^*)\, dk_z\, dk_z^*\, dw \\
&= 2\pi \frac{r_s}{\rho_N} \frac{\partial}{\partial r_s} \sum_{n=-\infty}^{\infty} e^{in(\phi'-\phi)} \int_{-\infty}^{\infty} iw \\
&\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ik_z z'} e^{ik_z^* z} G^{\text{out}}(r', r_s, k_z) G^{\text{in}}(r, r_s, k_z^*) \int_{-\infty}^{\infty} e^{iz_s(k_z - k_z^*)}\, dk_{z_s}\, dk_z\, dk_z^*\, dw \\
&= 2\pi \frac{r_s}{\rho_N} \frac{\partial}{\partial r_s} \sum_{n=-\infty}^{\infty} e^{in(\phi'-\phi)} \int_{-\infty}^{\infty} iw \\
&\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ik_z z'} e^{ik_z^* z} G^{\text{out}}(r', r_s, k_z) G^{\text{in}}(r, r_s, k_z^*) \delta(k_z - k_z^*)\, dk_z\, dk_z^*\, dw \\
&= 2\pi \frac{r_s}{\rho_N} \frac{\partial}{\partial r_s} \sum_{n=-\infty}^{\infty} e^{in(\phi'-\phi)} \\
&\times \int_{-\infty}^{\infty} iw \int_{-\infty}^{\infty} e^{-ik_z z'} e^{ik_z z} G^{\text{out}}(r', r_s, k_z) G^{\text{in}}(r, r_s, k_z)\, dk_z\, dw \\
&= 2\pi \frac{r_s}{\rho_N} \frac{\partial}{\partial r_s} \sum_{n=-\infty}^{\infty} e^{in(\phi'-\phi)} \\
&\times \int_{-\infty}^{\infty} e^{-ik_z(z'-z)} \int_{-\infty}^{\infty} iw G^{\text{out}}(r', r_s, k_z) G^{\text{in}}(r, r_s, k_z)\, dw\, dk_z. \qquad (6.53)
\end{aligned}$$

We know that Green's function depends on frequency variable w. For simplicity, we eliminate this variable in representation of Green's function. Now, we use w dependence of Green's function. Let

$$H(w) = G^{\text{out}}(r', r_s, k_z) G^{\text{in}}(r, r_s, k_z),$$

then

$$\begin{aligned}
&\int_{-\infty}^{\infty} iwG^{\text{out}}(r', r_s, k_z)G^{\text{in}}(r, r_s, k_z)\ dw \\
&= \int_{-\infty}^{\infty} iwH(w)\, dw && (6.54)\\
&= \int_{-\infty}^{0} iwH(w)\, dw + \int_{0}^{\infty} iwH(w)\, dw \\
&= \int_{\infty}^{0} i - w'H(-w') -\ dw' + \int_{0}^{\infty} iwH(w)\, dw \\
&= \int_{0}^{\infty} -iwH(-w)\, dw + \int_{0}^{\infty} iwH(w)\, dw \\
&= \int_{0}^{\infty} -iwH(w)^*\, dw + \int_{0}^{\infty} iwH(w)\, dw \\
&= \int_{0}^{\infty} iw\left(H(w) - H(w)^*\right)\ dw \\
&= \int_{0}^{\infty} iw2\boldsymbol{Im}(H(w)) && (6.55)
\end{aligned}$$

by the substitution $w = -w'$ in the integral $\int_{-\infty}^{0} iwH(w)\, dw$ and using the definition of negative frequency for wave functions. This result shows that the real part of the integrand has no contribution to the integral. Now, we substitute radial part of Green's function in (6.53). On S_N, the second variable $\mathbf{r_s}$ in G^{out} and G^{in} is an element of Region N. But $\mathbf{r}$ and $\mathbf{r}'$ are free to be in any region. Thus, we have N^2 cases for the combination of product terms depending on locations of points $\mathbf{r}$ and $\mathbf{r}'$: $G^{\text{out}}_{iN}(r', r_s)G^{\text{in}}_{jN}(r, r_s)$, for $1 \leq i, j \leq N$. We show that the products are purely real functions:

$$\begin{aligned}
G^{\text{out}}_{iN}(r', r_s)G^{\text{in}}_{jN}(r, r_s) &= \frac{R^i_n(r', r_s)\left(-iH^1_n(\mathbf{k}_N r_s)\right)}{\beta_n}.\frac{R^j_n(r, r_s)\left(-iH^2_n(\mathbf{k}_N r_s)\right)}{\overline{\beta_n}} \\
&= -\frac{2}{\pi\rho_1 a}\frac{R^i_n(r', r_s)R^j_n(r, r_s) \parallel H^1_n(\mathbf{k}_N r_s) \parallel}{\parallel \beta_n \parallel}.
\end{aligned}$$

In the derivation of Green's function, we proved that the functions $R^i_n(r, r_s)$ and $R^j_n(r, r_s)$ for any $1 \leq i, j \leq N$ are real and the modulus of any complex valued functions are real valued functions, which implies that all above products are real. Therefore, the integral (6.53) is equal to zero by (6.55).

Consequently, we obtain the function $P(\mathbf{r}, \mathbf{r}')$ as follows

$$\begin{aligned}
P(\mathbf{r}, \mathbf{r}') =& \int_{-\infty}^{\infty} iw \int_{S_N} G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) \nabla_s G^{\text{in}}(\mathbf{r}, \mathbf{r_s}).\mathbf{n_s}\, dS\, dw \\
=& \frac{1}{2} \int_{-\infty}^{\infty} iw \int_{S_N} \nabla_s \left(G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) G^{\text{in}}(\mathbf{r}, \mathbf{r_s}) \right) .\mathbf{n_s}\, dS\, dw \\
&+ \frac{1}{2} \int_{-\infty}^{\infty} iw \int_{V} \delta(\mathbf{r} - \mathbf{r_s}) G^{\text{out}}(\mathbf{r}', \mathbf{r_s}) - \delta(\mathbf{r}' - \mathbf{r_s}) G^{\text{in}}(\mathbf{r}, \mathbf{r_s})\, dV_s\, dw \\
=& \frac{\pi}{2} \frac{c^2(\mathbf{r}')\rho(\mathbf{r}) + c^2(\mathbf{r})\rho(\mathbf{r}')}{\rho(\mathbf{r})\rho(\mathbf{r}')} \delta(\mathbf{r} - \mathbf{r}').
\end{aligned}$$

Hence,

$$\begin{aligned}
q(\mathbf{r}) =& \int_{V'} p_0(\mathbf{r}') P(\mathbf{r}, \mathbf{r}')\, dV' \\
=& \int_{V'} p_0(\mathbf{r}') \frac{\pi}{2} \frac{c^2(\mathbf{r}')\rho(\mathbf{r}) + c^2(\mathbf{r})\rho(\mathbf{r}')}{\rho(\mathbf{r})\rho(\mathbf{r}')} \delta(\mathbf{r} - \mathbf{r}')\, dV' \\
=& \frac{\pi c^2(\mathbf{r})}{\rho(\mathbf{r})} p_0(\mathbf{r})
\end{aligned}$$

where $\rho(\mathbf{r})$ is a density function.

