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# FUZZY MODULARITY AND FUZZY COMPLEMENTS IN FUZZY LATTICES

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ABSTRACT. In this paper, we study the concept of fuzzy modularity in fuzzy lattices. We also define a fuzzy Birkhoff lattice and study fuzzy complements in fuzzy lattices. We prove that the notions of a right and a left complement coincide in a fuzzy lattice.

Keywords: Fuzzy lattice, fuzzy modular pair, fuzzy left complement, fuzzy right complement, fuzzy Birkhoff lattice.

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## 1. INTRODUCTION

The concept of a complement is well-known in lattice theory. Maeda, F. and Maeda, S. [3] introduced and studied the concept of a left complement and a right complement in lattices. Zadeh [8], [9] introduced the concept of a fuzzy set and a fuzzy relation. Many researchers have investigated fuzzy lattices and related concepts. Chon [2] developed some properties of fuzzy lattices. Mezzomo et. al. [4] considered bounded fuzzy lattices. Mezzomo et. al. [5] have developed ideal theory in fuzzy lattices. Amroune and Davvaz [1] have considered fuzzy ordered sets. As a generalization of the concept of a fuzzy modular lattice, Wasadikar and Khubchandani [6] defined a fuzzy modular pair and a semi-modular fuzzy lattice.

In this paper, we define a fuzzy Birkhoff lattice and study $FM$-symmetry in fuzzy lattices. We introduce the concept of a left and a right complement in a fuzzy lattice. In section 3, we prove that every $FM$-symmetric fuzzy lattice is a fuzzy Birkhoff lattice. In section 4, we define a right and a left complement in a fuzzy lattice. We also prove results where the notion of left and right complement coincides in a fuzzy lattice.

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## 2. Preliminaries

Throughout this paper, $(X, A)$ denotes a fuzzy lattice, where $A$ is a fuzzy binary ordering relation on a nonempty set $X$.

Zadeh [9] introduced the concept of a binary relation, equivalence relation and partial order relation in fuzzy sets. For the definitions of a fuzzy binary relation, fuzzy equivalence relation, fuzzy partial order relation, we refer to Chon [2].

Several researchers have studied fuzzy lattices. Chon [2] and some others use the terms *upper bound, lower bound* and use the notations $a \vee b$ and $a \wedge b$ to denote the supremum and the infimum of two elements $a, b$ in a fuzzy lattice $X$ in the fuzzy sense. Since the set $X$ itself may be a lattice, we use the notations $a \vee_F b$ and $a \wedge_F b$ to denote the fuzzy supremum and the fuzzy infimum of $a, b \in X$.

**Definition 2.1.** [2, Defifniton 3.1] *Let $(X, A)$ be a fuzzy poset and let $Y \subseteq X$. An element $b \in X$ is said to be a fuzzy upper bound for $Y$ iff $A(a, b) > 0$ for all $a \in Y$. A fuzzy upper bound $b_0$ for $Y$ is called a least upper bound (or supremum) of $Y$ iff $A(b_0, b) > 0$ for every fuzzy upper bound $b$ for $Y$. We then write $b_0 = \sup_F Y = \vee_F Y$. If $Y = \{a, b\}$, then we write $\vee_F Y = a \vee_F b$.*

*Similarly, an element $c \in X$ is said to be a fuzzy lower bound for $Y$ iff $A(c, a) > 0$, for all $a \in Y$. A fuzzy lower bound $c_0$ for $Y$ is called a fuzzy greatest lower bound (or infimum) of $Y$ iff $A(c, c_0) > 0$ for every fuzzy lower bound $c$ for $Y$. We then write $c_0 = \inf_F Y = \wedge_F Y$. If $Y = \{a, b\}$, then we write $\wedge_F Y = a \wedge_F b$.*

Since $A$ is fuzzy antisymmetric, the fuzzy least upper (fuzzy greatest lower) bound, if it exists, is unique.

**Definition 2.2.** [2, Defifniton 3.2] *Let $(X, A)$ be a fuzzy poset. Then, $(X, A)$ is called a fuzzy lattice if and only if $a \vee_F b$ and $a \wedge_F b$ exist, for all $a, b \in X$.*

**Definition 2.3.** [4, Definition 3.4] *A fuzzy lattice $(X, A)$ is said to be bounded if there exist elements $\perp$ and $\top$ in $X$, such that $A(\perp, a) > 0$ and $A(a, \top) > 0$, for every $a \in X$. In this case, $\perp$ and $\top$ are respectively, called bottom and top elements of $X$.*

In the following example, we illustrate these concepts. The fuzzy poset $(X, A)$ in this example is a fuzzy lattice.

**Example 2.1.** *Let $X = \{a, b, c, d, e\}$ and let $A : X \times X \longrightarrow [0, 1]$ be a fuzzy relation defined as follows:*
$A(a, a) = A(b, b) = A(c, c) = A(d, d) = A(e, e) = 1.0,$
$A(a, b) = 0.40,\ A(a, c) = 0.50,\ A(a, d) = 0.80,\ A(a, e) = 0.94,$
$A(b, a) = 0.0,\ A(b, c) = 0.0,\ A(b, d) = 0.60,\ A(b, e) = 0.90,$
$A(c, a) = 0.0,\ A(c, b) = 0.0,\ A(c, d) = 0.0,\ A(c, e) = 0.70,$
$A(d, a) = 0.0,\ A(d, b) = 0.0,\ A(d, c) = 0.0,\ A(d, e) = 0.40,$
$A(e, a) = 0.0,\ A(e, b) = 0.0,\ A(e, c) = 0.0,\ A(e, d) = 0.0.$
*Then $A$ is a fuzzy partial order relation.*
*We note that $(X, A)$ is a fuzzy lattice.*
*The fuzzy join and fuzzy meet tables are as follows:*

