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SPECTRAL INCLUSION BETWEEN A REGULARIZED QUASI-SEMIGROUPS AND THEIR GENERATORS

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Abstract. The notion of a regularized quasi-semigroups (or C-quasi-semigroups) of a bounded linear operators, as a generalization of C_0-quasi-semigroups of a bounded linear operators, was introduced by M. Janfada in 2010. In this paper, we will show some results concerning a regularized quasi-semigroups and we are going to show a spectral inclusion of a different spectra of a C-quasi-semigroups of a bounded linear operators on a Banach space and their infinitesimal generators.

Keywords: C-quasi-semigroup, spectrum, residual, essential, ascent.

AMS Subject Classification: 47A10, 47D06.

1. Introduction

We consider the time-independent abstract Cauchy problems :

$$x'(t) = Ax(t), \quad t \geq 0, \tag{1}$$

in a Banach space X and A an operator defined on the dense domain $D(A) \subset X$. If A is the generator of a C_0 -semigroup of bounded linear operators of X, then the theory of semigroups is a powerful tool for solving (1) (see [8] and [11]). In 1979, R. Derndinger and R. Nagel [6] showed that if $(T(t))_{t\geq 0}$ is a C_0-semi-group and A its generator, then $e^{t\sigma(A)} \subseteq \sigma(T(t))\backslash\{0\}$, $e^{t\sigma_p(A)} \subseteq \sigma_p(T(t))\backslash\{0\}$ et $e^{t\sigma_r(A)} \subseteq \sigma_r(T(t))\backslash\{0\}$, in 2001, A. El Koutri and A. Taoudi in [7] proved that $e^{t\sigma_K(A)} \subseteq \sigma_K(T(t))\backslash\{0\}$ and Recently, in [13] and [14], A. Tajmouati, F. Alhomaidi and H. Boua are studied different spectra of a C_0-Semi-group and its generator.

In 1953, Tosio Kato [10] considered the following evolution equation:

$$x'(t) = A(s+t)x(t) + f(t), \quad 0 \leq t \leq T, \quad x(0) = x_0 \tag{2}$$

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and the associated homogeneous equation

$$x'(t) = A(s+t)x(t), \;\; 0 \leq t \leq T, \;\; x(0) = x_0 \tag{3}$$

with $x(.)$ is an unknown function of the real interval $[0,T]$ into a Banach space X, and $A(s)$ is a closed operator given on X of domain $D(A(s)) = \mathcal{D}$ independent of s and dense in X. The solution of (2) is formally given by $x(t) = R(s;t)x_0$. A two parameter family $\{R(t;s)\}_{t;s\geq 0}$ on X is called a C_0-quasi-semigroup and $A(s)$ its generator.

In [15], [16] and [17] we obtained a spectral inclusion between a C_0-Quasi-semigroup and its generator for different party of ordinary spectrum.

Now, we consider the time-dependent abstract Cauchy problems

$$x'(t) = A(s+t)x(t), \;\; t,s \geq 0, \;\; x(0) = Cx_0 \tag{4}$$

Here $x(.)$ is an unknown function from the real interval $[0,T]$ into a Banach space X, C is an injective bounded linear operator on a Banach space X and $A(s)$ is a given, closed, linear operator in X with domain $\mathcal{D}(A(s)) = \mathcal{D}$, independent of s and dense in X. The solution of (4) is formally given by $x(t) = K(s,t)x_0$, a two parameter family $\{K(s,t)\}_{s,t\geq 0}$ on X is called a C-quasi-semigroups. Then, we have the existence of a solution for the Cauchy problem without any qualitative information on it. A classical approach to information on the solution $x(t)$ consists in directly studying the spectrum of the quasi-semigroup $K(s,t)$. In many applications, we only have the explicit expression of the generator $A(s)$. Hence, the need to have a relation betwen the spectrum of the quasi-semigroup $K(s,t)$ and the spectrum of its generator $A(s)$.

2. Preliminaries

Throughout this paper, $\mathcal{B}(X)$ denotes the algebra of all bounded linear operators on a Banach space X and T will be a closed linear operator on X with domain $D(T)$. We denote by $Rg(T)$, $Rg^{\infty}(T) := \bigcap_{n\geq 1} Rg(T^n)$, $N(T)$, $\rho(T)$ and $\sigma(T)$ respectively the range, the hyper range, the kernel, the resolvent and the spectrum of T, where

$$\sigma(T) = \mathbb{C}\backslash\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bijective}\}.$$

The function resolvent of T is defined for all $\lambda \in \rho(T)$ by $\mathcal{R}(\lambda, T) = (\lambda I - T)^{-1}$.
For a closed operator T we define the point spectrum, the approximate point spectrum and the residual spectrum by

- $\sigma_p(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not injective }\}$.
- $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not injective or } Rg(\lambda I - T) \text{ is not closed in } X\}$.
- $\sigma_r(T) = \{\lambda \in \mathbb{C} : Rg(\lambda I - T) \text{ is not dense in } X\}$.

From [1, p.79], we have $\lambda \in \sigma_{ap}(T)$ if and only if there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset D(T)$, such that $\|x_n\| = 1$ and $\lim_{n\to\infty} \|(T - \lambda I)x_n\| = 0$.
The ascent and descent of an operator T are defined respectively by,

$$a(T) = \inf\{k \in \mathbb{N} : N(T^k) = N(T^{k+1})\} \; ; \; d(T) = \inf\{k \in \mathbb{N} : Rg(T^k) = Rg(T^{k+1})\}.$$

with the convention $\inf(\varnothing) = \infty$.