At the beginning, we suppose

$$q(\mathbf{r}) = \int_{-\infty}^{\infty} \int_{S_N} P(\mathbf{r_s}, w) \frac{1}{\rho(\mathbf{r_s})} \frac{\partial G_{iN}{}^{\text{in}}(\mathbf{r}, \mathbf{r_s})}{\partial \mathbf{n_s}}\, dS\, dw, \tag{6.56}$$

and therefore the source distribution $p_0(\mathbf{r})$ is given by

$$p_0(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\pi c^2(\mathbf{r})} \int_{-\infty}^{\infty} \int_{S_N} P(\mathbf{r_s}, w) \frac{1}{\rho(\mathbf{r_s})} \frac{\partial G_{iN}{}^{\text{in}}(\mathbf{r}, \mathbf{r_s})}{\partial \mathbf{n_s}}\, dS\, dw. \tag{6.57}$$

CHAPTER 7

7. GENERAL INTEGRAL REPRESENTATION OF INVERSE SOLUTION OF THERMOACOUSTIC WAVE EQUATION FOR ANY NONHOMOGENEOUS MEDIUM

If $\rho(\mathbf{r})$ is a density and $\kappa(\mathbf{r})$ is a compressibilty function of the medium to be imaged respectively, then thermoacoustic imaging process is represented mathematically by the following nonhomogeneous wave equation:

$$\nabla.\left(\frac{1}{\rho(\mathbf{r})}\nabla p(\mathbf{r},t)\right) - \kappa(\mathbf{r})\frac{\partial^2 p}{\partial t^2} = -\frac{1}{\rho(\mathbf{r})}p_0(\mathbf{r}).\delta'(t) \tag{7.1}$$

called thermoacoustic wave equation (Kuchment & Kunyansky, 2008; M. Xu & Wang, 2005; Ammari, 2008; L. V. Wang & Wu, 2007), Here, $p(\mathbf{r},t)$ is the acoustic pressure at position **r** and time t and $-p_0(\mathbf{r}).\delta'(t)$ is the source term.

Suppose there is a nonhomogeneous medium M in R^3 containing thermoacoustic sources and acoustic waves are known on an arbitrary smooth the surface S covering M. Our aim is to determine source distribuiton in the medium from the information known on surface S. Let $P(\mathbf{r},w)$ be pressure function's Fourier transform. Then the thermoacoustic wave equation can be expressed in a frequency domain as given below

$$\nabla.\left(\frac{1}{\rho(\mathbf{r})}\nabla P(\mathbf{r},w)\right) + w^2\kappa(\mathbf{r})P(\mathbf{r},w) = iw\frac{1}{\rho(\mathbf{r})}p_0(\mathbf{r}), \tag{7.2}$$

for any nonhomogeneous medium with continuity conditions

$$P(\mathbf{r},w)|_{\mathbf{r}\in D^-} = P(\mathbf{r},w)|_{\mathbf{r}\in D^+} \tag{7.3}$$

$$\frac{1}{\rho(\mathbf{r})}\left.\frac{\partial P(\mathbf{r},w)}{\partial n}\right|_{\mathbf{r}\in D^-} = \frac{1}{\rho(\mathbf{r})}\left.\frac{\partial P(\mathbf{r},w)}{\partial n}\right|_{\mathbf{r}\in D^+} \tag{7.4}$$

on each boundary D appearing in the space.

Radiation conditions must hold for a pressure function $P(\mathbf{r},w)$:

$$P(\mathbf{r},w) = \mathcal{O}\left(\frac{1}{|\mathbf{r}|}\right) \quad \text{as} \quad |\mathbf{r}| \to \infty \tag{7.5}$$

$$\frac{\partial P}{\partial |\mathbf{r}|} - ikP = \mathcal{O}\left(\frac{1}{|\mathbf{r}|^2}\right) \quad \text{as} \quad |\mathbf{r}| \to \infty. \tag{7.6}$$

In addition, because of the nature of the problem, the pressure function $p(\mathbf{r},t)$ must satisfy the following initial conditions

$$p(\mathbf{r},0^+) = c^2(\mathbf{r})p_0(\mathbf{r}) \tag{7.7}$$

$$p(\mathbf{r},t) = 0 \quad \text{if} \quad t < 0 \tag{7.8}$$

$$\frac{\partial p(\mathbf{r},0)}{\partial t} = 0. \tag{7.9}$$

7.1 Forward Solution of Thermoacoustic Wave Equation for Nonhomogeneous Media

Previously, we stated the forward solution of thermoacoustic wave equation. Here, we again give the proof of forward solution since it has a key role for an inverse solution. Suppose $G^{out}(\mathbf{r},\mathbf{r}',w)$ is the outgoing Green's function in frequency domain describing existing nonhomogeneous smooth medium. Here, $\mathbf{r}$ and $\mathbf{r}'$ are locations of observation point and source point, respectively. The pressure function $P(\mathbf{r},w)$ and the Green's function $G^{out}(\mathbf{r}, \mathbf{r'}, w)$ satisfy the following equations, respectively:

$$\nabla . \left(\frac{1}{\rho(\mathbf{r})} \nabla P(\mathbf{r}, w) \right) + w^2 \kappa(\mathbf{r}) P(\mathbf{r}, w) = iw \frac{1}{\rho(\mathbf{r})} p_0(\mathbf{r}), \tag{7.10}$$

$$\nabla . \left(\frac{1}{\rho(\mathbf{r})} \nabla G^{out}(\mathbf{r}, \mathbf{r'}, w) \right) + w^2 \kappa(\mathbf{r}) G^{out}(\mathbf{r}, w) = -\delta(\mathbf{r} - \mathbf{r}') \tag{7.11}$$

where $\delta(.)$ is the Dirac delta function.