| $\vee_F$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $a$ | $b$ | $c$ | $d$ | $e$ |
| $b$ | $b$ | $b$ | $e$ | $d$ | $e$ |
| $c$ | $c$ | $e$ | $c$ | $e$ | $e$ |
| $d$ | $d$ | $d$ | $e$ | $d$ | $e$ |
| $e$ | $e$ | $e$ | $e$ | $e$ | $e$ |

| $\wedge_F$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $b$ | $a$ | $b$ | $b$ |
| $c$ | $a$ | $a$ | $c$ | $a$ | $c$ |
| $d$ | $a$ | $b$ | $a$ | $d$ | $d$ |
| $e$ | $a$ | $b$ | $c$ | $d$ | $e$ |

We recall some results from Chon [2].

**Proposition 2.1.** [2, Proposition 3.3] *Let $(X, A)$ be a fuzzy lattice. For $a, b, c \in X$. The following statements hold:*

(i) $A(a, a \vee_F b) > 0$, $A(b, a \vee_F b) > 0$, $A(a \wedge_F b, a) > 0$, $A(a \wedge_F b, b) > 0$.
(ii) $A(a, c) > 0$ *and* $A(b, c) > 0$ *implies* $A(a \vee_F b, c) > 0$.
(iii) $A(c, a) > 0$ *and* $A(c, b) > 0$ *implies* $A(c, a \wedge_F b) > 0$.
(iv) $A(a, b) > 0$ *iff* $a \vee_F b = b$.
(v) $A(a, b) > 0$ *iff* $a \wedge_F b = a$.
(vi) *If* $A(b, c) > 0$, *then* $A(a \wedge_F b, a \wedge_F c) > 0$ *and* $A(a \vee_F b, a \vee_F c) > 0$.
(vii) *If* $A(a \vee_F b, c) > 0$, *then* $A(a, c) > 0$ *and* $A(b, c) > 0$.
(viii) *If* $A(a, b \wedge_F c) > 0$, *then* $A(a, b) > 0$ *and* $A(a, c) > 0$.

**Proposition 2.2.** [2, Proposition 3.4] *Let $(X, A)$ be a fuzzy lattice and let $x$, $y$, $z \in X$. Then*
*(i)* $a \vee_F a = a$, $a \wedge_F a = a$;
*(ii)* $a \vee_F b = b \vee_F a$, $a \wedge_F b = b \wedge_F a$;
*(iii)* $(a \vee_F b) \vee_F c = a \vee_F (b \vee_F c)$, $(a \wedge_F b) \wedge_F c = a \wedge_F (b \wedge_F c)$;
*(iv)* $(a \vee_F b) \wedge_F a = a$, $(a \wedge_F b) \vee_F a = a$.

The following corollary can be proved by using (vi) of Proposition 2.1.

**Corollary 2.1.** *Let $(X, A)$ be a fuzzy lattice and $a, b, c, d \in X$. If $A(c, a) > 0$ and $A(d, b) > 0$, then $A(c \wedge_F d, a \wedge_F b) > 0$ and $A(c \vee_F d, a \vee_F b) > 0$*

**Proposition 2.3.** [2, Proposition 3.11] *(Modular inequality) Let $(X, A)$ be a fuzzy lattice and let $a, b, c \in X$. Then $A(c, b) > 0$ implies $A(c \vee_F (a \wedge_F b), (c \vee_F a) \wedge_F b) > 0$.*

**Definition 2.4.** [2, Definition 3.12] *A fuzzy lattice $(X, A)$ is called modular if $A(c, b) > 0$ implies $c \vee_F (a \wedge_F b) = (c \vee_F a) \wedge_F b$, for $a, b, c \in X$.*

An immediate consequence of the modular inequality is the following result.

**Corollary 2.2.** [2, Proposition 3.12] *A fuzzy lattice $(X, A)$ is modular iff $A(c, b) > 0$ implies $A((c \vee_F a) \wedge_F b, c \vee_F (a \wedge_F b)) > 0$ for $a, b, c \in X$.*

Maeda and Maeda [3] have investigated modular pairs in lattices. Wasadikar and Khubchandani [6] have introduced the notion of a fuzzy modular pair in a fuzzy lattice.

**Definition 2.5.** [6] *Let $\mathcal{L} = (X, A)$ be a fuzzy lattice. Let $a, b \in X$. We say that $(a, b)$ is a fuzzy meet-modular pair and we write $(a, b)_F M_m$ if whenever $A(c, b) > 0$, then $(c \vee_F a) \wedge_F b = c \vee_F (a \wedge_F b)$.*

*We say that $(a, b)$ is a fuzzy join-modular pair and we write $(a, b)_F M_j$ if whenever $A(b,\ c) > 0$, then $(c \wedge_F a) \vee_F b = c \wedge_F (a \vee_F b)$.*

*We write $(a, b)_F \overline{M_j}$ or $(a, b)_F \overline{M_m}$ when the pair $(a, b)$ is not a fuzzy join-modular or fuzzy meet-modular pair respectively.*

We illustrate this concept by an example.

