TWMS J. App. and Eng. Math. V.12, N.4, 2022, pp. 1247-1258

SECOND HANKEL DETERMINANT FOR A CLASS OF ANALYTIC FUNCTIONS OF THE MITTAG-LEFFLER-TYPE BOREL DISTRIBUTION RELATED WITH LEGENDRE POLYNOMIALS

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Abstract. In this paper, we obtain the Fekete-Szegö inequalities for the functions of complex order connected with the Mittag-Leffler-type Borel distribution based upon the Legendre polynomials. Also, find upper bounds of the second Hankel determinant $\left|a_2a_4 - a_3^2\right|$ for functions belonging to the class $\mathcal{M}_\gamma^\eta(\lambda, \alpha, \beta, x)$.

Keywords: Fekete-Szegö inequality, Second Hankel determinant, the Mittag-Leffler-functions, Borel distribution, Legendre polynomials, Complex order.

AMS Subject Classification: Primary 30C45, 11B37, Secondary 47B35, 47B38.

1. Introduction

Denote $\mathcal{A}$ the family of analytic functions whose members are

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, (\Delta = \{z : |z| < 1,\ z \in \mathbb{C}\} \tag{1}$$

with the normalization condition $f(0) = 0 = f'(0) - 1$, and $\mathcal{S}$ be the subclass of $\mathcal{A}$, which are univalent functions. Furthermore, let $\mathcal{P}$ be the family of functions $p(z) \in \mathcal{A}$

If f and g are analytic functions in Δ, we say that f is subordinate to g, written $f \prec g$ if there exists a Schwarz function w, which is analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \Delta$, such that $f(z) = g(w(z))$. Furthermore, if the function g is univalent in Δ, then we have the following equivalence (see [5] and [21]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta).$$

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§ Manuscript received: August 29, 2020; accepted: November 11, 2020.

Let $\mathbf{E}_\alpha(z)$ and $\mathbf{E}_{\alpha,\beta}(z)$ be the function defined by

$$\mathbf{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad (z \in \mathbb{C}, \mathrm{Re}(\alpha) > 0\ ) \tag{2}$$

and

$$\mathbf{E}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad (\alpha,\beta \in \mathbb{C}, \ \ \Re(\alpha) > 0,\ \Re(\beta) > 0).$$

It can be written in other form

$$\mathbf{E}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + \sum_{k=2}^{\infty} \frac{z^{k-1}}{\Gamma(\alpha(k-1)+\beta)}, \qquad (\alpha,\beta \in \mathbb{C}, \ \ \Re(\alpha) > 0,\ \Re(\beta) > 0).$$

The function $\mathbf{E}_\alpha(z)$ was introduced by Mittag-Leffler [24] and is, therefore, known as the Mittag-Leffler function. A more general function $\mathbf{E}_{\alpha,\beta}$ generalizing $E_\alpha(z)$ was introduced by Wiman [29] and defined by

$$\mathbf{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad (z,\alpha,\beta \in \mathbb{C}, \mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0\ ). \tag{3}$$

Observe that the function $\mathbf{E}_{\alpha,\beta}$ contains many well-known functions as its special case, for example,
$\mathbf{E}_{1,1}(z) = e^z$, $\mathbf{E}_{1,2}(z) = \frac{e^z-1}{z}$, $\mathbf{E}_{2,1}(z^2) = \cosh z$, $\mathbf{E}_{2,1}(-z^2) = \cos z$, $\mathbf{E}_{2,2}(z^2) = \frac{\sinh z}{z}$, $\mathbf{E}_{2,2}(-z^2) = \frac{\sin z}{z}$, $\mathbf{E}_4(z) = \frac{1}{2}[\cos z^{1/4} + \cosh z^{1/4}]$ and $\mathbf{E}_3(z) = \frac{1}{2}[e^{z^{1/3}} + 2e^{-\frac{1}{2}z^{1/3}} \cos(\frac{\sqrt{3}}{2} z^{1/3})]$. The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found e.g. in [2, 4, 12, 13, 14, 18]. Observe that Mittag-Leffler function $\mathbf{E}_{\alpha,\beta}(z)$ does not belong to the family $\mathcal{A}$. Thus,it is natural to consider the following normalization of Mittag-Leffler functions as below :

$$E_{\alpha,\beta}(z) = z\Gamma(\beta)\mathbf{E}_{\alpha,\beta}(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)} z^k, \tag{4}$$

it holds for complex parameters α,β and $z \in \mathbb{C}$. In this paper,we shall restrict our attention to the case of real-valued α,β and $z \in \Delta$.