If we multiply the first equation by $G^{out}(\mathbf{r}, \mathbf{r}', w)$ and the second equation by $P(\mathbf{r}, w)$ and subtract each other, we obtain

$$\begin{aligned} &\nabla . \left(\frac{1}{\rho(\mathbf{r})} \nabla P(\mathbf{r}, w) \right) G^{out}(\mathbf{r}, \mathbf{r}', w) - \nabla . \left(\frac{1}{\rho(r)} \nabla G^{out}(\mathbf{r}, \mathbf{r}', w) \right) P(\mathbf{r}, w) \\ &= iw \frac{1}{\rho(\mathbf{r})} p_0(\mathbf{r}) G^{out}(\mathbf{r}, \mathbf{r}', w) + \delta(\mathbf{r} - \mathbf{r}') P(\mathbf{r}, w). \end{aligned} \tag{7.12}$$

Now, we take the volume integral of both sides over the unit ball $B(\mathbf{r})$ with radius r big enough, so as it contains all possible sources in space:

$$\begin{aligned} &\int_{B(\mathbf{r})} \nabla . \left(\frac{1}{\rho(\mathbf{r})} \nabla P(\mathbf{r}, w) \right) G^{out}(\mathbf{r}, \mathbf{r}', w) - \nabla . \left(\frac{1}{\rho(r)} \nabla G^{out}(\mathbf{r}, \mathbf{r}', w) \right) P(\mathbf{r}, w) \, dV \\ &= \int_{B(\mathbf{r})} iw \frac{1}{\rho(\mathbf{r})} p_0(\mathbf{r}) G^{out}(\mathbf{r}, \mathbf{r}', w) + \delta(\mathbf{r} - \mathbf{r}') P(\mathbf{r}, w) \, dV \end{aligned}$$

which implies

$$\begin{aligned} &\int_{B(\mathbf{r})} \nabla . \left(\frac{1}{\rho(\mathbf{r})} \nabla P(\mathbf{r}, w) \right) G^{out}(\mathbf{r}, \mathbf{r}', w) - \nabla . \left(\frac{1}{\rho(\mathbf{r})} \nabla G^{out}(\mathbf{r}, \mathbf{r}', w) \right) P(\mathbf{r}, w) \, dV \\ &= P(\mathbf{r}', w) + \int_{B(\mathbf{r})} iw \frac{1}{\rho(\mathbf{r})} p_0(\mathbf{r}) G^{out}(\mathbf{r}, \mathbf{r}', w) \, dV. \end{aligned} \tag{7.13}$$

By adding and subtracting the term $\dfrac{1}{\rho(\mathbf{r})}\nabla P(\mathbf{r},w)\nabla G^{out}(\mathbf{r},\mathbf{r}',w)$ in left hand side of above equation, we obtain

$$
\begin{aligned}
&\int_{B(\mathbf{r})} \nabla.\left(\frac{1}{\rho(\mathbf{r})}\nabla P(\mathbf{r},w)\right) G^{out}(\mathbf{r},\mathbf{r}',w) - \nabla.\left(\frac{1}{\rho(r)}\nabla G^{out}(\mathbf{r},\mathbf{r'},w)\right) P(\mathbf{r},w)\,dV \\
=&\int_{B(\mathbf{r})} \nabla.\left(\frac{1}{\rho(\mathbf{r})}\nabla P(\mathbf{r},w)\right) G^{out}(\mathbf{r},\mathbf{r}',w) + \frac{1}{\rho(\mathbf{r})}\nabla P(\mathbf{r},w)\nabla G^{out}(\mathbf{r},\mathbf{r}',w) \\
&- \nabla P(r,w)\frac{1}{\rho(\mathbf{r})}\nabla G^{out}(\mathbf{r},\mathbf{r}',w) - \nabla.\left(\frac{1}{\rho(\mathbf{r})}\nabla G^{out}(\mathbf{r},\mathbf{r}',w)\right) P(\mathbf{r},w)\,dV \\
=&\int_{B(\mathbf{r})} \nabla\left(\frac{1}{\rho(\mathbf{r})}\nabla P(\mathbf{r},w) G^{out}(\mathbf{r},\mathbf{r}',w)\right) - \nabla\left(P(\mathbf{r},w)\nabla\frac{1}{\rho(\mathbf{r})}G^{out}(\mathbf{r},\mathbf{r}',w)\right)\,dV.
\end{aligned}
$$

The integrand appearing in the last expression is continuous since the pressure function and Green's function must satisfy continuity boundary conditions (7.3) and (7.4). Hence by using Green's Theorem and the equality (7.19), we get

$$
\begin{aligned}
&\int_{S(\mathbf{r})} \left(\frac{1}{\rho(\mathbf{r})}\nabla P(\mathbf{r},w) G^{out}(\mathbf{r},w) - P(\mathbf{r},w)\frac{1}{\rho(\mathbf{r})}\nabla G(\mathbf{r},\mathbf{r'},w)\right).\mathbf{n}\,dS \\
&= P(\mathbf{r}',w) + iw\int_{B(\mathbf{r})}\frac{1}{\rho(\mathbf{r})}p_0(\mathbf{r})G^{out}(\mathbf{r},\mathbf{r}',w)\,dV
\end{aligned}
$$

where $S(\mathbf{r})$ is the surface of unit ball.

Now, if we add and substract the term $\dfrac{1}{\rho(\mathbf{r})}ikP(r,w)G^{out}(\mathbf{r},\mathbf{r}',w)$ in left hand side of above expression and let the radius of unit ball goes to infinity, the left hand side goes to zero by radiation condition (7.5). Thus, we obtain the forward solution

$$
P(\mathbf{r}',w) = -iw\int_V \frac{1}{\rho(\mathbf{r})}p_0(\mathbf{r})G^{out}(\mathbf{r},\mathbf{r}',w)\,dV \tag{7.14}
$$

in which V is the support of source function.

7.2 Inverse Solution of Thermoacoustic Wave Equation for Nonhomogeneous Media

In an inverse problem, the source function $p_0(r)$ is to be determined by pressure function $P(\mathbf{r}, w)$ known on a measurement surface S. Now, we suppose that $P(\mathbf{r}, w)$ is known on an arbitrary smooth surface S and G^{in} is the incoming Green's function satisfying the Helmholtz equation (7.2) and the radiation condition (7.6).

To obtain the inverse solution, we follow the the similar steps used in deriving the forward solution: The pressure function and incoming Green's function satisfies the following equations, respectively:

$$\nabla . \left(\frac{1}{\rho(\mathbf{r})} \nabla P(\mathbf{r}, w) \right) + w^2 \kappa(\mathbf{r}) P(\mathbf{r}, w) = iw \frac{1}{\rho(\mathbf{r})} p_0(\mathbf{r}) \tag{7.15}$$

$$\nabla . \left(\frac{1}{\rho(\mathbf{r})} \nabla G^{in}(\mathbf{r}, \mathbf{r}', w) \right) + w^2 \kappa(\mathbf{r}) G^{in}(\mathbf{r}, \mathbf{r}', w) = -\delta(\mathbf{r} - \mathbf{r}'). \tag{7.16}$$

If we multiply first equation by $G^{in}(\mathbf{r}, \mathbf{r}', w)$ and the second equation by $P(\mathbf{r}, w)$ and subtract each other, we obtain

$$\nabla . \left(\frac{1}{\rho(\mathbf{r})} \nabla P(\mathbf{r}, w) \right) G^{in}(\mathbf{r}, \mathbf{r}', w) - \nabla . \left(\frac{1}{\rho(\mathbf{r})} \nabla G^{in}(\mathbf{r}, \mathbf{r}', w) \right) P(\mathbf{r}, w) \tag{7.17}$$

$$= iw \frac{1}{\rho(\mathbf{r})} p_0(\mathbf{r}) G^{in}(\mathbf{r}, \mathbf{r}', w) + \delta(\mathbf{r} - \mathbf{r}') P(\mathbf{r}, w). \tag{7.18}$$