**Example 2.2.** *Consider the fuzzy lattice in Example 2.1.*
*We note that* $A(a,c) = 0.50 > 0$ *and*
$(a \vee_F b) \wedge_F c = b \wedge_F c = a = a \vee_F a = a \vee_F (b \wedge_F c)$.
*Thus* $(b,c)_F M_m$ *holds.*

*We note that* $A(b,d) = 0.60 > 0$. *But* $(b \vee_F c) \wedge_F d = e \wedge_F d = d$ *and*
$b \vee_F (c \wedge_F d) = b \vee_F a = b \neq d$.
*Hence* $(c,d)_F M_m$ *does not hold.*

**Definition 2.6.** [6, Definition 3.2] *Let* $a, b \in X$. *We say that* $(a,b)$ *is a fuzzy independent pair and we write* $(a,b) \perp_F M_m$ *if* $(a,b)_F M_m$ *and* $a \wedge_F b = \perp$ *hold.*

**Definition 2.7.** [6, Definition 4.2] *A fuzzy lattice* $(X, A)$ *with* $\perp$ *is called* $\perp_F$*-symmetric when in* $(X,A)$, $(a,b) \perp_F M_m$ *implies* $(b,a)_F M_m$.

**Lemma 2.1.** [6, Lemma 3.2] *If* $(a,b)_F M_m$, $A(a \wedge_F b, c) > 0$, *then* $(a \wedge_F c, b)_F M_m$.

The covering relation in lattices is well studied in Maeda and Maeda [3]. Wasadikar and Khubchandani [6] have introduced this concept in the context of a fuzzy lattice.

**Definition 2.8.** [6, Definition 4.4] *Let* $\mathcal{L} = (X, A)$ *be a fuzzy lattice. Let* $a,\ b \in X$. *We say that a fuzzy covers b and write* $b \prec_F a$, *if* $0 < A(b,a) < 1$ *and* $A(b,c) > 0$ *and* $A(c,a) > 0$ *imply* $c = a$ *or* $c = b$.

## 3. Fuzzy Modularity in fuzzy Birkhoff Lattice

In this section, we define a fuzzy Birkhoff lattice, a $FM$-symmetric fuzzy lattice and obtain a characterization theorem. Also, some lemmas and propositions are derived.

**Lemma 3.1.** *Suppose that* $b, c \in X$. *Then* $(b,c)_F M_m$ *if and only if* $A(b \wedge_F c, a) > 0$ *and* $A(a,c) > 0$ *imply that* $(a \vee_F b) \wedge_F c = a$.

*Proof.* Assume that $(b,c)_F M_m$ holds.
Let $a \in X$ be such that

$$A(b \wedge_F c, a) > 0 \text{ and } A(a,c) > 0.$$

From $A(a,c) > 0$ and $(b,c)_F M_m$ we have

$$(a \vee_F b) \wedge_F c = a \vee_F (b \wedge_F c). \tag{3.1}$$

As $A(b \wedge_F c, a) > 0$, so by (iv) of Proposition 2.1, we have

$$(b \wedge_F c) \vee_F a = a.$$

So, (3.1) reduces to $(a \vee_F b) \wedge_F c = a$.

Conversely, we show that $(b,c)_F M_m$ holds.
Let $d \in X$ be such that

$$A(d,c) > 0.$$

Put $d_1 = d \vee_F (b \wedge_F c)$.
By (iv) of Proposition 2.1 we get

$$A(b \wedge_F c, d_1) > 0. \tag{3.2}$$

As $A(d,c) > 0$ so, by (vi) of Proposition 2.1, we have

$$A(d \vee_F (b \wedge_F c), c \vee_F (b \wedge_F c)) > 0.$$

By absorption law we get

$$A(d \vee_F (b \wedge_F c), c) > 0,$$

that is,

$$A(d_1, c) > 0. \tag{3.3}$$

From (3.2) and (3.3) we have

$$(d_1 \vee_F b) \wedge_F c = d_1. \tag{3.4}$$

We have

$$\begin{aligned} d \vee_F (b \wedge_F c) &= d_1, \\ &= [d_1 \vee_F b] \wedge_F c, \text{ by (3.4)} \\ &= [d \vee_F (b \wedge_F c) \vee_F b] \wedge_F c, \\ &= (d \vee_F b) \wedge_F c. \textit{ by absorption law} \end{aligned}$$

that is,

$$(d \vee_F b) \wedge_F c = d \vee_F (b \wedge_F c).$$

Hence $(b, c)_F M_m$ holds. □

**Lemma 3.2.** *Let $b, c \in X$. If $(b, c)_F M_m$, then $(b_1, c_1)_F M_m$ for all $b_1, c_1 \in X$ with $A(b \wedge_F c, b_1) > 0$, $A(b \wedge_F c, c_1) > 0$, $A(b_1, b) > 0$ and $A(c_1, c) > 0$.*

*Proof.* Let $b_1, c_1 \in X$ be such that

$$A(b \wedge_F c, b_1) > 0,\ A(b \wedge_F c, c_1) > 0,\ A(b_1, b) > 0 \text{ and } A(c_1, c) > 0.$$