A discrete random variable x is said to have a Borel distribution if it takes the values $1, 2, 3, \cdots$ with the probabilities $\frac{e^{-\lambda}}{1!}, \frac{2\lambda e^{-2\lambda}}{2!}, \frac{9\lambda^2 e^{-3\lambda}}{3!}, \cdots$, respectively, where λ is called the parameter.
Very recently, Wanas and Khuttar [28] introduced the Borel distribution (BD) whose probability mass function is

$$P(x=\rho) = \frac{(\rho\lambda)^{\rho-1} e^{-\lambda\rho}}{\rho!}, \quad \rho = 1, 2, 3, \cdots.$$

Wanas and Khuttar introduced a series $\mathcal{M}(\lambda; z)$ whose coefficients are probabilities of the Borel distribution (BD)

$$\mathcal{M}(\lambda; z) = z + \sum_{k=2}^{\infty} \frac{[\lambda(k-1)]^{k-2} e^{-\lambda(k-1)}}{(k-1)!} z^k,\ (0 < \lambda \leq 1). \tag{5}$$

The probability mass function of the Mittag-Leffler-type Borel distribution is given by

$$\mathcal{P}\left(\lambda,\alpha,\beta;\rho\right)=\frac{\left(\lambda\rho\right)^{\rho-1}}{E_{\alpha,\beta}\left(\lambda\rho\right)\Gamma\left(\alpha\rho+\beta\right)},\quad \rho=0,1,2,\cdots,$$

where

$$E_{\alpha,\beta}\left(z\right)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma\left(\alpha k+\beta\right)},\qquad \left(\alpha,\beta\in\mathbb{C},\ \ \Re\left(\alpha\right)>0,\ \Re\left(\beta\right)>0\right).$$

Thus by using(4) and (5) and by convolution operator, we define the Mittag-Leffler-type Borel distribution series as below

$$\mathcal{B}\left(\lambda,\alpha,\beta\right)=z+\sum_{k=2}^{\infty}\frac{\left(\lambda\left(k-1\right)\right)!\left[\lambda\left(k-1\right)\right]^{k-2}e^{-\lambda(k-1)}}{(k-1)!E_{\alpha,\beta}\left(\lambda\left(k-1\right)\right)\Gamma\left(\alpha\left(k-1\right)+\beta\right)}z^{k},\ \left(0<\lambda\leq 1\right).$$

Next, we introduce the convolution operator

$$\begin{aligned}\mathcal{B}\left(\lambda,\alpha,\beta\right)f(z) &= z+\sum_{k=2}^{\infty}\frac{\left(\lambda\left(k-1\right)\right)!\left[\lambda\left(k-1\right)\right]^{k-2}e^{-\lambda(k-1)}}{(k-1)!E_{\alpha,\beta}\left(\lambda\left(k-1\right)\right)\Gamma\left(\alpha\left(k-1\right)+\beta\right)}a_{k}z^{k},\\ &= z+\sum_{k=2}^{\infty}\phi_{k}a_{k}z^{k},\end{aligned}\tag{6}$$

where $\alpha,\beta\in\mathbb{C},\ \Re\left(\alpha\right)>0,\ \Re\left(\beta\right)>0,\ 0<\lambda\leq 1$ and

$$\phi_{k}=\frac{\left(\lambda\left(k-1\right)\right)!\left[\lambda\left(k-1\right)\right]^{k-2}e^{-\lambda(k-1)}}{(k-1)!E_{\alpha,\beta}\left(\lambda\left(k-1\right)\right)\Gamma\left(\alpha\left(k-1\right)+\beta\right)}.\tag{7}$$

Legendre polynomials, which are exceptional cases of Legendre functions, are familiarized in 1784 by the French mathematician A. M. Legendre (1752-1833). Legendre functions are a vital and important in problems including spherical coordinates. As well, the Legendre polynomials, $P_k(x),(|x|<1)$, are designated via the following generating function(see [19]) :

$$G(x,z)=\frac{1}{\sqrt{1-2xz+z^{2}}}=\sum_{k=0}^{\infty}P_{k}(x)z^{k}.\tag{8}$$

Legendre polynomials are the everywhere regular solutions of Legendre's differential equation that we can write as follows:

$$(1-x^{2})\frac{d^{2}}{dx^{2}}P_{k}(x)-2x\frac{d}{dx}P_{k}(x)+mP_{k}(x)=0$$

where $m=k(k+1)$ *and* $k=0,1,2,\cdots$. Taking $x=1$ in (8) and by using geometric series, we see that $P_k(1)=1$, so that the Legendre polynomials are normalized.Thus Let $G(x,z)$ denote the class of analytic functions on U which are normalized by the conditions $G(x,0)=0$ and $G'(x,0)=1$.