Now, rather than an arbitrary unit ball used in deriving forward solution, we take the integral of both sides over the volume V covered by measurement surface S for inverse solution:

$$\int_V \nabla . \left(\frac{1}{\rho(\mathbf{r})} \nabla P(\mathbf{r}, w) \right) G^{in}(\mathbf{r}, \mathbf{r}', w) - \nabla . \left(\frac{1}{\rho(\mathbf{r})} \nabla G^{in}(\mathbf{r}, \mathbf{r}', w) \right) P(\mathbf{r}, w) \, dV$$
$$= \int_V iw \frac{1}{\rho(\mathbf{r})} p_0(\mathbf{r}) G^{in}(\mathbf{r}, w) + \delta(\mathbf{r} - \mathbf{r}') P(\mathbf{r}, w) \, dV$$

which implies

$$\int_V \nabla.\left(\frac{1}{\rho(\mathbf{r})}\nabla P(\mathbf{r},w)\right)G^{in}(\mathbf{r},\mathbf{r}',w) - \nabla.\left(\frac{1}{\rho(\mathbf{r})}\nabla G^{in}(\mathbf{r},\mathbf{r}',w)\right)P(\mathbf{r},w)\,dV$$
$$= P(\mathbf{r}',w) + iw\int_V \frac{1}{\rho(\mathbf{r})}p_0(\mathbf{r})G^{in}(\mathbf{r},\mathbf{r}',w)\,dV. \qquad (7.19)$$

By adding and subtracting the term $\frac{1}{\rho(r)}\nabla P(\mathbf{r},w)\nabla G^{in}(\mathbf{r},\mathbf{r}',w)$ in left hand side of above equation, we obtain

$$\int_V \nabla.\left(\frac{1}{\rho(\mathbf{r})}\nabla P(\mathbf{r},w)\right)G^{in}(\mathbf{r},w) - \nabla.\left(\frac{1}{\rho(\mathbf{r})}\nabla G^{in}(\mathbf{r},\mathbf{r}',w)\right)P(\mathbf{r},w)\,dV$$
$$= \int_V \nabla.\left(\frac{1}{\rho(\mathbf{r})}\nabla P(\mathbf{r},w)\right)G^{out}(\mathbf{r},\mathbf{r}',w) + \frac{1}{\rho(\mathbf{r})}\nabla P(\mathbf{r},w)\nabla G^{in}(\mathbf{r},\mathbf{r}',w)$$
$$- \nabla P(\mathbf{r},w)\frac{1}{\rho(\mathbf{r})}\nabla G^{in}(\mathbf{r},\mathbf{r}',w) - \nabla.\left(\frac{1}{\rho(r)}\nabla G^{in}(\mathbf{r},\mathbf{r}',w)\right)P(\mathbf{r},w)\,dV$$
$$= \int_V \nabla\left(\frac{1}{\rho(\mathbf{r})}\nabla P(\mathbf{r},w)G^{in}(\mathbf{r},\mathbf{r}',w)\right) - \nabla\left(P(\mathbf{r},w)\nabla\frac{1}{\rho(\mathbf{r})}G^{in}(\mathbf{r},\mathbf{r}',w)\right)\,dV.$$

The integrand appearing in the last expression is continuous, since the pressure function and Green's function must satisfy continuity boundary conditions (7.3) and (7.4). Hence by using Green's theorem and the equality (7.19), we get

$$\int_S \left(\frac{1}{\rho(\mathbf{r})}\nabla P(\mathbf{r},w)G^{in}(\mathbf{r},\mathbf{r}',w) - P(r,w)\frac{1}{\rho(\mathbf{r})}\nabla G^{in}(\mathbf{r},\mathbf{r}',w)\right).\mathbf{n}\,dS$$
$$= P(\mathbf{r}',w) + iw\int_V \frac{1}{\rho(\mathbf{r})}p_0(\mathbf{r})G^{in}(\mathbf{r},\mathbf{r}',w)\,dV$$

where S is the measurement surface. The main difference in solution of forward and inverse solution starts at this step. Now, we know that incoming Green's function is a complex conjugate of outgoing Green's function, that is $G^{in} = (G^{out})^*$ and also

$P(\mathbf{r}, \mathbf{r}', -w) = P^*(\mathbf{r}, \mathbf{r}', w)$ for any frequency value w. Therefore, we obtain the following equalities

$$\begin{aligned}
&\int_S \left(\frac{1}{\rho(\mathbf{r})} \nabla P(\mathbf{r}, w) G^{out}(\mathbf{r}, \mathbf{r}', w) - P(r, w) \frac{1}{\rho(\mathbf{r})} \nabla G^{in}(\mathbf{r}, \mathbf{r}', w) \right) .\mathbf{n}\, dS \\
&= P(\mathbf{r}', w) + iw \int_V \frac{1}{\rho(\mathbf{r})} p_0(\mathbf{r}) G^{in}(\mathbf{r}, \mathbf{r}', w)\, dV \\
&= P(\mathbf{r}', w) + \left(-iw \int_V \frac{1}{\rho(\mathbf{r})} p_0(\mathbf{r}) G^{out}(\mathbf{r}, \mathbf{r}', w)\, dV \right)^* \\
&= P(\mathbf{r}', w) + P(\mathbf{r}', w)^*
\end{aligned}$$

by expression of forward solution. Now, we multiply both sides by e^{-iwt} and take the integral of both sides with respcet to variable w:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_S \left(\frac{1}{\rho(\mathbf{r})} \nabla P(\mathbf{r}, w) G^{in}(\mathbf{r}, \mathbf{r}', w) - P(\mathbf{r}, w) \frac{1}{\rho(\mathbf{r})} \nabla G^{in}(\mathbf{r}, \mathbf{r}', w) \right) e^{-iwt} .\mathbf{n}\, dS\, dw \\
&= \int_{-\infty}^{\infty} P(\mathbf{r}', w) e^{-iwt}\, dw + \int_{-\infty}^{\infty} P(\mathbf{r}', w)^* e^{-iwt}\, dw \\
&= \int_{-\infty}^{\infty} P(\mathbf{r}', w) e^{-iwt}\, dw - \int_{\infty}^{-\infty} P(\mathbf{r}', -w)^* e^{iwt}\, dw \\
&= \int_{-\infty}^{\infty} P(\mathbf{r}', w) e^{-iwt}\, dw + \int_{-\infty}^{\infty} P(\mathbf{r}', w) e^{iwt}\, dw
\end{aligned}$$