The ascent spectrum and descent spectrum are defined respectively by,

$$\sigma_a(T) = \{\lambda \in \mathbb{C} : a(\lambda I - T) = \infty\}, \; \sigma_d(T) = \{\lambda \in \mathbb{C} : d(\lambda I - T) = \infty\}.$$

A closed operator T is called Fredholm if $\alpha(T) = \dim N(T)$ and $\beta(T) = co\dim Rg(T)$ are finite. The essential spectrum is defined by,

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not Fredholm}\}.$$

Similarly, we can define the spectra $\sigma(C,T)$, $\sigma_p(C,T)$, $\sigma_{ap}(C,T)$, $\sigma_e(C,T)$, $\sigma_r(C,T)$, $\sigma_a(C,T)$ and $\sigma_d(C,T)$, replacing the identity operator I by an injective operator $C \in B(X)$.

Let $C \in \mathcal{B}(X)$ be injective. The family $(S(t))_{t\geq 0} \subseteq \mathcal{B}(X)$ is a C-semigroup [5] if it has the following properties:

(1) S(0)=C,
(2) S(t)S(s)=CS(t+s),
(3) The map $t \to S(t)x$ from $[0,+\infty[$ into X is continuous for all $x \in X$.

In this case, its generator A is defined by

$$D(A) = \{x \in X \, : \lim_{t\to 0^+} \frac{S(t)x - Cx}{t} \text{ exists and it's in } Rg(C)\},$$

with

$$Ax = C^{-1}\big[\lim_{t\to 0^+} \frac{S(t)x - Cx}{t}\big], \text{ for all } x \in D(A).$$

In particular, the C_0-semigroups are the I-semigroups where I is the identity operator.

The theory of quasi-semigroups of bounded linear operators, as a generalization of semigroups of operators, was introduced by H. Leiva and D. Barcenas [2], [3], [4] and recently Sutrima et al. [12], have shown some relations between a C_0-quasi-semigroup and its generator related to the time-dependent evolution equation.
A two parameter commutative family $\{R(t,s)\}_{t,s\geq 0} \subseteq \mathcal{B}(X)$ is called a strongly continuous quasi-semigroup (or C_0-quasi-semigroup) of operators [2] if for every $t,s,r \geq 0$ and $x \in X$, we have

(1) $R(t,0) = I$, the identity operator on X,
(2) $R(t,s+r) = R(t+r,s)R(t,r)$,
(3) $\lim_{(t,s)\longrightarrow(t_0,s_0)} ||R(t,s)x - R(t_0,s_0)x|| = 0, \quad x \in X,$
(4) there exists a continuous increasing mapping $M : [0,+\infty[\longrightarrow [1,+\infty[$ such that,
$$||R(t,s)|| \leq M(t+s).$$

For a C_0-quasi-semigroup $\{R(t,s)\}_{t,s\geq 0}$ on a Banach space X, let $\mathcal{D}$ be the set of all $x \in X$ for which the following limits exist,

$$\lim_{s\to 0^+} \frac{R(0,s)x - x}{s} \text{ and } \lim_{s\to 0^+} \frac{R(t,s)x - x}{s} = \lim_{s\to 0^+} \frac{R(t-s,s)x - x}{s}, \ t > 0.$$

In this case, for $t \geq 0$, we define an operator $A(t)$ on $\mathcal{D}$ as

$$A(t)x = \lim_{s\to 0^+} \frac{R(t,s)x - x}{s}.$$

The family $\{A(t)\}_{t\geq 0}$ is called the infinitesimal generator of the C_0-quasi-semigroups $\{R(t,s)\}_{t,s\geq 0}$. The generator $A(t)$ of a C_0-quasi-semigroup is not necessary closed or densely defined [12, Examples 2.3 and 3.3].

In [9] M. Janfada introduced the notion of regularized quasi-semigroup of a bounded linear operators on a Banach spaces, as a generalization of regularized semigroups of operators.

Definition 2.1. [9, Definition 2.1]

Suppose that C is an injective bounded linear operator on a Banach space X. A commutative two parameter family $\{K(t,s)\}_{t,s\geq 0} \subseteq \mathcal{B}(X)$ is called a regularized quasi-semigroups (or C-quasi-semigroups) if for every $t,s,r \geq 0$ and $x \in X$, we have

(1) $K(t,0) = C$;
(2) $CK(t,s+r) = K(t+r,s)K(t,r)$;
(3) $\{K(t,s)\}_{t,s\geq 0}$ *is strongly continuous, that is,*

$$\lim_{(t,s)\longrightarrow(t_0,s_0)} ||K(t,s)x - K(t_0,s_0)x|| = 0,\ x \in X;$$

(4) *there exists a continuous and increasing mapping* $M : [0,+\infty[\longrightarrow [0,+\infty[$ *such that, for any* $t,s > 0$, $||K(t,s)|| \leq M(t+s)$.