Definition 1.1. *Let $P_k(x)$ is Legendre polynomials of the first kind of order $k=0,1,2,\cdots$, the recurrence formula is*

$$P_{k+1}(x)=\frac{2k+1}{k+1}xP_{k}(x)-\frac{k}{k+1}P_{k-1}(x),\tag{9}$$

with

$$P_0(x)=1 \quad \text{and} \quad P_1(x)=x.$$

In 1976, Noonan and Thomas [26] discussed the q^{th} Hankel determinant of a locally univalent analytic function $f(z)$ for $q \geq 1$ and $n \geq 1$ which is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

For our present discussion, we consider the Hankel determinant in the case $q = 2$ and $n = 2$, i.e. $H_2(2) = a_2a_4 - a_3^2$. This is popularly known as the second Hankel determinant of f.
Stimulated by the recent works on radii problems for some classes of analytic functions and coefficient results associated with Legendre polynomials in the articles[6, 8, 9],in this paper we define a new class $\mathcal{M}_\gamma^\eta(\lambda, \alpha, \beta, x)$ given in Definition 1.2.Based on Earlier works on sharp upper bounds of $H_2(2)$ for different classes of analytic functions(see[1, 3, 10, 11, 16, 22, 23, 25])we investigate the Fekete-Szegö inequalities for the functions in the class. We also obtain an upper bound to the functional $H_2(2)$ for $f \in \mathcal{M}_\gamma^\eta(\lambda, \alpha, \beta, x)$.

Now, we define the following class $\mathcal{M}_\gamma^\eta(\lambda, \alpha, \beta, x)$ $(0 \leq \gamma \leq 1,\ \eta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\},\ \alpha, \beta \in \mathbb{C},\ \Re(\alpha) > 0,\ \Re(\beta) > 0,\ 0 < \lambda \leq 1,\ |x| < 1)$ as follows:

Definition 1.2. *Let a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{M}_\gamma^\eta(\lambda, \alpha, \beta, x)$ if*

$$1 + \frac{1}{\eta}\left((1-\gamma)\tfrac{\mathcal{B}(\lambda,\alpha,\beta)f(z)}{z} + \gamma\left(\mathcal{B}(\lambda, \alpha, \beta) f(z)\right)' - 1\right) \prec G(x, z) \tag{10}$$

where $\eta \in \mathbb{C}^*;\ 0 \leq \gamma \leq 1;\ 0 < \lambda \leq 1;\ |x| < 1;\ z \in \Delta$.

Example 1.1. *Let a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{M}_0^\eta(\alpha, \beta, x) \equiv \mathcal{N}^\eta(\alpha, \beta, x)$ if*

$$1 + \frac{1}{\eta}\left(\tfrac{\mathcal{B}(\lambda,\alpha,\beta)f(z)}{z} - 1\right) \prec G(x, z) \tag{11}$$

where $\eta \in \mathbb{C}^*;\ 0 < \lambda \leq 1;\ |x| < 1;\ z \in \Delta$.

Example 1.2. *Let a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{M}_1^\eta(\lambda, \alpha, \beta, x) \equiv \mathcal{R}^\eta(\alpha, \beta, x)$ if*

$$1 + \frac{1}{\eta}\left(\left(\mathcal{B}(\lambda, \alpha, \beta) f(z)\right)' - 1\right) \prec G(x, z) \tag{12}$$

where $\eta \in \mathbb{C}^*;\ 0 < \lambda \leq 1;\ |x| < 1;\ z \in \Delta$.

Example 1.3. *Let a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{M}_0^1(\alpha, \beta, x) \equiv \mathcal{N}(\alpha, \beta, x)$ if*

$$\left(\tfrac{\mathcal{B}(\lambda,\alpha,\beta)f(z)}{z}\right) \prec G(x, z) \tag{13}$$

where $0 < \lambda \leq 1;\ |x| < 1;\ z \in \Delta$.

Example 1.4. *Let a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{M}_1^1(\lambda, \alpha, \beta, x) \equiv \mathcal{R}(\alpha, \beta, x)$ if*

$$\left(\mathcal{B}(\lambda, \alpha, \beta) f(z)\right)' \prec G(x, z) \tag{14}$$

where $0 < \lambda \leq 1;\ |x| < 1;\ z \in \Delta$.

2. Preliminary Results

To prove our results, we need the following lemmas.

Lemma 2.1. [27] *Let*

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \prec 1 + \sum_{n=1}^{\infty} C_n z^n = H(z) \quad (z \in \Delta). \tag{15}$$

If the function H is univalent in Δ and $H(\Delta)$ is a convex set, then

$$|c_n| \leq |C_1| . \tag{16}$$

Lemma 2.2. [7] *Let a function $p \in \mathcal{P}$ be given by*

$$p(z) = 1 + c_1 z + c_2 z^2 + ... \quad (z \in \Delta), \tag{17}$$

then, we have

$$|c_n| \leq 2 \quad (n \in \mathbb{N}). \tag{18}$$

The result is sharp.

Lemma 2.3. [17, 20] *Let $p \in \mathcal{P}$ be given by the power series (17), then for any complex number ν, then*

$$\left| c_2 - \nu c_1^2 \right| \leq 2 \max\{1; |2\nu - 1|\}. \tag{19}$$

The result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \quad \textit{and} \quad p(z) = \frac{1+z}{1-z} \quad (z \in \Delta).$$

Lemma 2.4. [15].*Let a function $p \in \mathcal{P}$ be given by the power series (17), then*

$$2c_2 = c_1^2 + \kappa(4 - c_1^2) \tag{20}$$

for some κ, $|\kappa| \leq 1$, and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1\kappa - c_1(4 - c_1^2)\kappa^2 + 2(4 - c_1^2)\left(1 - |\kappa|^2\right) z, \tag{21}$$

for some z, $|z| \leq 1$.