by using the substitution $w = -w$ in the integral including $P(\mathbf{r}', w)^*$. If we evaluate $t = 0$ at both sides, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_S \left(\frac{1}{\rho(\mathbf{r})} \nabla P(\mathbf{r}, w) G^{in}(\mathbf{r}, \mathbf{r}', w) - P(\mathbf{r}, w) \frac{1}{\rho(\mathbf{r})} \nabla G^{in}(\mathbf{r}, \mathbf{r}', w) \right) .\mathbf{n}\, dS\, dw \\
&= \int_{-\infty}^{\infty} P(\mathbf{r}', w)\, dw + \int_{-\infty}^{\infty} P(\mathbf{r}', w)\, dw \\
&= 2 \int_{-\infty}^{\infty} P(\mathbf{r}', w)\, dw \\
&= 2\pi \left(p(\mathbf{r}, 0^-) + p(\mathbf{r}, 0^+) \right) \\
&= 2\pi c^2(\mathbf{r}) p_0(\mathbf{r}')
\end{aligned}$$

by inverse Fourier transform properties and discontinuity at $t = 0$. Thus, we derive an inverse solution as

$$
\begin{aligned}
p_0(\mathbf{r}') &= \frac{1}{2\pi c^2(\mathbf{r})} \\
&\times \int_{-\infty}^{\infty} \int_S \left(\frac{1}{\rho(\mathbf{r})} \nabla P(\mathbf{r}, w) G^{in}(\mathbf{r}, \mathbf{r}', w) - P(\mathbf{r}, w) \frac{1}{\rho(\mathbf{r})} \nabla G^{in}(\mathbf{r}, \mathbf{r}', w) \right) .\mathbf{n}\, dS_{\mathbf{r}}\, dw
\end{aligned}
\tag{7.20}
$$

where $P(\mathbf{r}, w)$ is a pressure function known on the measurement surface S, $\rho(r)$ is a density function and $G^{in}(\mathbf{r}, \mathbf{r}', w)$ is the Green's function corresponding to nonhomogeneous media.

CHAPTER 8

8. RESULTS AND DISCUSSION

8.1 Numerical Simulations

We test and compare our layered solution with conventional solution based on ho-

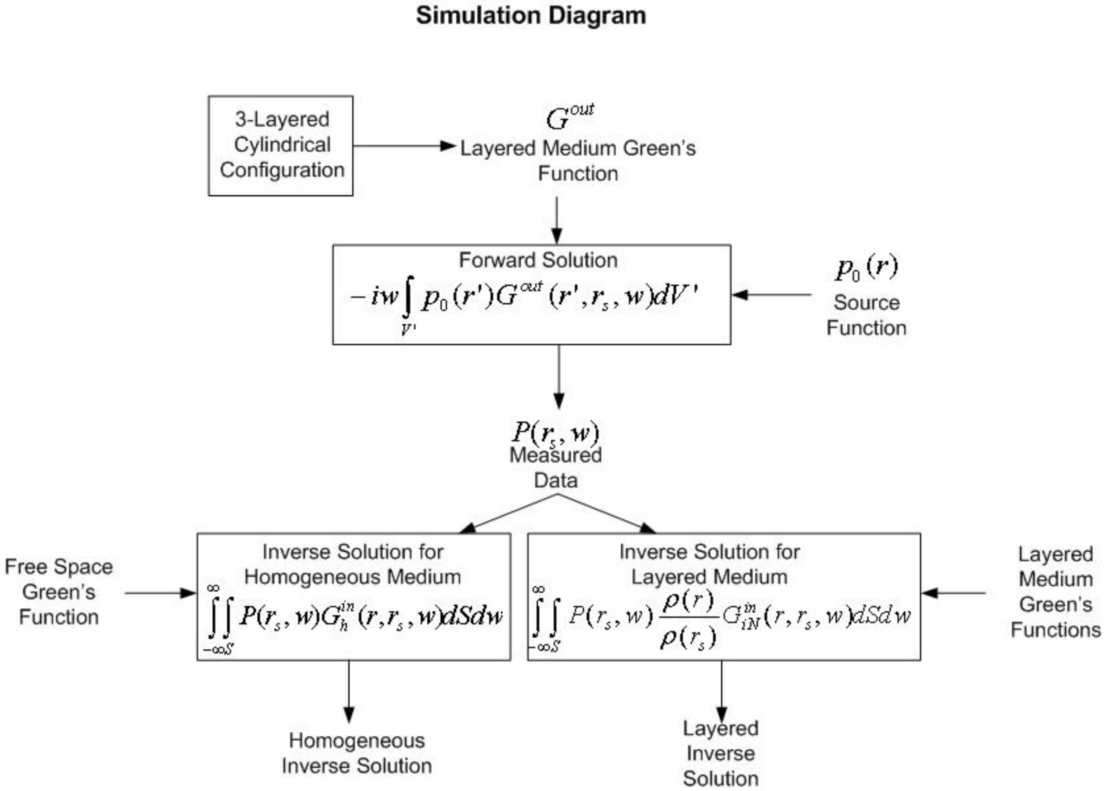

Figure 8.1 Simulation Diagram

mogeneous medium assumption by performing simulations using numerical test phantoms. We firstly study on two dimensional two layered phantoms and after that we study on a cross section of three layered cylindrical region phantoms as an example of N-layered structure in three dimensions. In the simulations, we generate synthetic data by using layered medium Green's function in forward solution (4.9) of thermoacoustic wave equation. For this purpose, we choose 3 MHz temporal frequency band between 1.5 MHz and 4.5 MHz, and collect data by the 512 element transducer located on a circle $r = 7.5mm$ in outmost layer. Then we reconstruct the thermoacoustic source distribution from this data using the existing homogeneous inverse solution (3.9) including free space Green's functions and our layered inverse solution (3.10) including layered medium Green's functions. Here we firstly present an illustrative test results for a cross section of three layered cylindrical region in Figure 8.2 and Figure 8.3, where the numerical phantom and the reconstructed inverse source distributions (thermoacoustic images of point targets) are displayed. Three layered phantoms are depicted at first panels of figures. In Figure 8.2, first layer is the region $0mm \leq r \leq 2.5mm$, second layer is the region $2.5mm \leq r \leq 5mm$ and third layer is the region $r \geq 5mm$. Densities and acoustic speeds for layers are choosen as 1.06 g/m^3, 0.95 g/m^3, 1 g/m^3 and 1000 m/s, 1500 m/s, 2000 m/s from inner to outer. This phantom consists of thermoacoustic point sources at each layer, their polar coordinates are (1,25 mm, 0), (3,75 mm, $5\pi/4$) and (6,25 mm, $2\pi/3$). In Figure 8.3, we model breast as three main layers: Glandular tissue is the region $0mm \leq r \leq 6.75mm$, fat tissue is the region $6.75mm \leq r \leq 7.35mm$ and skin is the region $7.35mm \leq r \leq 7.5mm$ considering ratios of actual thicknesses of these breast layers. Densities and acoustic speeds for breast layers are taken as 1 g/m^3, 0.95 g/m^3, 1.15 g/m^3 and 1480 m/s, 1450 m/s, 1730 m/s respectively. This phantom consists of thermoacoustic point sources at glandular tissue layer, their polar coordinates are (3,4 mm, 0) and (5,4 mm, 0). The middle panel shows us that reflections and refractions on layer boundaries cause smearing and morphological deformation of image because of homogeneous assumption in inversion algorithm. Hence, the test results ensure us that the homogeneous medium assumption, as expected, produces incorrect source locations and poor point-spread-functions with severe side-lobes associated with the original point sources. Our layered solution produces source locations correctly and point-spread-functions with relatively narrower main-lobe and lower side-lobes.