Let $a_1 \in X$ be such that

$$A(b_1 \wedge_F c_1, a) > 0,\ A(a, c_1) > 0.$$

Given $A(b \wedge_F c, b_1) > 0$, $A(b \wedge_F c, c_1) > 0$,
hence by (iii) of Proposition 2.1, we get

$$A(b \wedge_F c, b_1 \wedge_F c_1) > 0. \tag{3.5}$$

As $A(b_1 \wedge_F c_1, b_1) > 0$ and $A(b_1, b) > 0$ by fuzzy transitivity of $A$ we have

$$A(b_1 \wedge_F c_1, b) > 0.$$

Also, $A(b_1 \wedge_F c_1, c_1) > 0$, $A(c_1, c) > 0$ by fuzzy transitivity of $A$ we have

$$A(b_1 \wedge_F c_1, c) > 0.$$

Thus, $b_1 \wedge_F c_1$ is a lower bound of $\{b, c\}$ and $b \wedge_F c$ is the greatest lower bound of $\{b, c\}$. We have

$$A(b_1 \wedge_F c_1, b \wedge_F c) > 0. \tag{3.6}$$

From (3.5) and (3.6) by fuzzy antisymmetry of $A$ we get

$$b_1 \wedge_F c_1 = b \wedge_F c.$$

As $A(a, c_1) > 0$ and $A(c_1, c) > 0$ by fuzzy transitivity of $A$ we have

$$A(a, c) > 0.$$

Therefore, we get $A(b \wedge_F c, a) > 0$ and $A(a, c) > 0$ which imply by $(b, c)_F M_m$ that

$$(a \vee_F b) \wedge_F c = a.$$

As $A(b_1, b) > 0$ by (vi) of Proposition 2.1, we have

$$A(a \vee_F b_1, a \vee_F b) > 0$$

again by (vi) of Proposition 2.1, we have

$$A((a \vee_F b_1) \wedge_F c_1, (a \vee_F b) \wedge_F c_1) > 0.$$

As $A(c_1, c) > 0$ so, by (v) of Proposition 2.1, we get

$$c \wedge_F c_1 = c_1.$$

Therefore,

$$A((a \vee_F b_1) \wedge_F c_1, (a \vee_F b) \wedge_F c \wedge_F c_1) > 0,$$

$A((a \vee_F b_1) \wedge_F c_1, a \wedge_F c_1) > 0$ so we get

$$A((a \vee_F b_1) \wedge_F c_1, a) > 0 \tag{3.7}$$

and

$$A(a, a \vee_F (b_1 \wedge_F c_1)) > 0. \tag{3.8}$$

From (3.7) and (3.8) by fuzzy transitivity of $A$ we get

$$A((a \vee_F b_1) \wedge_F c_1, a \vee_F (b_1 \wedge_F c_1)) > 0. \tag{3.9}$$

We know that

$$A(a \vee_F (b_1 \wedge_F c_1), (a \vee_F b_1) \wedge_F c_1) > 0 \tag{3.10}$$

always holds.
By (3.9) and (3.10) by fuzzy antisymmetry of $A$ we get

$$a \vee_F (b_1 \wedge_F c_1) = (a \vee_F b_1) \wedge_F c_1.$$

Hence $(b_1, c_1)_F M_m$. □

**Corollary 3.1.** *Let $(X, A)$ be a fuzzy lattice. Then $(a \wedge_F b, a)_F M_m$, $(a \wedge_F b, b)_F M_m$, $(a, a \vee_F b)_F M_m$, $(b, a \vee_F b)_F M_m$ and $(a \wedge_F b, a \vee_F b)_F M_m$.*

**Corollary 3.2.** *Each fuzzy chain is a fuzzy modular lattice.*

**Definition 3.1.** *A fuzzy lattice $\mathcal{L} = (X, A)$ is called a fuzzy Birkhoff lattice if in $\mathcal{L}$, $a \wedge_F b \prec_F a$, $b$ implies that $a$, $b \prec_F a \vee_F b$ for $a$, $b \in X$.*

**Definition 3.2.** *A fuzzy lattice $\mathcal{L} = (X, A)$ is called FM-symmetric if in $\mathcal{L}$, $(a, b)_F M_m$ implies $(b, a)_F M_m$.*

**Theorem 3.1.** *Every $FM$-symmetric fuzzy lattice is a fuzzy Birkhoff lattice.*

*Proof.* Let $\mathcal{L} = (X, A)$ be a $FM$-symmetric fuzzy lattice. Let $a, b \in X$.
Suppose that $a \wedge_F b \prec_F b$.
We know that $A(a, a \vee_F b) > 0$, $A(b, a \vee_F b) > 0$.
To prove $a \prec_F a \vee_F b$ holds.
Suppose that $c \in X$ satisfies $A(a, c) > 0$ and

$$A(c, a \vee_F b) > 0. \tag{3.11}$$

From $A(a, c) > 0$, we have $A(a \wedge_F b, c \wedge_F b) > 0$. Also $A(c \wedge_F b, b) > 0$ holds.
Since $a \wedge_F b \prec_F b$ holds, we have either $a \wedge_F b = c \wedge_F b$ or $c \wedge_F b = b$.
If $c \wedge_F b = b$, then by (v) of Proposition 2.1, we have

$$A(b, c) > 0.$$

As $A(a, c) > 0$ by using (vi) of Proposition 2.1, we have
$A(a \vee_F b, c \vee_F b) > 0$ we get

$$A(a \vee_F b, c) > 0. \tag{3.12}$$

Hence from (3.11) and (3.12) by fuzzy antisymmetry of $A$ we get

$$c = a \vee_F b.$$

Suppose that $a \wedge_F b = c \wedge_F b$.
We now show that $(c, b)_F M_m$ holds.
Let $d \in X$ be such that

$$A(c \wedge_F b, d) > 0 \ \text{ and } A(d, b) > 0.$$

Therefore,

$$A(a \wedge_F b, d) > 0 \text{ and } A(d, b) > 0.$$

As $a \wedge_F b \prec_F b$ we get

$$a \wedge_F b = d \text{ or } d = b.$$

Case (1): Let $a \wedge_F b = d$.
Then

$$\begin{aligned} (d \vee_F c) \wedge_F b &= ((a \wedge_F b) \vee_F c) \wedge_F b, \\ &= ((c \wedge_F b) \vee_F c) \wedge_F b, \ \text{ as } a \wedge_F b = c \wedge_F b \\ &= c \wedge_F b, \ \text{ by absorption law} \\ &= a \wedge_F b, \ \text{ as } a \wedge_F b = c \wedge_F b \\ &= d. \end{aligned}$$