Lemma 2.5. [15] *The power series for $p(z)$ given in* (17) *converges in Δ to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n = 1, 2, 3, \cdots \tag{22}$$

and $c_{-k} = \overline{c_k}$, are all nonnegative. They are strictly positive except for

$$p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k z}), \ \rho_k > 0, \ t_k \text{ real}$$

and $t_k \neq t_j$ for $k \neq j$ in this case $D_n > 0$ for $n < m - 1$ and $D_n = 0$ for $n \geq m$.

3. Main results

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\eta \in \mathbb{C}^*,\ \alpha,\beta \in \mathbb{C},\ \Re(\alpha) > 0,\ \Re(\beta) > 0,\ 0 \leq \gamma \leq 1,\ 0 < \lambda \leq 1,\ |x| < 1$ and $z \in \Delta$, the powers are understood as principle values.

We give the following result related to the coefficient of $f(z) \in \mathcal{M}_\gamma^\eta(\lambda,\alpha,\beta,x)$

Theorem 3.1. *Let $f(z)$ given by (1) belongs to the class $\mathcal{M}_\gamma^\eta(\lambda,\alpha,\beta,x)$ and $\eta \in \mathbb{C}^*$, then*

$$|a_k| \leq \frac{|\eta x|\,(k-1)!E_{\alpha,\beta}(\lambda(k-1))\,\Gamma(\alpha(k-1)+\beta)}{[1+\gamma(k-1)]\,(\lambda(k-1))!\,[\lambda(k-1)]^{k-2}\,e^{-\lambda(k-1)}},\quad (k \in \mathbb{N}\setminus\{1\}). \tag{23}$$

Proof. If $f(z)$ of the form (1) belongs to the class $\mathcal{M}_\gamma^\eta(\lambda,\alpha,\beta,x)$, then

$$1+\frac{1}{\eta}\left((1-\gamma)\tfrac{\mathcal{B}(\lambda,\alpha,\beta)f(z)}{z}+\gamma\left(\mathcal{B}(\lambda,\alpha,\beta)f(z)\right)'-1\right) \prec G(x,z)$$

where$\eta \in \mathbb{C}^*,\ 0 \leq \gamma \leq 1,\ 0 < \lambda \leq 1,\ |x| < 1,\ z \in \Delta$, and $G(x,z)$ is convex univalent in Δ, we have

$$\begin{aligned}
&1+\frac{1}{\eta}\left((1-\gamma)\tfrac{\mathcal{B}(\lambda,\alpha,\beta)f(z)}{z}+\gamma\left(\mathcal{B}(\lambda,\alpha,\beta)f(z)\right)'-1\right)\\
=\ &1+\frac{1}{\eta}\sum_{k=2}^{\infty}(1+k\gamma-\gamma)\phi_k a_k z^{k-1}\\
=\ &1+\frac{1}{\eta}\sum_{k=2}^{\infty}(1+k\gamma-\gamma)\frac{(\lambda(k-1))!\,[\lambda(k-1)]^{k-2}\,e^{-\lambda(k-1)}}{(k-1)!E_{\alpha,\beta}(\lambda(k-1))\,\Gamma(\alpha(k-1)+\beta)}a_k z^{k-1}\\
=\ &1+\sum_{k=1}^{\infty}\frac{(1+k\gamma)}{\eta}\frac{(\lambda k)!\,[\lambda k]^{k-1}\,e^{-\lambda k}}{k!E_{\alpha,\beta}(\lambda k)\,\Gamma(\alpha k+\beta)}a_{k+1}z^k.
\end{aligned}$$

By Definition 1.2, we get

$$\begin{aligned}
&1+\sum_{k=1}^{\infty}\frac{(1+k\gamma)}{\eta}\frac{(\lambda k)!\,[\lambda k]^{k-1}\,e^{-\lambda k}}{k!E_{\alpha,\beta}(\lambda k)\,\Gamma(\alpha k+\beta)}a_{k+1}z^k\\
\prec\ &1+xz-\frac{1}{2}(3x^2-1)z^2+\frac{1}{2}(5x^3-3x)z^3+\cdots\ (z \in \Delta).
\end{aligned} \tag{24}$$

Now, by applying Lemma 2.1, we get

$$|a_{k+1}| \leq \frac{|x\eta|}{(1+k\gamma)}\frac{k!E_{\alpha,\beta}(\lambda k)\,\Gamma(\alpha k+\beta)}{(\lambda k)!\,[\lambda k]^{k-1}\,e^{-\lambda k}}.$$

This completes the proof of Theorem 3.1.

□

In the next two theorems, we obtain the result concerning Fekete-Szego inequality and upper bound of Hankel determinant for the class $\mathcal{M}_\gamma^\eta(\lambda,\alpha,\beta,x)$.