For a C-quasi-semigroup $\{K(t,s)\}_{t,s\geq 0}$ on a Banach space X, let $\mathcal{D}$ be the set of all $x \in X$ for which the following limits exist in the range of C:

$$\lim_{s\to 0^+} \frac{K(0,s)x - Cx}{s} \text{ and } \lim_{s\to 0^+} \frac{K(t,s)x - Cx}{s} = \lim_{s\to 0^+} \frac{K(t-s,s)x - Cx}{s},\ t > 0.$$

In this case, for $t \geq 0$, we define an operator $A(t)$ on $\mathcal{D}$ as

$$A(t)x = C^{-1} \lim_{s\to 0^+} \frac{K(t,s)x - Cx}{s}.$$

The family $\{A(t)\}_{t\geq 0}$ is called the infinitesimal generator of the regularized quasi-semigroup $\{K(t,s)\}_{t,s\geq 0}$.
In particular, the C_0-quasi-semigroups are the I-quasi-semigroups where I is the identity operator.

Example 2.1. [9, Examples 2.2, 2.4 and 2.5]

(1) *Let* $\{S(t)\}_{t\geq 0}$ *be a strongly continuous exponentially bounded* C*-semigroup of operators on a Banach space* X*, with the generator* A*.*
For $t,s \geq 0$*, define* $K(t,s) = S(s)$*, then* $\{K(t,s)\}_{t,s\geq 0}$ *is a* C*-quasi-semigroup with* $\mathcal{D} = D(A)$ *and its generator is* $A(t) = A$ *for all* $t \geq 0$*.*
(2) *Let* $\{T(t)\}_{t\geq 0}$ *be a strongly continuous semigroup of operators on a Banach space* X*, with the generator* A*. If* $C \in B(X)$ *is injective and commutes with* $T(t)$*,* $t \geq 0$*, then* $K(t,s) = Ce^{T(s+t)-T(t)}$*, for* $t,s \geq 0$ *is a* C*-quasi-semigroup with* $\mathcal{D} = D(A)$ *and its generator is* $A(t) = AT(t)$ *for all* $t \geq 0$*.*
(3) *Let* $\{S(t)\}_{t\geq 0}$ *be a strongly continuous exponentially bounded* C*-semigroup of operators on a Banach space* X*, with the generator* A*.*
For $t,s \geq 0$*, define* $K(t,s) = T(g(t+s) - g(t))$*. where* $g(t) = \int_0^t a(u)du$ *and* $a \in \mathcal{C}([0,+\infty[)$ *with* $a(t) > 0$*. Then* $\{K(t,s)\}_{t,s\geq 0}$ *is a* C*-quasi-semigroup with* $\mathcal{D} = D(A)$ *and its generator for all* $t \geq 0$

$$A(t) = a(t)A.$$

Theorem 2.1. [9, Theorems 2.6] *Let* $\{K(t,s)\}_{t,s\geq 0}$ *be a* C*-quasi-semigroup on a Banach space* X *with generator* $(A(t))_{t\geq 0}$*. Then we have*

(1) *If* $x \in \mathcal{D}$*,* $t \geq 0$ *and* $t_0, s_0 \geq 0$*, then* $K(t_0,s_0)x \in \mathcal{D}$ *and*

$$K(t_0,s_0)A(t)x = A(t)K(t_0,s_0)x.$$

(2) *For each* $x_0 \in \mathcal{D}$*,*

$$\frac{\partial}{\partial s}K(t,s)Cx_0 = A(t+s)K(t,s)Cx_0 = K(t,s)A(t+s)Cx_0.$$

(3) *If* $A(.)$ *is locally integrable, then for each* $x_0 \in \mathcal{D}$ *and* $s \geq 0$*,*

$$K(t,s)x_0 = Cx_0 + \int_0^s A(t+h)K(t,h)x_0 dh,\ t \geq 0$$

(4) *If* $f : [0, +\infty[\longrightarrow X$ *is a continuous function, then for every* $s \in [0, +\infty[$,
$\lim_{r\to 0^+} \int_s^{s+r} K(t, h)f(h)dh = K(t, s)f(s)$.

(5) *Let* $C' \in B(X)$ *be injective and for any* $t, s \geq 0$, $C'K(t, s) = K(t, s)C'$*.Then* $U(t, s) = C'K(t, s)$ *is a* CC'*-quasi-semigroup with the generator* $(A(t))_{t\geq 0}$.

(6) *Suppose that* $\{R(t, s)\}_{t,s\geq 0}$ *be a* C_0*-quasi-semigroup of operators on a Banach space* X *with the generator* $(A(t))_{t\geq 0}$ *and* $C \in B(X)$ *commutes with every* $R(t, s)$, $t, s \geq 0$. *Then* $K(t, s) = CR(t, s)$ *is a* C*-quasi-semigroup of operators on* X *with the generator* $(A(t))_{t\geq 0}$.

3. Main results

Inspired by the spectral studies of C_0-semigroups in the works [7],[8], [11], [13] and [14] and the inclusion spectrum for C_0-quasi-semigroups in papers [15], [16] and [17] and also the spectral mapping theorems for C-semigroups did by Song Xiaoqiu in [18]. We show a spectral inclusion of different spectra for C-quasi-semigroups and their generators.

We start by the important result.