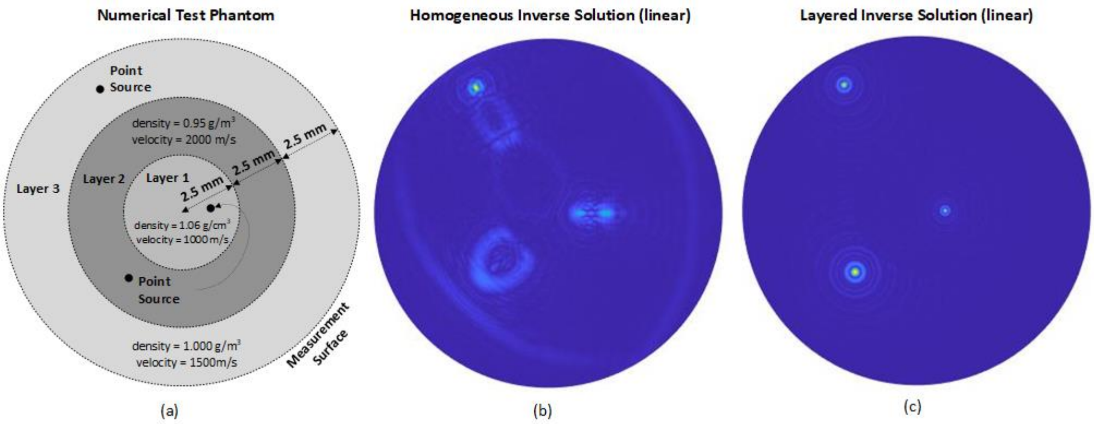

Figure 8.2 (a) Test phantom (b) Numerical simulation obtained under homogeneous medium assumption (c) Numerical simulation obtained for layered medium, showing correct source locations

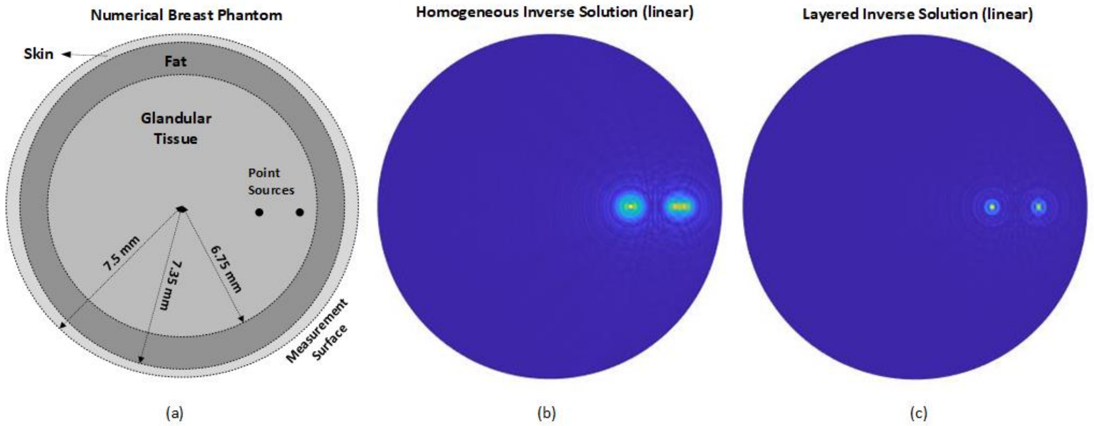

Figure 8.3 (a) Breast phantom (b) Numerical simulation obtained under homogeneous medium assumption (c) Numerical simulation obtained for layered medium, showing correct source locations

We also test our inverse solution for the capability in measuring the strength of sources by using same layer properties, frequency band and sampling rates with above simulation. We locate two point sources at coordinates (1,25 mm, 0) and (1,25 mm, π) with amplitude values 1 and 10, respectively. In the simulations, again, we generate synthetic data by using layered medium Green's function in forward solution (4.9) of thermoacoustic wave equation and reconstruct the thermoacoustic source distribution from this data using layered inverse solution (6.34) including layered medium Green's functions. The test phantom and numerical simulation results are depicted in

Figure 8.4. Simulation results show the inverse solution is accurate in distinguishing different source strengths.

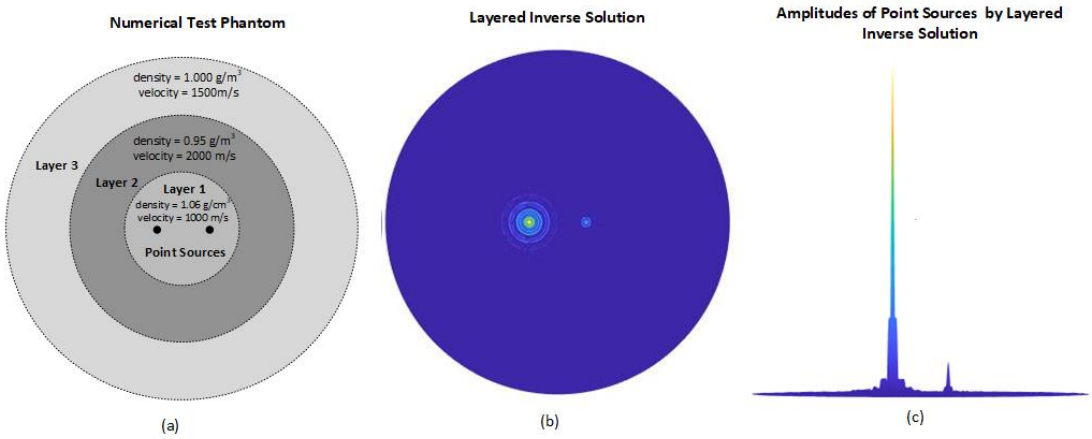

Figure 8.4 (a) Test phantom (b) Numerical simulation obtained by layered medium inverse solution (c) Numerical simulation showing source strengths

Now, we mention about two layered phantoms test results comparing with three layered phantoms. In simulations, we first get an inconsistency in radiation pattern of point spread function for two layer and three layer mediums seen in Figure 8.5 and Figure 8.6:

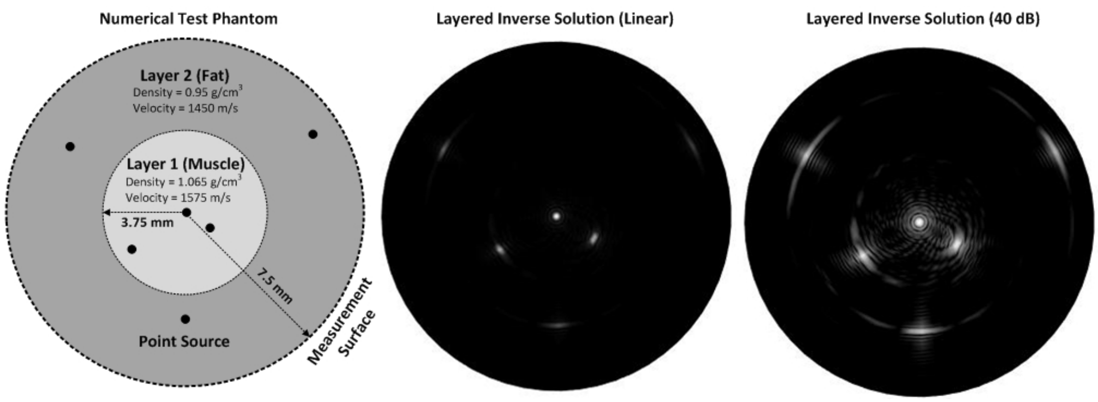

Figure 8.5 Radiation Pattern in Two Layer Media with fixed N value

In two layer's simulation, point spread function seems to like arcly radiated, but in three layer's simulation, it seems to like circularly radiated. Green's function in polar and cylindrical coordinates consists of infinite sum of Bessel functions. The main differences between two programs simulating two and three layered mediums are the

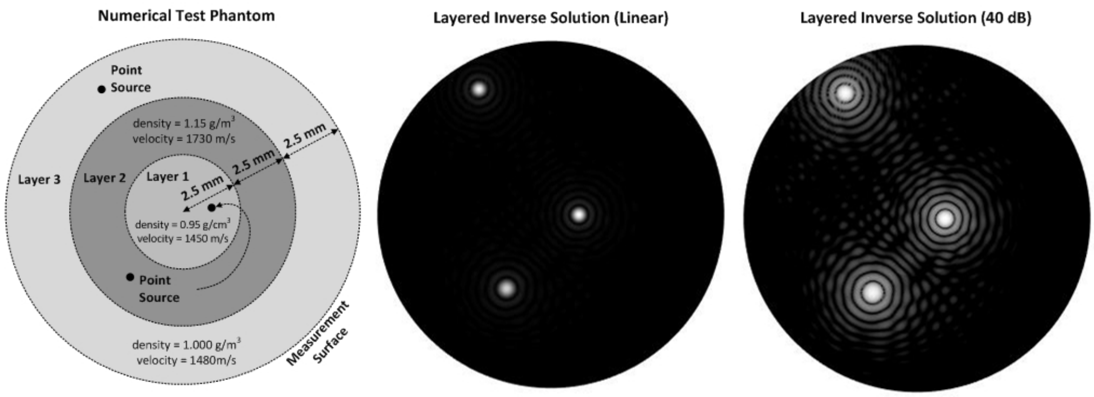

Figure 8.6 Radiation Pattern in Three Layer Media

limit N of sum of Bessel functions used in Matlab program. In two layer, the number N is fixed. But in three layer, N is determined with respect to the frequency value. In simulations, we realize that the fixed number used in two layered medium is not enough to get a good approximation of an infinite sum. At this point, in our layered solution in Matlab program we take all parameters equal for each layer so that obtaining homogeneous medium solution and look for a finite number N which satisfies a good approximation to Bessel's addition theorem. Then we run our two layer programs with new N numbers and conclude that point spread function radiated circularly. Some simulation results are given in Figure 8.7:

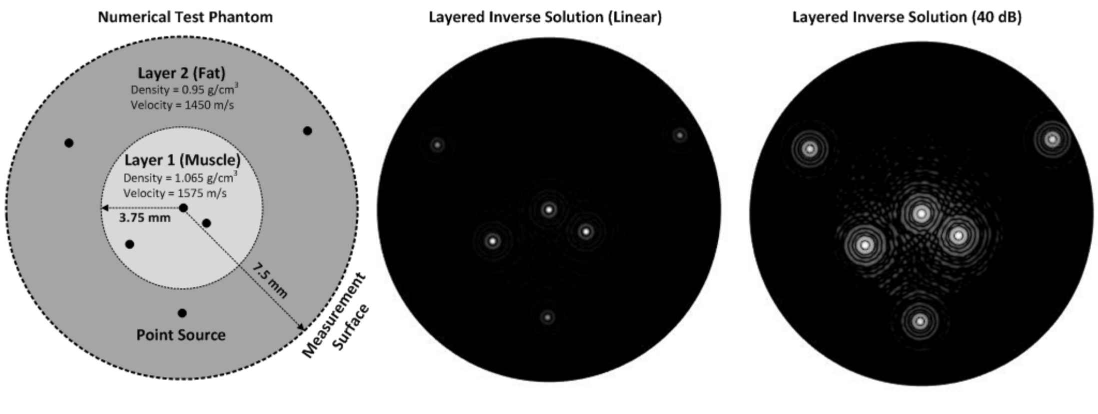

Figure 8.7 Radiation Pattern in Two Layer Media with adapted N value

Also, we simulate the phantoms containing volume sources. The numerical test results in Figure 8.8 show that the layered solution enable reconstruction of both low and high contrast cyst-like structures with more accurate features compared to the homogenous solution.

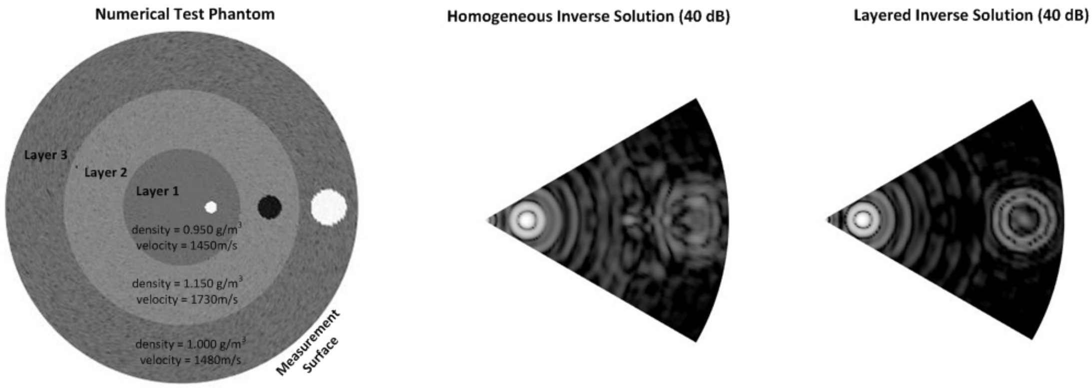

Figure 8.8 Volume Sources Simulations

The limitation of the proposed inverse solution for thermoacoustic imaging is need of tissue properties and structures as apriori information. But, these informations can be obtained from acquired thermoacoustic data or additional transmission ultrasound scan. The error in apriori information of tissue structures will reduce the image quality but this effect can be minimized by some iterative methods.