Theorem 3.2. *Let $f(z)$ given by (1) belongs to the class $\mathcal{M}_\gamma^\eta(\lambda,\alpha,\beta,x)$, $0 \leq \gamma \leq 1$, $-1 \leq B < A \leq 1$ and $\eta \in \mathbb{C}^*$, then*

$$\begin{aligned}
\left|a_3-\mu a_2^2\right| \quad &\leq \quad \frac{|\eta x|E_{\alpha,\beta}(2\lambda)\,\Gamma(2\alpha+\beta)}{\lambda(1+2\gamma)(2\lambda)!e^{-2\lambda}}\\
&\times \quad \max\left\{1,\left|\frac{1}{2x}-\frac{3}{2}x+\frac{\mu\eta\lambda x(1+2\gamma)(2\lambda)!E_{\alpha,\beta}^2(\lambda)\,\Gamma^2(\alpha+\beta)}{2(1+\gamma)^2(\lambda!)^2E_{\alpha,\beta}(2\lambda)\,\Gamma(2\alpha+\beta)}\right|\right\}
\end{aligned} \tag{25}$$

This result is sharp.

Proof. Let $f(z) \in \mathcal{M}_{\gamma}^{\eta}(\lambda, \alpha, \beta, x)$, then there is a Schwarz function $w(z)$ in Δ with $w(0) = 0$ and $|w(z)| < 1$ in Δ and such that

$$1 + \frac{1}{\eta}\left((1-\gamma)\tfrac{\mathcal{B}(\lambda,\alpha,\beta)f(z)}{z} + \gamma\left(\mathcal{B}(\lambda,\alpha,\beta)f(z)\right)' - 1\right) = \Phi(w(z)) \qquad (z \in \Delta), \tag{26}$$

where

$$\begin{aligned} \Phi(z) &= \frac{1}{\sqrt{1-2xz+z^2}} = 1 + xz + \frac{1}{2}(3x^2-1)z^2 + \frac{1}{2}(5x^3-3x)z^3 + \cdots, \\ &= 1 + P_1(x)z + P_2(x)z^2 + P_3(x)z^3 + P_4(x)z^4 + \cdots (z \in \Delta). \end{aligned} \tag{27}$$

If the function $p_1(z)$ is analytic and has positive real part in Δ and $p_1(0) = 1$, then

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots (z \in \Delta). \tag{28}$$

Since $w(z)$ is a Schwarz function. Define

$$\begin{aligned} h(z) &= 1 + \frac{1}{\eta}\left((1-\gamma)\tfrac{\mathcal{B}(\lambda,\alpha,\beta)f(z)}{z} + \gamma\left(\mathcal{B}(\lambda,\alpha,\beta)f(z)\right)' - 1\right) \\ &= 1 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots (z \in \Delta). \end{aligned} \tag{29}$$

In view of the equations (26) and (28), we have

$$p(z) = \Phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right).$$

$$\frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 + \frac{c_1^3}{4} - c_1 c_2\right)z^3 + \cdots\right]. \tag{30}$$

Therefore, we have

$$\begin{aligned} \Phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) &= 1 + \frac{1}{2}P_1(x)c_1 z + \left[\frac{1}{2}P_1(x)\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}P_2(x)c_1^2\right]z^2 \\ &+ \left(\frac{P_1(x)}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) + \frac{P_2(x)c_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{P_3(x)c_1^3}{8}\right)z^3 + \cdots, \end{aligned} \tag{31}$$

and from this equation and (29), we obtain

$$d_1 = \frac{1}{2}P_1(x)c_1, \quad d_2 = \frac{1}{2}P_1(x)\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}P_2(x)c_1^2. \tag{32}$$

Sinceand

$$d_3 = \frac{P_1(x)}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) + \frac{P_2(x)c_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{P_3(x)c_1^3}{8}. \tag{33}$$

Then, from (27), we see that

$$d_1 = \frac{(1+\gamma)\lambda! e^{-\lambda} a_2}{\eta E_{\alpha,\beta}(\lambda)\Gamma(\alpha+\beta)}, \tag{34}$$

$$d_2 = \frac{\lambda(1+2\gamma)(2\lambda)! e^{-2\lambda} a_3}{\eta E_{\alpha,\beta}(2\lambda)\Gamma(2\alpha+\beta)}, \tag{35}$$

and

$$d_3 = \frac{3\lambda^2 (1+3\gamma)(3\lambda)! e^{-3\lambda} a_4}{2\eta E_{\alpha,\beta}(3\lambda)\Gamma(3\alpha+\beta)} \tag{36}$$

Now from (27),(29) and (34), we have the following

$$a_2 = \frac{\eta x E_{\alpha,\beta}(\lambda)\Gamma(\alpha+\beta) c_1}{2(1+\gamma)\lambda! e^{-\lambda}}, \tag{37}$$

Thus by Lemma 2.2

$$|a_2| \leq \frac{|\eta x| E_{\alpha,\beta}(\lambda)\Gamma(\alpha+\beta)}{(1+\gamma)\lambda! e^{-\lambda}}.$$