Theorem 3.1. *Let* $A(t)$ *be the generator of the* C*-quasi-semigroup* $\{K(t, s)\}_{t,s\geq 0}$ *such that* $A(t)$ *is closed and densely defined, and let* $C \in B(X)$ *be injective. Then for all* $t \geq s \geq 0$ *and all* $\lambda \in \mathbb{C}$*, we have*

(1) *For all* $x \in \mathcal{D}$,

$$D_\lambda(t, s)(\lambda I - A(t))x = [e^{\lambda s}C - K(t - s, s)]x.$$

(2) *For all* $x \in X$*, we have* $D_\lambda(t, s)x \in \mathcal{D}$ *and*

$$(\lambda I - A(t))D_\lambda(t, s)x = [e^{\lambda s}C - K(t - s, s)]x.$$

where $D_\lambda(t, s)x = \displaystyle\int_0^s e^{\lambda(s-h)}K(t - h, h)xdh$ *is a bounded and linear operator.*

Proof. (1) First we note that from 2) of the definition of a C-quasi-semigroup $\{K(t, s)\}_{t,s\geq 0}$ with $r = 0$, we have $CK(t, s) = K(t, s)C$.
By Theorem 3.1 in [9] and theorem 2.1, $K(s, t)Cx_0$ is a unique solution of the problem $x'(t) = A(s + t)x(t)$, $t, s \geq 0$, $x(0) = C^2x_0$, moreover, for all $t > h > 0$ and for all $x \in \mathcal{D}$, $\frac{\partial}{\partial h}K(t - h, h)Cx = A(t)K(t - h, h)Cx = K(t - h, h)A(t)Cx$.
So, $\dfrac{\partial}{\partial h}(CK(t - h, h))x = CA(t)K(t - h, h)x = CK(t - h, h)A(t)x$.

Therefore, we conclude that

$$\begin{aligned}
D_\lambda(t, s)[A(t)x] &= \int_0^s e^{\lambda(s-h)}K(t - h, h)[A(t)x]dh \\
&= \int_0^s e^{\lambda(s-h)}C^{-1}CK(t - h, h)[A(t)x]dh \\
&= \int_0^s e^{\lambda(s-h)}C^{-1}\Big[\frac{\partial}{\partial h}(CK(t - h, h))\Big]xdh \\
&= \Big[e^{\lambda(s-h)}C^{-1}CK(t - h, h)x\Big]_0^s + \lambda\int_0^s e^{\lambda(s-h)}C^{-1}CK(t - h, h)xdh \\
&= \Big[e^{\lambda(s-h)}K(t - h, h)x\Big]_0^s + \lambda\int_0^s e^{\lambda(s-h)}K(t - h, h)xdh \\
&= K(t - s, s)x - e^{\lambda s}Cx + \lambda D_\lambda(t, s)x. \qquad (5)
\end{aligned}$$

Finally, we obtain for all $x \in \mathcal{D}$

$$D_\lambda(t,s)(\lambda I - A(t))x = [e^{\lambda s}C - K(t-s,s)]x.$$

(2) Let $\mu \in \rho(A(t))$. From [12, Theorem 3.4] and the commutativity of $\{K(t,s)\}_{t,s\geq 0}$, we have for all $x \in X$, $\mathcal{R}(\mu, A(t))K(t,s)x = K(t,s)\mathcal{R}(\mu, A(t))x$, such that the resolvent $\mathcal{R}(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ Hence, for all $x \in X$ we conclude

$$\begin{aligned}
\mathcal{R}(\mu, A(t))D_\lambda(t,s)x &= \mathcal{R}(\mu, A(t))\int_0^s e^{\lambda(s-h)}K(t-h,h)xdh \\
&= \int_0^s e^{\lambda(s-h)}\mathcal{R}(\mu, A(t))K(t-h,h)xdh \\
&= \int_0^s e^{\lambda(s-h)}K(t-h,h)\mathcal{R}(\mu, A(t))xdh \\
&= D_\lambda(t,s)\mathcal{R}(\mu, A(t))x.
\end{aligned}$$

Therefore, we obtain for all $x \in X$,

$$\begin{aligned}
D_\lambda(t,s)x &= \int_0^s e^{\lambda(s-h)}K(t-h,h)xdh \\
&= \int_0^s e^{\lambda(s-h)}K(t-h,h)(\mu - A(t))\mathcal{R}(\mu, A(t))xdh \\
&= \mu\int_0^s e^{\lambda(s-h)}K(t-h,h)\mathcal{R}(\mu, A(t))xdh - \int_0^s e^{\lambda(s-h)}K(t-h,h)A(t)\mathcal{R}(\mu, A(t))xdh \\
&= \mu\int_0^s e^{\lambda(s-h)}\mathcal{R}(\mu, A(t))K(t-h,h)xdh - \int_0^s e^{\lambda(s-h)}K(t-h,h)A(t)\mathcal{R}(\mu, A(t))xdh \\
&= \mu R(\mu, A(t))\int_0^s e^{\lambda(s-h)}K(t-h,h)xdh - \int_0^s e^{\lambda(s-h)}K(t-h,h)[A(t)\mathcal{R}(\mu, A(t))x]dh \\
&= \mu\mathcal{R}(\mu, A(t))D_\lambda(t,s)x - D_\lambda(t,s)[A(t)\mathcal{R}(\mu, A(t))x]
\end{aligned}$$

and according to (5) we obtained,

$$\begin{aligned}
D_\lambda(t,s)x &= \mu\mathcal{R}(\mu, A(t))D_\lambda(t,s)x - \Big[K(t-s,s)\mathcal{R}(\mu, A(t))x - e^{\lambda s}C\mathcal{R}(\mu, A(t))x \\
&+ \lambda D_\lambda(t,s)\mathcal{R}(\mu, A(t))x\Big] \\
&= \mu\mathcal{R}(\mu, A(t))D_\lambda(t,s)x - \mathcal{R}(\mu, A(t))K(t-s,s)x + e^{\lambda s}C\mathcal{R}(\mu, A(t))x \\
&- \lambda\mathcal{R}(\mu, A(t))D_\lambda(t,s)x \\
&= \mathcal{R}(\mu, A(t))\Big[\mu D_\lambda(t,s)x - K(t-s,s)x + e^{\lambda s}Cx - \lambda D_\lambda(t,s)x\Big].
\end{aligned}$$