Case (2): Let $d = b$.
Then

$$\begin{aligned} (d \vee_F c) \wedge_F b &= (b \vee_F c) \wedge_F b, \\ &= b, \ \text{ by absorption law} \\ &= d. \end{aligned}$$

Hence by Lemma 3.1, we get $(c, b)_F M_m$.
As $(X, A)$ is a $FM$-symmetric fuzzy lattice we conclude that $(b, c)_F M_m$ holds.
Since

$$A(b \wedge_F c, a) > 0 \text{ and } A(a, c) > 0,$$

we have,

$$\begin{aligned} c &= (a \vee_F b) \wedge_F c, \\ &= a \vee_F (b \wedge_F c), \ \text{ using } \ (b, c)_F M_m \\ &= a. \end{aligned}$$

We get

$$a = c \text{ or } c = a \vee_F b.$$

Therefore, $a \prec_F a \vee_F b$.
Similarly, we can prove that $b \prec_F a \vee_F b$. □

## 4. Right and left complements in fuzzy lattices

In this section, we define a right and a left complement in a fuzzy lattice. We also prove a result where the notions of a left and a right complement coincide in $(X, A)$.

**Definition 4.1.** *Let $a, b, b_1 \in X$. Then $b_1$ is called a right complement within $b$ of $a$ in $a \vee_F b$ if, $A(b_1, b) > 0$, $a \vee_F b_1 = a \vee_F b$ and $(a, b_1) \perp_F M_m$ hold.*

*We say that $b_1$ is a left complement within $b$ of $a$ in $a \vee_F b$ if $A(b_1, b) > 0$, $a \vee_F b_1 = a \vee_F b$, $(b_1, a) \perp_F M_m$.*

**Example 4.1.** *Let* $X = \{\bot, a, b, c, d, e, f, g, \top\}$ *and let* $A : X \times X \longrightarrow [0, 1]$ *be a fuzzy relation defined as follows:*
$A(\bot, \bot) = A(a, a) = A(b, b) = A(c, c) = A(d, d) = A(e, e) = A(f, f) = A(g, g) = 1.0$
$A(\top, \top) = 1.0,$

$A(\bot, a) = 0.2,\ A(\bot, b) = 0.2,\ A(\bot, c) = 0.2,\ A(\bot, d) = 0.2,\ A(\bot, e) = 0.2,\ A(\bot, f) = 0.2,$
$A(\bot, g) = 0.2,\ A(\bot, \top) = 0.8,$

$A(a, \bot) = 0,\ A(a, b) = 0,\ A(a, c) = 0,\ A(a, d) = 0.6,\ A(a, e) = 0.6,\ A(a, f) = 0.6,$
$A(a, g) = 0.6,\ A(a, \top) = 0.8,$

$A(b, \bot) = 0,\ A(b, a) = 0,\ A(b, c) = 0.6,\ A(b, d) = 0.6,\ A(b, e) = 0,\ A(b, f) = 0.6,$
$A(b, g) = 0.6,\ A(b, \top) = 0.8,$

$A(c, \bot) = 0,\ A(c, a) = 0,\ A(c, b) = 0,\ A(c, d) = 0,\ A(c, e) = 0,\ A(c, f) = 0,\ A(c, g) = 0.6,$
$A(c, \top) = 0.8,$

$A(d, \bot) = 0,\ A(d, a) = 0,\ A(d, b) = 0,\ A(d, c) = 0,\ A(d, e) = 0,\ A(d, f) = 0.6,$
$A(d, g) = 0.6,\ A(d, \top) = 0.8,$

$A(e, \bot) = 0,\ A(e, a) = 0,\ A(e, b) = 0,\ A(e, c) = 0,\ A(e, d) = 0,\ A(e, f) = 0.6,\ A(e, g) = 0,$
$A(e, \top) = 0.8,$

$A(f, \bot) = 0,\ A(f, a) = 0,\ A(f, b) = 0,\ A(f, c) = 0,\ A(f, d) = 0,\ A(f, e) = 0,\ A(f, g) = 0,$
$A(f, \top) = 0.8,$

$A(g, \bot) = 0,\ A(g, a) = 0,\ A(g, b) = 0,\ A(g, c) = 0,\ A(g, d) = 0,\ A(g, e) = 0,\ A(g, f) = 0,$
$A(g, \top) = 0.8,$

$A(\top, \bot) = 0,\ A(\top, a) = 0,\ A(\top, b) = 0,\ A(\top, c) = 0,\ A(\top, d) = 0,\ A(\top, e) = 0,$