Now

$$\begin{aligned} a_3 &= \frac{\eta x E_{\alpha,\beta}(2\lambda)\Gamma(2\alpha+\beta)}{4\lambda(1+2\gamma)(2\lambda)! e^{-2\lambda}} \left\{ 2c_2 - c_1^2 \left(\frac{2x+1}{2x} - \frac{3}{2}x \right) \right\} \\ &= \frac{\eta x E_{\alpha,\beta}(2\lambda)\Gamma(2\alpha+\beta)}{2\lambda(1+2\gamma)(2\lambda)! e^{-2\lambda}} \left\{ c_2 - \frac{c_1^2}{2} \left(\frac{2x+1}{2x} - \frac{3}{2}x \right) \right\}, \end{aligned} \tag{38}$$

thus by Lemma 2.3, we have $\left|c_2 - \nu c_1^2\right| \leq \max\{1; |2\nu - 1|\}$, thus

$$|a_3| \leq \frac{|\eta x| E_{\alpha,\beta}(2\lambda)\Gamma(2\alpha+\beta)}{2\lambda(1+2\gamma)(2\lambda)! e^{-2\lambda}} \max\{1; |2\nu-1|\}$$

where $\nu = \frac{1}{2}\left(\frac{2x+1}{2x} - \frac{3}{2}x\right)$ Hence

$$|a_3| \leq \frac{|\eta x| E_{\alpha,\beta}(2\lambda)\Gamma(2\alpha+\beta)}{2\lambda(1+2\gamma)(2\lambda)! e^{-2\lambda}} \max\{1; \left|\frac{1}{2x} - \frac{3}{2}x\right|\}.$$

Now we note that

$$a_4 = \frac{\eta x E_{\alpha,\beta}(3\lambda)\Gamma(3\alpha+\beta)}{24\lambda^2(1+3\gamma)(3\lambda)! e^{-3\lambda}} \left\{ 8xc_3 + 4c_1c_2\left(3x^2 - 2x - 1\right) + c_1^3(5x^3 - 6x^2 - x + 2) \right\}. \tag{39}$$

Therefore, we have

$$a_3 - \mu a_2^2 = \frac{\eta x E_{\alpha,\beta}(2\lambda)\Gamma(2\alpha+\beta)}{2\lambda(1+2\gamma)(2\lambda)! e^{-2\lambda}} \left\{ c_2 - \nu c_1^2 \right\}, \tag{40}$$

where

$$\nu = \frac{1}{2}\left[\frac{2x+1}{2x} - \frac{3}{2}x + \frac{\mu\eta\lambda x(1+2\gamma)(2\lambda)! E_{\alpha,\beta}^2(\lambda)\Gamma^2(\alpha+\beta)}{2(1+\gamma)^2(\lambda!)^2 E_{\alpha,\beta}(2\lambda)\Gamma(2\alpha+\beta)} \right]. \tag{41}$$

Our result now follows by an application of Lemma 2.3.This completes the proof of Theorem 3.2.
The result is sharp for the functions

$$1 + \frac{1}{\eta}\left((1-\gamma)\frac{\mathcal{B}(\lambda,\alpha,\beta)f(z)}{z} + \gamma\left(\mathcal{B}(\lambda,\alpha,\beta) f(z)\right)' - 1 \right) = \Phi(z^2)$$

$$\begin{aligned} \Phi(z^2) &= \frac{1}{\sqrt{1 - 2xz^2 + z^4}} = 1 + xz^2 + \frac{1}{2}(3x^2-1)z^4 + \frac{1}{2}(5x^3 - 3x)z^6 + \cdots, \\ &= 1 + P_1(x)z^2 + P_2(x)z^4 + P_3(x)z^6 + P_4(x)z^8 + \cdots (z \in \Delta). \end{aligned}$$

Here $d_1 = 0 \Rightarrow a_2 = 0$, also we get $c_1 = 0$ and

$$d_2 = P_1(x) \Rightarrow a_3 = \frac{\eta x E_{\alpha,\beta}(2\lambda)\Gamma(2\alpha+\beta)}{\lambda(1+2\gamma)(2\lambda)! e^{-2\lambda}}$$

Thus by (40)

$$|a_3-\mu a_2^2|\leq \frac{|\eta x|E_{\alpha,\beta}\left(2\lambda\right)\Gamma\left(2\alpha+\beta\right)}{\lambda\left(1+2\gamma\right)(2\lambda)!e^{-2\lambda}}.$$

Actually this is $|a_3|$, hence the result is sharp for $w(z)=z^2$ which is $\Phi(z^2)$. □

Theorem 3.3. *If* $f(z)\in\mathcal{M}_\gamma^\eta\left(\lambda,\alpha,\beta,x\right),$ *then*

$$|a_2a_4-a_3^2|\leq\left(\frac{\eta x E_{\alpha,\beta}\left(2\lambda\right)\Gamma\left(2\alpha+\beta\right)}{\lambda\left(1+2\gamma\right)(2\lambda)!e^{-2\lambda}}\right)^2. \tag{42}$$