CHAPTER 9

9. CONCLUSION

In this study, we have considered the inverse source problem for thermoacoustic wave equation in layered circular and cylindrical models. We have derived an exact analytic inverse solution in frequency domain under boundary conditions. Also, the derived solution was tested in a three-layer numerical tissue models. The solution presented here is a suitable approach for cross-sectional imaging of cylindrical and spherical structures (such as breast) to get better image quality. The general integral solution may not ease the application of tomography, since more complex media means more complex Green's function. But it represents an exact inverse solution to the thermoacoustic wave equation.

REFERENCES

Agranovsky, M., & Kuchment, P. (2007). Uniqueness of reconstruction and an inversion procedure for thermoacoustic and photoacoustic tomography with variable sound speed. *Inverse Problems*, *23*(5), 2089-2102.

Ammari, H. (2008). *An introduction to mathematics of emerging biomedical imaging*. Berlin: Springer.

Anastasio, M. A., Zhang, J., & Pan, X. (2005). Image reconstruction in thermoacoustic tomography with compensation for acoustic heterogeneities. *Proceedings of the SPIE*, *5750*, 298-304.

Baranski, S., & Czerski, P. (1976). *Interaction of microwaves with living systems. biologic material-microwave properties*. Warsaw: Wiley.

Bell, A. G. (1880). On the production and reproduction of sound by light. *Am. J. Sci.*, *20*(3), 305-324.

Boniol, P., Ferlay, J., Autier, P., Heanue, M., Colombet, M., & Boyle, P. (2007). Estimates of the cancer incidence and mortality in europe in 2006. *Annals of Oncology*, *18*, 581-592.

Boyle, P., & Ferlay, J. (2005). Cancer incidence and mortality in europe 2004. *Annals of Oncology*, *18*, 481-488.

Council, N. R. (1996). *Mathematics and physics of emerging biomedical imaging*. Washington: The National Academies Press.

Foster, K. R., & Arkhipov, N. (1974). Microwave hearing: evidence for thermoacoustic auditory stimulation by pulsed microwaves. *Science*, *185*(147), 256-258.

Guy, C., & Fytche, D. (2005). *An introduction to the principles of medical imaging*. London: Imperial College Press.

Hristova, Y., Kuchment, P., & Nyugen, L. (2008). Reconstruction and time reversal in thermoacoustic tomograhy in acoustically homogeneous and inhomogeneous media. *Inverse Problems*, *24*(5), 055006.

İdemen, M., & Alkumru, A. (2012). On an inverse source problem connected with photo-acoustic and thermo-acoustic tomographies. *Wave Motion*, *49*(6), 595-604.

Karabutov, A. A., & Gusev, V. E. (1993). *Laser optoacoustics*. Newyork: Amer. Inst. Phys.

Kruger, R. A., Kiser, W. L., Miller, K. D., Reynolds, H. E., Reinecke, D. R., Kruger, G. A., & Hofacker, P. J. (2000). Thermoacoustic CT: Imaging principles. In *Proc. spie 3916, biomedical optoacoustics.* SPIE.

Kuchment, P., & Kunyansky, L. (2008). Mathematics of thermoacoustic tomography. *European J. of Appl. Math*, *19*, 191-224.

Lihong, V., Xu, M., & Wang, L. V. (2006). Photoacoustic imaging in biomedicine. *Rewiev of Scientific Instruments*, *77*(4), 041101-1-041101-22.

Liu, S., Lu, Y., Zhu, X., & Jin, H. (2017). Measurement matrix uncertainty model-based microwave induced thermoacoustic sparse reconstruction in acoustically heterogeneous media. *Applied Physics Letters*, *119*(26), 263701.

Olsen, R. G. (1982). *Generation of acoustic images from the absorption of pulsed microwave energy*. Newyork: Plenum Publishing.

Qian, J., Stefanov, P., Uhlman, G., & Zhao, H. (2011). A new numerical algorithn for thermoacoustic and photoacoustic tomography with variable sound speed. *Comput. Appl. Math*, 10-81.

Schoonover, R. W., & Anastasio, M. A. (2011). Image reconstruction in photoacoustic tomography involving layered acoustic media. *J. Opt. Soc. Am. A Opt Image Sci Vis.*, *28*(6), 1114-1120.

Siegel, R. L., Miller, K. D., Fuchs, H. E., & Jemal, A. (2021). Cancer statistics, 2021. *Cancer J. Clin.*, *71*, 7-33.

Stakgold, I. (1979). *Green's function and boundary value problems*. New York: Wiley.

Stefanov, P., & Uhlman, G. (2009). Thermoacustis tomography with variable sound speed. *Inverse Problems*, *25*(7), 075011.

Wang, B., Zhao, Z., Liu, S., Nie, Z., & Liu, Q. (2017). Mitigating acoustic heterogeneous effects in microwave-induced breast thermoacoustic tomography using multi-physical k-means clustering. *Applied Physics Letters*, *111*(22), 223701.

Wang, J., Zhao, Z., Song, J., Chen, G., Nie, Z., & Liu, Q.-H. (2015). Reducing the effects of acoustic heterogeneity with an iterative reconstruction method from experimental data in microwave induced thermoacoustic tomography. *Medical*

Physics, *42*(5), 2103-2112.

Wang, L. V. (2003). Ultrasound-mediated biophotonic imaging: A review of acousto-optical tomography and photo-acoustic tomography. *IOS Press*, *19*, 123-138.

Wang, L. V., & Wu, H. (2007). *Biomedical optics principles and imaging*. USA: Wiley.

Wang, L. V., & Yang, X. (2007). Boundary conditions in photoacoustic tomography and image reconstruction. *Journal of Biomedical Optics*, *12*(1), 014027-1-014027-10.

Xu, M., & Wang, L. V. (2002). Time domain reconstruction for thermoacoustic tomography in spherical geometry. *IEEE Transactions on Medical Imaging*, *21*(7), 814-822.

Xu, M., & Wang, L. V. (2005). Universal back-projection algorithm for photoacoustic computed tomography. *Physical Review E*, *71*(1), 016706-1-016706-7.

Xu, Y., & Wang, L. V. (2003). Effects of acoustic heterogeneity in breast thermoacoustic tomography. *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control*, *50*(9), 1134-1146.

Xu, Y., Xu, M., & Wang, L. V. (2002a). Exact frequency-domain reconstruction for thermoacoustic tomography-II: Cylindrical geometry. *IEEE Transactions on Medical Imaging*, *21*(7), 829-833.

Xu, Y., Xu, M., & Wang, L. V. (2002b). Exact frequency-domain reconstruction for thermoacoustic tomography-I: Planar geometry. *IEEE Transactions on Medical Imaging*, *21*(7), 823-828.

RESUME