Therefore, for all $x \in X$ we deduce $D_\lambda(t,s)x \in \mathcal{D}$ and we have

$$(\mu I - A(t))D_\lambda(t,s)x = \mu D_\lambda(t,s)x - K(t-s,s)x + e^{\lambda s}Cx - \lambda D_\lambda(t,s)x.$$

Finally, if $\mu \to \lambda$, we obtain for all $x \in X$,

$$(\lambda I - A(t))D_\lambda(t,s)x = [e^{\lambda s}C - K(t-s,s)]x.$$

□

For $t \geq 0$, we fix $\mathcal{D}^0 = \mathcal{D}(A(t)^0) = X$, $A(t)^0 = I$, and for $n \in \mathbb{N}$ we define by recurrence:

$$\mathcal{D}^n = \mathcal{D}(A(t)^n) := \{x \in \mathcal{D}(A(t)^{n-1}) : A(t)^{n-1}x \in \mathcal{D}(A(t))\},$$

$$A(t)^n x = A(t)A(t)^{n-1}x \quad \text{pour} \quad x \in \mathcal{D}(A(t)^n),$$

We introduce :

$$X = D(A(t)^0) \supseteq D(A(t)) \supseteq D(A(t)^2) \supseteq ... \supseteq D(A(t)^n).$$

Corollary 3.1. *Let $A(t)$ be the generator of the C-quasi-semigroup $\{K(t,s)\}_{t,s\geq 0}$ such that $A(t)$ is closed and densely defined, and let $C \in B(X)$ be injective. Then for all $t \geq s \geq 0$, $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}\setminus\{0\}$, we obtain*

(1) *For all $x \in X$,*
$$(\lambda I - A(t))^n [D_\lambda(t,s)]^n x = [e^{\lambda s} C - K(t-s,s)]^n x.$$
(2) *For all $x \in \mathcal{D}^n$,*
$$[D_\lambda(t,s)]^n (\lambda I - [A(t)]^n) x = [e^{\lambda s} C - K(t-s,s)]^n x.$$
(3) $N[\lambda I - A(t)] \subseteq N[e^{\lambda s} C - K(t-s,s)]$.
(4) $Rg[e^{\lambda s} C - K(t-s,s)] \subseteq Rg[\lambda I - A(t)]$.
(5) $N[\lambda I - A(t)]^n \subseteq N[e^{\lambda s} C - K(t-s,s)]^n$.
(6) $Rg[e^{\lambda s} C - K(t-s,s)]^n \subseteq Rg[\lambda I - A(t)]^n$.
(7) $Rg^\infty[e^{\lambda s} C - K(t-s,s)] \subseteq Rg^\infty[\lambda I - A(t)]$.

Proof. It's automatic by Theorem 3.1. □

The following theorem characterizes the ordinary, point, approximate point, essential and residual spectra of a C-quasi-semigroup.

Theorem 3.2. *Let $A(t)$ be the generator of the C-quasi-semigroup $\{K(t,s)\}_{t,s\geq 0}$ such that $A(t)$ is closed and densely defined, and let $C \in B(X)$ be injective. Then for all $t \geq s \geq 0$, we get*

(1) $e^{\sigma(A(t))s} \subset \sigma(C, K(t-s,s))\setminus\{0\}$
(2) $e^{\sigma_p(A(t))s} \subset \sigma_p(C, K(t-s,s))\setminus\{0\}$
(3) $e^{\sigma_{ap}(A(t))s} \subset \sigma_{ap}(C, K(t-s,s))\setminus\{0\}$
(4) $e^{\sigma_e(A(t))s} \subset \sigma_e(C, K(t-s,s))\setminus\{0\}$
(5) $e^{\sigma_r(A(t))s} \subset \sigma_r(C, K(t-s,s))\setminus\{0\}$.

Proof. it's immediately by the Theorem 3.1 and Corollary 3.1 □

Remark 3.1. *Note that the inclusion $\{e^{\lambda s}, \lambda \in \sigma_*(A(t))\} \subset \sigma_*(C, K(t-s,s))\setminus\{0\}$, where $\sigma_* = \sigma$, σ_{ap}, σ_e is strict as shown in the following example.*

Example 3.1. *Let $\{K(t,s)\}_{t,s\geq 0} = T(s)$ where $\{T(s)\}_{s\geq 0}$ is the translation group on the space $C_{2\pi}(\mathbb{R})$ of all 2π periodic continuous functions on $\mathbb{R}$ and denote its generator by A (see* [8, Paragraph I.4.15]). From [8, Examples 2.6.iv] we have, $\sigma(A(t)) = \sigma(A) = i\mathbb{Z}$, then $e^{\sigma(A(t))s}$ is at most countable, therefore $e^{\sigma_*(A(t))s}$ are also.
The spectra of the operators $T(s)$ are always contained in $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ and contain the eigenvalues e^{iks} for $k \in \mathbb{Z}$. Since $\sigma(T(s))$ is closed, it follows from [8, Theorem IV.3.16] that $\sigma(T(s)) = \Gamma$ whenever $s/2\pi \notin \mathbb{Q}$, then $\sigma(T(s))$ is not countable,
so $\sigma_*(I, K(t-s,s))\setminus\{0\}$ are also.