Proof. Since $f(z)\in\mathcal{M}_\gamma^\eta\left(\lambda,\alpha,\beta,x\right)$, and, from (37),(38) (39), it can be established that

$$\begin{aligned}|a_2a_4-a_3^2| &= \frac{1}{48\lambda^2\left(1+\gamma\right)\left(1+3\gamma\right)\lambda!\ \ (3\lambda)!e^{-4\lambda}}\\ &\times\Big|\,\eta^2x^2E_{\alpha,\beta}\left(\lambda\right)\Gamma\left(\alpha+\beta\right)E_{\alpha,\beta}\left(3\lambda\right)\Gamma\left(3\alpha+\beta\right)c_1\\ &\left\{8xc_3+4c_1c_2\left(3x^2-2x-1\right)+c_1^3(5x^3-6x^2-x+2)\right\}\\ &-\left(\frac{\eta x E_{\alpha,\beta}\left(2\lambda\right)\Gamma\left(2\alpha+\beta\right)}{4\lambda\left(1+2\gamma\right)(2\lambda)!e^{-2\lambda}}\left\{2c_2-c_1^2\left(\frac{2x+1}{2x}-\frac{3}{2}x\right)\right\}\right)^2\Big|.\end{aligned} \tag{43}$$

For the sake of brevity we consider

$$M=\frac{\eta^2x^2E_{\alpha,\beta}\left(\lambda\right)E_{\alpha,\beta}\left(3\lambda\right)\Gamma\left(\alpha+\beta\right)\Gamma\left(3\alpha+\beta\right)}{48\lambda^2\left(1+\gamma\right)\left(1+3\gamma\right)\lambda!\ \ (3\lambda)!e^{-4\lambda}}>0, \tag{44}$$

and

$$N=\left(\frac{\eta x E_{\alpha,\beta}\left(2\lambda\right)\Gamma\left(2\alpha+\beta\right)}{4\lambda\left(1+2\gamma\right)(2\lambda)!e^{-2\lambda}}\right)^2>0. \tag{45}$$

Thus, we have

$$\begin{aligned}|a_2a_4-a_3^2| &= \big|Mc_1\left\{8xc_3+4c_1c_2\left(3x^2-2x-1\right)+c_1^3(5x^3-6x^2-x+2)\right\}\\ &-N\left(2c_2-c_1^2\left(\frac{2x+1}{2x}-\frac{3}{2}x\right)\right)^2\Big|.\end{aligned} \tag{46}$$

Suppose $c_1=c$ and $c\in[0,2]$. We make use of Lemma 2.5 to obtain the proper bound on (43). We may assume without restriction that $c_1>0$. We begin by rewriting (22) for the cases $n=2$ and $n=3$,

$$D_2\ =\ \begin{vmatrix}2 & c_1 & c_2\\ c_1 & 2 & c_1\\ \overline{c_2} & c_1 & 2\end{vmatrix}\ =\ 8+2\text{Re}\ \{c_1^2c_2\}-2|c_2|^2-4c_1^2\geq 0, \tag{47}$$

which is equivalent to

$$2c_2=c_1^2+\kappa(4-c_1^2) \tag{48}$$

for some x, $|x|\leq 1$. Then $D_3\geq 0$ is equivalent to

$$|\ (4c_3-4c_1c_2+c_1^3)(4-c_1^2)+c_1(2c_2-c_1^2)^2\ |\leq 2(4-c_1^2)^2-2|2c_2-c_1^2|^2 \tag{49}$$

and from (20) with (49), we have,

$$4c_3\ =\ c_1^3+2(4-c_1^2)c_1\kappa-c_1(4-c_1^2)\kappa^2+2(4-c_1^2)(1-|\kappa|^2)z, \tag{50}$$

for some value of z, $|z| \leq 1$. Using (48) along with (50) , (49) we obtain

$$\begin{aligned}
|a_2a_4 - a_3^2| &= \Big|M\left\{8xc_1c_3 + 4c_1^2c_2\left(3x^2-2x-1\right) + c_1^4(5x^3-6x^2-x+2)\right\} \\
&- N\left(2c_2 - c_1^2\left(\frac{2x+1}{2x} - \frac{3}{2}x\right)\right)^2\Big| \\
&\leq \left|M\left\{8xc_1c_3 + 4c_1^2c_2\left(3x^2-2x-1\right) + c_1^4(5x^3-6x^2-x+2)\right\}\right| \\
&+ \left|N(2c_2 - c_1^2(\tfrac{2x+1}{2x} - \tfrac{3}{2}x))^2\right|.
\end{aligned}$$