$A(\top, f) = 0,\ A(\top, g) = 0.$

*Then* $A$ *is a fuzzy partial order relation.*

*We note that* $(X, A)$ *is a fuzzy lattice.*

*This fuzzy relation is summarized in the following table:*

| $A$ | $\bot$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $\top$ |
|---|---|---|---|---|---|---|---|---|---|
| $\bot$ | 1.0 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 | 0.8 |
| $a$ | 0.0 | 1.0 | 0.0 | 0.0 | 0.6 | 0.6 | 0.6 | 0.6 | 0.8 |
| $b$ | 0.0 | 0.0 | 1.0 | 0.6 | 0.6 | 0.0 | 0.6 | 0.6 | 0.8 |
| $c$ | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.6 | 0.8 |
| $d$ | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.6 | 0.6 | 0.8 |
| $e$ | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.6 | 0.0 | 0.8 |
| $f$ | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.8 |
| $g$ | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.8 |
| $\top$ | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 |

The fuzzy join and fuzzy meet tables are as follows:

| $\vee_F$ | $\bot$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $\top$ |
|---|---|---|---|---|---|---|---|---|---|
| $\bot$ | $\bot$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $\top$ |
| $a$ | $a$ | $a$ | $d$ | $g$ | $d$ | $e$ | $f$ | $g$ | $\top$ |
| $b$ | $b$ | $d$ | $b$ | $c$ | $d$ | $f$ | $f$ | $g$ | $\top$ |
| $c$ | $c$ | $g$ | $c$ | $c$ | $g$ | $\top$ | $\top$ | $g$ | $\top$ |
| $d$ | $d$ | $d$ | $d$ | $g$ | $d$ | $f$ | $f$ | $g$ | $\top$ |
| $e$ | $e$ | $e$ | $f$ | $\top$ | $f$ | $e$ | $f$ | $\top$ | $\top$ |
| $f$ | $f$ | $f$ | $f$ | $\top$ | $f$ | $f$ | $f$ | $\top$ | $\top$ |
| $g$ | $g$ | $g$ | $g$ | $g$ | $g$ | $\top$ | $\top$ | $g$ | $\top$ |
| $\top$ | $\top$ | $\top$ | $\top$ | $\top$ | $\top$ | $\top$ | $\top$ | $\top$ | $\top$ |

| $\wedge_F$ | $\bot$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $\top$ |
|---|---|---|---|---|---|---|---|---|---|
| $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ |
| $a$ | $\bot$ | $a$ | $\bot$ | $\bot$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $\bot$ | $\bot$ | $b$ | $b$ | $b$ | $\bot$ | $b$ | $b$ | $b$ |
| $c$ | $\bot$ | $\bot$ | $b$ | $c$ | $b$ | $\bot$ | $b$ | $c$ | $c$ |
| $d$ | $\bot$ | $a$ | $b$ | $b$ | $d$ | $a$ | $d$ | $d$ | $d$ |
| $e$ | $\bot$ | $a$ | $\bot$ | $\bot$ | $a$ | $e$ | $e$ | $a$ | $e$ |
| $f$ | $\bot$ | $a$ | $b$ | $b$ | $d$ | $e$ | $f$ | $d$ | $f$ |
| $g$ | $\bot$ | $a$ | $b$ | $c$ | $d$ | $a$ | $d$ | $g$ | $g$ |
| $\top$ | $\bot$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $\top$ |

As $A(e,f)>0$, $c\vee_F f=\top$, $e\vee_F c=\top$ and $c\wedge_F e=\bot$.
As $A(a,e)>0$ and $(a\vee_F c)\wedge_F e=g\wedge_F e=a=a\vee_F\bot=a\vee_F(c\wedge_F e)$.
Hence $(c,e)_F M_m$ holds.
As $c\wedge_F e=\bot$ and $(c,e)_F M_m$ hold.
Thus, $(c,e)\perp_F$ holds.
Hence $e$ is a right complement within $f$ of $c$ in $f\vee_F c$.

**Theorem 4.1.** *Let $(a,b)_F M_m$. Then $b_1$ is a right complement within $b$ of $a$ in $a\vee_F b$ iff $b_1$ is a right complement of $a\wedge_F b$ in $b$.*

*Proof.* Suppose that $b_1$ is a right complement within $b$ of $a$ in $a\vee_F b$.
Then we have

$$A(b_1,b)>0,\ b_1\vee_F a=b\vee_F a \text{ and } (a,b_1)\perp_F M_m,$$

that is, $a\wedge_F b_1=\bot$ and $(a,b_1)_F M_m$ hold.
Then

$$\begin{aligned} b_1\vee_F(a\wedge_F b) &= (b_1\vee_F a)\wedge_F b,\ \text{ since } (a,b)_F M_m\\ &= (b\vee_F a)\wedge_F b,\\ &= b,\ \text{by absorption law}\\ &= b\vee_F(a\wedge_F b). \end{aligned}$$

Therefore,

$$b_1\vee_F(a\wedge_F b)=b\vee_F(a\wedge_F b).$$

To show that $(a\wedge_F b,b_1)\perp_F M_m$ holds.
that is, to show: (i) $(a\wedge_F b)\wedge_F b_1=\bot$ (ii) $(a\wedge_F b,b_1)_F M_m$.
We have,

$$\begin{aligned} (a\wedge_F b)\wedge_F b_1 &= a\wedge_F(b\wedge_F b_1),\\ &= a\wedge_F b_1,\ \text{ because }\ A(b_1,b)>0\\ &= \bot. \end{aligned}$$

As $(a,b_1)_F M_m$ holds and $A(a\wedge_F b_1,b)>0$ then by Lemma 2.1, we have $(a\wedge_F b,b_1)_F M_m$.
Therefore, $(a\wedge_F b,b_1)\perp_F M_m$ holds.