To obtain the results concerning the ascent and descent spectra we need the following theorem.

Theorem 3.3. *Let $A(t)$ be the generator of the C-quasi-semigroup $\{K(t,s)\}_{t,s\geq 0}$ such that $A(t)$ is closed and densely defined, and let $C \in B(X)$ be injective. Then for all $t \geq s > 0$ and all $\lambda \in \mathbb{C}$, we have*

(1) *$(\lambda I - A(t))L_\lambda(t,s) + \varphi_\lambda(s)D_\lambda(t,s) = C$, where $L_\lambda(t,s) = \dfrac{1}{s}\displaystyle\int_0^s e^{-\lambda h} D_\lambda(t,h)dh$ and $\varphi_\lambda(s) = \frac{1}{s}e^{-\lambda s}$.*

Moreover, the operators $L_\lambda(t,s)$, $D_\lambda(t,s)$ and $(\lambda I - A(t))$ are mutually commuting. Also, C is commute with each one $D_\lambda(t,s)$ and $L_\lambda(t,s)$

(2) *For all $n \in \mathbb{N}\setminus\{0\}$, there exists an operator $F_{\lambda,n}(t,s) \in \mathcal{B}(X)$, such that*

$$(\lambda I - A(t))^n [L_\lambda(t,s)]^n + F_{\lambda,n}(t,s)D_\lambda(t,s) = C^n.$$

Moreover, the operator $F_{\lambda,n}(t,s)$ is commute with each one of $D_\lambda(t,s)$ and $L_\lambda(t,s)$.

(3) *For all $n \in \mathbb{N}\setminus\{0\}$, there exists an operator $B_{\lambda,n}(t,s) \in \mathcal{B}(X)$, such that*

$$(\lambda I - A(t))^n B_{\lambda,n}(t,s) + [F_{\lambda,n}(t,s)]^n [D_\lambda(t,s)]^n = C^{n^2}.$$

Moreover, the operator $B_{\lambda,n}(t,s)$ is commute with each one of $D_\lambda(t,s)$ and $F_{\lambda,n}(t,s)$.

Proof. (1) Let $\mu \in \rho(A(t))$, by Theorem 3.1, for all $x \in X$ we have $D_\lambda(t,h)x \in \mathcal{D}$ and hence, for all $t, s > 0$,

$$\begin{aligned} sL_\lambda(t,s)x &= \int_0^s e^{-\lambda h} D_\lambda(t,h)xdh \\ &= \int_0^s e^{-\lambda h}\mathcal{R}(\mu, A(t))(\mu - A(t))D_\lambda(t,h)xdh, \\ &= \mathcal{R}(\mu, A(t))[\mu \int_0^s e^{-\lambda h} D_\lambda(t,h)xdh - \int_0^s e^{-\lambda h} A(t)D_\lambda(t,h)xdh] \\ &= \mathcal{R}(\mu, A(t))[\mu s L_\lambda(t,s)x - \int_0^s e^{-\lambda h} A(t)D_\lambda(t,h)xdh] \end{aligned}$$

Therefore for all $x \in X$, we have $L_\lambda(t,s)x \in \mathcal{D}$ and

$$s(\mu - A(t))L_\lambda(t,s)x = \mu s L_\lambda(t,s)x - \int_0^s e^{-\lambda h} A(t)D_\lambda(t,h)xdh.$$

Thus

$$A(t)(sL_\lambda(t,s)x) = \int_0^s e^{-\lambda h} A(t)D_\lambda(t,h)xdh.$$

Hence, from Theorem 3.1, we conclude that

$$\begin{aligned} (\lambda I - A(t))(sL_\lambda(t,s)x) &= \lambda s L_\lambda(t,s)x - \int_0^s e^{-\lambda h} A(t)D_\lambda(t,h)xdh \\ &= \lambda s L_\lambda(t,s)sx - \int_0^s e^{-\lambda h}\big[\lambda D_\lambda(t,h)x - e^{\lambda h}Cx + K(t-h,h)x\big]dh \\ &= \lambda s L_\lambda(t,s)x - \lambda \int_0^s e^{-\lambda h} D_\lambda(t,h)xdh + \int_0^s Cxdh - \int_0^s e^{-\lambda h} K(t-h,h)xdh \\ &= \lambda s L_\lambda(t,s)x - \lambda s L_\lambda(t,s)x + sCx - e^{-\lambda s}\int_0^s e^{\lambda(s-h)} K(t-h,h)xdh \\ &= sCx - e^{-\lambda s} D_\lambda(t,s)x \\ &= \big[sC - s\varphi_\lambda(s)D_\lambda(t,s)\big]x. \end{aligned}$$

Therefore, we obtain $(\lambda I - A(t))L_\lambda(t,s) + \varphi_\lambda(s)D_\lambda(t,s) = C$.
On the other hand and since the family $\{K(t,s)\}_{t,s\geq 0}$ is commutative, then for all $t > s > h > 0$, we have $D_\lambda(t,h)K(t-s,s) = K(t-s,s)D_\lambda(t,h)$.
Hence, then for all $s, r, t \geq h > 0$, we have $D_\lambda(t,s)D_\lambda(t,r) = D_\lambda(t,r)D_\lambda(t,s)$.