By using Lemma 2.4, we have

$$\begin{aligned}
|a_2a_4 - a_3^2| &\leq M\Big|c^4\left(5x^3-3x\right) - 2x\left(4-c^2\right)c^2\chi^2 + 2\left(4-c^2\right)\left(3x^2-1\right)c^2\chi \\
&+ 4cx\left(4-c^2\right)\left(1-|\chi|^2\right)z\Big| + \\
&\quad N\left|\left(4-c^2\right)^2\chi^2 - 2c^2\chi\left(4-c^2\right)\left(\frac{1-3x^2}{2x}\right) + c^4\left(\frac{1-3x^2}{2x}\right)^2\right| \\
&\leq M\left[c^4\left(5x^3-3x\right) - 2x\left(4-c^2\right)c^2\rho^2 + 2\left(4-c^2\right)\left(3x^2-1\right)c^2\rho\right. \\
&+ \left. 4cx\left(4-c^2\right)\left(1-\rho^2\right)\right] + \\
&\quad N\left[\left(4-c^2\right)^2\rho^2 - 2c^2\rho\left(4-c^2\right)\left(\frac{1-3x^2}{2x}\right) + c^4\left(\frac{1-3x^2}{2x}\right)^2\right] \\
&= \mathcal{F}(\rho, c), \qquad (51)
\end{aligned}$$

where $\rho = |\chi| \leq 1$ and $|z| < 1$. We assume that the upper bound for (54) is attained at an interior point of the set $\{(\rho, c) : \ \rho \in [0,1], \ c \in [0,2]\}$, then

$$\begin{aligned}
\frac{\partial \mathcal{F}(\rho, c)}{\partial \rho} &= M\left[-4x\left(4-c^2\right)c^2\rho + 2\left(4-c^2\right)\left(3x^2-1\right)c^2 - 8cx\rho\left(4-c^2\right)\right] + \\
&\quad N\left[2\rho\left(4-c^2\right)^2 - 2c^2\left(4-c^2\right)\left(\tfrac{1-3x^2}{2x}\right)\right]. \qquad (52)
\end{aligned}$$

We note that $\frac{\partial \mathcal{F}(\rho,c)}{\partial \rho} > 0$ and consequently $\mathcal{F}$ is increasing and $max\ \mathcal{F}(\rho, c) = \mathcal{F}(1, c)$, which contradicts our assumption of having the maximum value at the interior of $\rho \in [0,1]$. Now let

$$\begin{aligned}
\mathcal{G}(c) &= \mathcal{F}(1,c) = M\left[c^4\left(5x^3-3x\right) - 2x\left(4-c^2\right)c^2 + 2\left(4-c^2\right)\left(3x^2-1\right)c^2\right] + \\
&\quad N\left[\left(4-c^2\right)^2 - 2c^2\left(4-c^2\right)\left(\frac{1-3x^2}{2x}\right) + c^4\left(\frac{1-3x^2}{2x}\right)^2\right] \\
&= M\left[c^4\left(5x^3-6x^2-x+2\right) + 8c^2\left(3x^2-x-1\right)\right] + \\
&\quad N\left[c^4\left(1+\tfrac{1-3x^2}{2x}\right)^2 - 8c^2\left(1+\tfrac{1-3x^2}{2x}\right) + 16\right], \qquad (53)
\end{aligned}$$

then

$$\begin{aligned}
\mathcal{G}'(c) &= M\left[4c^3\left(5x^3-6x^2-x+2\right) + 16c\left(3x^2-x-1\right)\right] + \\
&\quad N\left[4c^3\left(1+\tfrac{1-3x^2}{2x}\right)^2 - 16c\left(1+\tfrac{1-3x^2}{2x}\right)\right] = 0, \qquad (54)
\end{aligned}$$

therefore (54) implies $c = 0$, which is a contradiction. We note that

$$\begin{aligned} \mathcal{G}''(c) &= M\left[12c^2\left(5x^3-3x^2-x+1\right)+16\left(3x^2-x-1\right)\right]+ \\ & N\left[12c^2\left(1+\frac{1-3x^2}{2x}\right)^2-16\left(1+\frac{1-3x^2}{2x}\right)\right]<0. \end{aligned} \tag{55}$$

Thus any maximum points of $\mathcal{G}$ must be on the boundary of $c \in [0,2]$. However, $\mathcal{G}(c) \geq \mathcal{G}(2)$ and thus $\mathcal{G}$ has maximum value at $c = 0$. The upper bound for (51) corresponds to $\rho = 1$ and $c = 0$, in which case we get

$$|a_2a_4 - a_3^2| \leq 16N = \left(\frac{\eta x E_{\alpha,\beta}(2\lambda)\Gamma(2\alpha+\beta)}{\lambda(1+2\gamma)(2\lambda)!e^{-2\lambda}}\right)^2,$$

this completes the proof Theorem 3.3.

□

Remark 3.1. *By specializing the parameters $\gamma = 0$ and $\gamma = 1$ one can derive the coefficient estimate, Fekete-Szegö inequalities and second Hankel determinant inequalities as in Theorems 3.1, 3.2,and 3.3 respectively for the various other new interesting subclasses of $\mathcal{A}$ stated in Example 1.1 to 1.4. The details involved may be left as an exercise for the interested reader.*

Acknowledgement We record our sincere thanks to the referees for their valuable suggestions to state the results in present form.

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