Conversely, suppose that $b_1$ is right complement of $a\wedge_F b$ in $b$.
To show that $b_1$ is a right complement within $b$ of $a$ in $a\vee_F b$.
Suppose

$$(a\wedge_F b)\vee_F b_1=b=(a\wedge_F b)\vee_F b, \tag{4.1}$$

so, we have $A(b_1,b)>0$ and by (v) of Proposition 2.1, we have

$$b_1\wedge_F b=b_1.$$

Also, $(a\wedge_F b,b_1)\perp_F M_m$ implies

$$(a\wedge_F b)\wedge_F b_1=\bot \text{ and } (a\wedge_F b,b_1)_F M_m.$$

We have,

$$\begin{aligned} a\vee_F b_1 &= a\vee_F(a\wedge_F b)\vee_F b_1,\ \text{ putting } a=a\vee_F(a\wedge_F b)\\ &= a\vee_F b,\ \text{ from (4.1)} \end{aligned}$$

Therefore, we get $a\vee_F b_1=a\vee_F b$.
Clearly $a\wedge_F b_1=a\wedge_F(b_1\wedge_F b)=\bot$.
To show that $A((d\vee_F a)\wedge_F b_1, d\vee_F(a\wedge_F b_1))>0$ holds.
Let $A(d,b_1)>0$.

We have
$A((d \vee_F a) \wedge_F b_1, d \vee_F (a \wedge_F b_1))$
$= A((d \vee_F a) \wedge_F (b_1 \wedge_F b), d)$, as $a \wedge_F b_1 = \perp$ and $b_1 \wedge_F b = b_1$
$= A([(d \vee_F a) \wedge_F b] \wedge_F b_1, d)$,
$= A([d \vee_F (a \wedge_F b)] \wedge_F b_1, d)$, as $A(d, b_1) > 0$, $A(b_1, b) > 0$, imply $A(d, b) > 0$ and $(a, b)_F M_m$
$= A(d \vee_F [(a \wedge_F b) \wedge_F b_1], d)$, because $A(d, b_1) > 0$ and $(a \wedge_F b, b_1)_F M_m$
$= A(d \vee_F \perp, d)$, as $(a \wedge_F b) \wedge_F b_1 = \perp$
$= A(d, d)$,
$= 1 > 0$.
Therefore,

$$A((d \vee_F a) \wedge_F b_1, d \vee_F (a \wedge_F b_1)) > 0 \tag{4.2}$$

holds.
As

$$A(d \vee_F (a \wedge_F b_1), (d \vee_F a) \wedge_F b_1) > 0 \tag{4.3}$$

always holds.
From (4.2) and (4.3) by antisymmetry of $A$ we have

$$d \vee_F (a \wedge_F b_1) = (d \vee_F a) \wedge_F b_1.$$

So, $(a, b_1)_F M_m$ holds.
Hence $(a, b_1) \perp_F M_m$ holds.
Therefore, $b_1$ is a right complement within $b$ of $a$ in $a \vee_F b$. □

We see that the forward implication holds also for left complement, that is, if $b_1$ is a left complement within $b$ of $a$ in $a \vee_F b$, then $b_1$ is a left complement of $a \wedge_F b$ in $b$.

**Theorem 4.2.** *If $\mathcal{L} = (X, A)$ is left complemented, then $\mathcal{L}$ is right complemented. The notions of right and left complement coincide, and $\mathcal{L}$ is a $FM$-symmetric fuzzy lattice.*

*Proof.* We show that $\mathcal{L}$ is a $FM$-symmetric fuzzy lattice.
Suppose that $(b, c)_F M_m$ holds.
We have to show that $(c, b)_F M_m$ holds.
Since $\mathcal{L}$ is left complemented there exists a left complement $c_1$ within $c$ of $b$ in $b \vee_F c$.
Hence

$$A(c_1, c) > 0,\ c \vee_F b = c_1 \vee_F b \text{ and } (c_1, b) \perp_F M_m,$$

that is, $c_1 \wedge_F b = \perp$ and $(c_1, b)_F M_m$ hold.
To show that $(c, b)_F M_m$ holds.
Let $b_1 \in X$ be such that $A(b_1, b) > 0$.
By Theorem 4.1, $c_1$ is a left complement of $b \wedge_F c$ in $c$.
Hence

$$A(c_1, c) > 0,\ c_1 \vee_F (b \wedge_F c) = c \vee_F (b \wedge_F c) = c \text{ and } (c_1, b \wedge_F c) \perp_F M_m.$$

We have

$$\begin{aligned}
(b_1 \vee_F c) \wedge_F b &= [b_1 \vee_F c_1 \vee_F (b \wedge_F c)] \wedge_F b, \text{ as } c = c_1 \vee_F (b \wedge_F c) \\
&= [b_1 \vee_F (b \wedge_F c) \vee_F c_1] \wedge_F b \\
&= b_1 \vee_F (b \wedge_F c) \vee_F (c_1 \wedge_F b), \text{ as } (c_1, b) \perp_F M_m \text{ and } A(b_1, b) > 0 \\
&= b_1 \vee_F (b \wedge_F c) \vee_F \perp, \\
&= b_1 \vee_F (b \wedge_F c).
\end{aligned}$$