Thus, we deduce that

$$D_\lambda(t,s)L_\lambda(t,s)=L_\lambda(t,s)D_\lambda(t,s).$$

Since for all $x\in X$, $A(t)L_\lambda(t,s)x=\int_0^s e^{-\lambda h}A(t)D_\lambda(t,h)xdh$ and for all $x\in\mathcal{D}$, $A(t)D_\lambda(t,h)x=D_\lambda(t,h)A(t)x$, then we obtain for all $x\in\mathcal{D}$,

$$\begin{aligned}(\lambda I-A(t))L_\lambda(t,s)x &= \lambda L_\lambda(t,s)x-A(t)L_\lambda(t)x\\ &= \lambda L_\lambda(t,s)x-\int_0^s e^{-\lambda h}A(t)D_\lambda(t,h)xdh\\ &= \lambda L_\lambda(t,s)x-\int_0^s e^{-\lambda h}D_\lambda(t,h)A(t)xdh\\ &= \lambda L_\lambda(t,s)x-L_\lambda(t,s)A(t)x\\ &= L_\lambda(t,s)(\lambda I-A(t))x.\end{aligned}$$

From 2) of the definition of a C-quasi-semigroup $\{K(t,s)\}_{t,s\geq 0}$ with $r=0$, we have $CK(t-h,h)=K(t-h,h)C$ for all $t>h>0$.

Therefore, $CD_\lambda(t,s)=D_\lambda(t,s)C$ and $CL_\lambda(t,s)=L_\lambda(t,s)C$.

(2) For all $n\in\mathbb{N}\setminus\{0\}$, we obtain

$$\begin{aligned}[(\lambda I-A(t))L_\lambda(t,s)]^n &= [C-\varphi_\lambda(s)D_\lambda(t,s)]^n\\ &= \sum_{i=0}^{n} C_n^i C^{n-i}[-\varphi_\lambda(s)D_\lambda(t,s)]^i\\ &= C^n+\sum_{i=1}^{n} C_n^i C^{n-i}[-\varphi_\lambda(s)D_\lambda(t,s)]^i\\ &= C^n-D_\lambda(t,s)\sum_{i=1}^{n} C_n^i C^{n-i}[\varphi_\lambda(s)]^i[-D_\lambda(t,s)]^{i-1}\\ &= C^n-D_\lambda(t,s)F_{\lambda,n}(t,s),\end{aligned}$$

Where

$$F_{\lambda,n}(t,s)=\sum_{i=1}^{n} C_n^i C^{n-i}[\varphi_\lambda(s)]^i[-D_\lambda(t,s)]^{i-1}.$$

Therefore, we have

$$(\lambda I-A(t))^n[L_\lambda(t,s)]^n+D_\lambda(t,s)F_{\lambda,n}(t,s)=C^n.$$

Finally, for commutativity, it is clear that $F_{\lambda,n}(t,s)$ commute with each one of $D_\lambda(t,s)$ and $L_\lambda(t,s)$ since the operators $L_\lambda(t,s)$, $D_\lambda(t,s)$ and $(\lambda I-A(t))$ are mutually commuting and C is commute with each one $D_\lambda(t,s)$ and $L_\lambda(t,s)$ from (1).

(3) Since we have $D_\lambda(t,s)F_{\lambda,n}(t,s)=C^n-(\lambda I-A(t))^n[L_\lambda(t,s)]^n$, then for all $n\in\mathbb{N}^*$

$$\begin{aligned}[D_\lambda(t,s)F_{\lambda,n}(t,s)]^n &= \left[C^n-(\lambda I-A(t))^n[L_\lambda(t,s)]^n\right]^n\\ &= C^{n^2}-\sum_{i=1}^{n} C_n^i\left[C^n\right]^{n-i}\left[(\lambda I-A(t))^n[L_\lambda(t,s)]^n\right]^i\\ &= C^{n^2}-(\lambda I-A(t))^n\sum_{i=1}^{n} C_n^i\left[C^{n(n-i)}(\lambda I-A(t))^{n(i-1)}[L_\lambda(t,s)]^{ni}\right.\\ &= C^{n^2}-(\lambda I-A(t))^n B_{\lambda,n}(t,s),\end{aligned}$$

Where $B_{\lambda,n}(t,s) = \sum_{i=1}^{n} C_n^i C^{n(n-i)}(\lambda I - A(t))^{n(i-1)}[L_\lambda(t,s)]^{ni}$. Hence, we obtain

$$[D_\lambda(t,s)]^n[F_{\lambda,n}(t,s)]^n + (\lambda I - A(t))^n B_{\lambda,n}(t,s) = C^{n^2}.$$

Finally, the commutativity is clear.

□

Proposition 3.1. *Let $A(t)$ be a closed and densely defined generator of a $C-$quasi-semigroup $\{K(t,s)\}_{t,s\geq 0}$ on a Banach space X. For all $\lambda \in \mathbb{C}$, $n \in \mathbb{N}\setminus\{0\}$ and $t \geq s > 0$, we have*

(1) *If $d(e^{\lambda s}C - K(t-s,s)) = n$, then $d(\lambda I - A(t)) \leq n$.*
(2) *If $a(e^{\lambda s}C - K(t-s,s) = n$, then $a(\lambda I - A(t)) \leq n$.*

Proof.

(1) Let $y \in Rg[\lambda I - A(t)]^n$, then there exists $x \in \mathcal{D}^n$ (domain of $A(t)^n$) satisfying,

$$(\lambda I - A(t))^n x = y.$$

Since $d[e^{\lambda s}C - K(t-s,s)] = n$, then

$$Rg[e^{\lambda s}C - K(t-s,s)]^n = Rg[e^{\lambda s}C - K(t-s,s)]^{n+1}.$$

Hence, there exists $z \in X$ such that

$$[e^{\lambda s}C - K(t-s,s)]^n x = [e^{\lambda s}C - K(t-s,s)]^{n+1} z, \tag{6}$$

Let $u \in X$ such as, $C^{n^2}u = x$, thus $y = (\lambda I - A(t))^n C^{n^2} u$.

On the other hand, by (3) in Theorem 3.3, we have,

$$(\lambda I - A(t))^n B_{\lambda,n}(t,s)u + [F_{\lambda,n}(t,s)]^n[D_\lambda(t,s)]^n u = C^{n^2}u, \tag{7}$$

Thus we have,

$$\begin{aligned}
y &= (\lambda I - A(t))^n[(\lambda I - A(t))^n B_{\lambda,n}(t,s) + [F_{\lambda,n}(t,s)]^n[D_\lambda(t,s)]^n]u \\
&= (\lambda I - A(t))^n(\lambda - A(t))^n B_{\lambda,n}(t,s)u + [F_{\lambda,n}(t,s)]^n(\lambda I - A(t))^n[D_\lambda(t,s)^n]u \\
&= (\lambda I - A(t))^{2n} B_{\lambda,n}(t,s)u + [F_{\lambda,n}(t,s)]^n[e^{\lambda s}C - K(t-s,s)]^n u, \text{ (by Theorem 3.1)} \\
&= (\lambda I - A(t))^{2n} B_{\lambda,n}(t,s)u + [F_{\lambda,n}(t,s)]^n[e^{\lambda s}C - K(t-s,s)]^n C^{-n^2}x \\
&= (\lambda I - A(t))^{2n} B_{\lambda,n}(t,s)u + [F_{\lambda,n}(t,s)]^n C^{-n^2}[e^{\lambda s}C - K(t-s,s)]^n x \\
&= (\lambda I - A(t))^{2n} K_{\lambda,n}(t,s)u + [D_{\lambda,n}(t,s)]^n C^{-n^2}[[e^{\lambda s}C - K(t-s,s)]^{n+1}z], \text{ by (6)} \\
&= (\lambda I - A(t))^{2n} K_{\lambda,n}(t,s)u + [D_{\lambda,n}(t,s)]^n[[e^{\lambda s}C - K(t-s,s)]^{n+1}C^{-n^2}z] \\
&= (\lambda I - A(t))^{2n} K_{\lambda,n}(t,s)u + [D_{\lambda,n}(t,s)]^n[(\lambda I - A(t))^{n+1}[D_\lambda(t,s)]^{n+1}C^{-n^2}z] \\
&= (\lambda I - A(t))^{n+1}[(\lambda I - A(t))^{n-1}K_{\lambda,n}(t,s)u + [D_{\lambda,n}(t,s)]^n[D_\lambda(t,s)]^{n+1}C^{-n^2}z].
\end{aligned}$$

Therefore, we conclude that $y \in Rg[\lambda I - A(t)]^{n+1}$ and hence,

$$Rg[\lambda I - A(t)]^n = Rg[\lambda I - A(t)]^{n+1}.$$

Finally, we conclude that

$$d(\lambda I - A(t)) \leq n.$$

(2) Let $x \in N(\lambda I - A(t))^{n+1}$ and we suppose that $a[e^{\lambda s}C - K(t-s,s)] = n$, then we obtain

$$N[e^{\lambda s}C - K(t-s,s)]^n = N[e^{\lambda s}C - K(t-s,s)]^{n+1}.$$

From corollary 3.1, we have

$$N(\lambda I - A(t))^{n+1} \subseteq N[e^{\lambda s}C - K(t-s,s)]^{n+1},$$

hence

$$x \in N[e^{\lambda s}C - K(t-s,s)]^{n}.$$

Moreover, by Theorem 3.1 and (7) we have

$C^{n^2}(\lambda I - A(t))^n x = [(\lambda I - A(t))^n B_{\lambda,n}(t,s) + [F_{\lambda,n}(t,s)]^n [D_\lambda(t,s)]^n](\lambda I - A(t))^n x$
$= (\lambda I - A(t))^{n-1} B_{\lambda,n}(t,s)(\lambda I - A(t))^{n+1} x + [F_{\lambda,n}(t,s)]^n [e^{\lambda s}C - K(t-s,s)]^n x$
$= 0.$

Therefore, we obtain $x \in N(\lambda I - A(t))^n$ and hence, $a(\lambda I - A(t)) \leq n$.

□

Corollary 3.2. *Let $A(t)$ be a closed and densely defined generator of a $C-$quasi-semigroup $\{K(t,s)\}_{t,s\geq 0}$ on a Banach space X. For all $\lambda \in \mathbb{C}$ and all $t, s > 0$, we have*

(1) $e^{\sigma_a(A(t))s} \subseteq \sigma_a(C, K(t-s,s))\backslash\{0\}$.
(2) $e^{\sigma_d(A(t))s} \subseteq \sigma_d(C, K(t-s,s))\backslash\{0\}$.

Proof. Immediately comes from Proposition 3.1. □

4. Conclusion

In this paper, we have proved some results concerning the C-quasi-semigroups and we have showed a spectral inclusion of a different spectra of a regularized quasi-semigroups of a bounded linear operators on a Banach space and their infinitesimal generators, and we will end this article with the open question concerning the equality of a different spectra of this family of operators and its infinitesimal generator.

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