Therefore, $(b_1 \vee_F c) \wedge_F b = b_1 \vee_F (b \wedge_F c)$.
Hence $(c, b)_F M_m$ holds.
Thus, $\mathcal{L}$ is a $FM$-symmetric fuzzy lattice. □

**Corollary 4.1.** *If $\mathcal{L} = (X, A)$ is $\perp_F$-symmetric and complemented, then $\mathcal{L} = (X, A)$ is a $FM$-symmetric fuzzy lattice.*

*Proof.* Obvious as $\mathcal{L} = (X, A)$ is $\perp_F$-symmetric. □

**Theorem 4.3.** *Let $\mathcal{L} = (X, A)$ be a left complemented lattice. Let $a, b \in X$. If every complement $b_1$ within $b$ of $a$ in $a \vee_F b$ is a complement of $a \wedge_F b$ in $b$, then $(a, b)_F M_m$.*

*Proof.* Let $A(d, b) > 0$ and let $d_1$ be a left complement within $d$ of $a$ in $a \vee_F d$.
Then $A(d_1, d) > 0$, $d_1 \vee_F a = d \vee_F a$ and $(d_1, a) \perp_F M_m$ holds.
Now, let $b_2 \in X$ be a complement within $b$ of $d \vee_F a$ in $b \vee_F a$.
Then $A(b_2, b) > 0$ and $b_2 \vee_F d_1 \vee_F a = b_2 \vee_F d \vee_F a = b \vee_F a$ and $(b_2, d \vee_F a) \perp_F M_m$.
Hence $(b_2, d_1 \vee_F a) \perp_F M_m$ holds.
So, $(b_2, d_1 \vee_F a)_F M_m$ holds and $b_2 \wedge_F (d_1 \vee_F a) = \perp$.
Put

$$b_1 = b_2 \vee_F d_1. \tag{4.4}$$

Then $A(b_1, b) > 0$, $b_1 \vee_F a = b \vee_F a$, $(b_1, a) \perp_F M_m$,
that is, $b_1$ is a complement within $b$ of $a$ in $a \vee_F b$.
Given that $b_1$ is a complement of $a \wedge_F b$ in $b$,
hence

$$b_1 \vee_F (a \wedge_F b) = b \vee_F (a \wedge_F b) = b. \tag{4.5}$$

As $A(a \wedge_F b, a) > 0$ so by (vi) of Proposition 2.1, we have

$$A((a \wedge_F b) \vee_F d_1, a \vee_F d_1) > 0.$$

We have

$$\{[(a \wedge_F b) \vee_F d_1] \vee_F b_2\} \wedge_F (d_1 \vee_F a) = [(a \wedge_F b) \vee_F d_1] \vee_F [b_2 \wedge_F (d_1 \vee_F a)]. \tag{4.6}$$

To show that $(a, b)_F M_m$ holds.
Let $d \in X$ be such that $A(d, b) > 0$.
We know that

$$A(d \vee_F (a \wedge_F b), (d \vee_F a) \wedge_F b) > 0 \tag{4.7}$$

always holds.
To show that $A((d \vee_F a) \wedge_F b, d \vee_F (a \wedge_F b)) > 0$ holds.
We have
$A((d \vee_F a) \wedge_F b, d \vee_F (a \wedge_F b))$
$= A((d_1 \vee_F a) \wedge_F b, d \vee_F (a \wedge_F b))$, because $d_1 \vee_F a = d \vee_F a$
$= A((d_1 \vee_F a) \wedge_F [b_1 \vee_F (a \wedge_F b)], d \vee_F (a \wedge_F b))$, by (4.5)
$= A((d_1 \vee_F a) \wedge_F \{b_2 \vee_F d_1 \vee_F (a \wedge_F b)\}, d \vee_F (a \wedge_F b))$, by (4.4)
$= A((d_1 \vee_F a) \wedge_F \{b_2 \vee_F [d_1 \vee_F (a \wedge_F b)]\}, d \vee_F (a \wedge_F b))$,
$= A([(d_1 \vee_F a) \wedge_F b_2] \vee_F [d_1 \vee_F (a \wedge_F b)], d \vee_F (a \wedge_F b))$, by (4.6)
$= A(\perp \vee_F [d_1 \vee_F (a \wedge_F b)], d \vee_F (a \wedge_F b))$, as $(b_2, d_1 \vee_F a) \perp_F M_m$.
Thus

$$A((d \vee_F a) \wedge_F b, d \vee_F (a \wedge_F b)) = A(d_1 \vee_F (a \wedge_F b), d \vee_F (a \wedge_F b)). \tag{4.8}$$

As $A(d_1, d) > 0$ we have $A(d_1 \vee_F (a \wedge_F b), d \vee_F (a \wedge_F b)) > 0$.
Hence by (4.8) we have

$$A((d \vee_F a) \wedge_F b, d \vee_F (a \wedge_F b)) > 0. \tag{4.9}$$

From (4.7) and (4.9) by antisymmetry of $A$ we have $(d \vee_F a) \wedge_F b = d \vee_F (a \wedge_F b)$. Thus, $(a, b)_F M_m$ holds. □

## 5. Conclusion and Future work

In this article we have presented a novel approach to fuzzy Birkhoff lattices and complements in a fuzzy lattice $(X, A)$. This theory will be useful to develop the theory of modular pairs, distributive pairs, symmetricity in fuzzy lattices and fuzzy partially ordered sets.

Future work. One of the most promising idea could be the investigation of fuzzy semi-orthogonality in fuzzy lattices and study some properties.

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