TWMS J. App. and Eng. Math. V.13, N.1, 2023, pp. 276-284

A NEW CLASS OF NON NORMAL OPERATORS ON HILBERT SPACES

AISSA NASLI BAKIR[1],§

Abstract. We introduce the class of (M, k)-quasi-parahyponormal operators on a separable Hilbert space as an extension of the classes of parahyponormal and k-quasi-parahyponormal operators given in [13]. The matrix representation of an (M, k)-quasi-parahyponormal operator, the finite ascent, the SVEP and other several properties of such class of operators are also presented.

Keywords: M-parahyponormal operator, k-quasi-parahyponormal operator, SVEP, finite ascent.

AMS Subject Classification(2010): 47A30, 47B47, 47B20.

1. Introduction

Let $B(H)$ denote the Banach algebra of bounded linear operators on an infinite dimensional separable complex Hilbert space H. For an operator $T \in B(H)$, $ker(T)$ and $ran(T)$ denote respectively, the null space and the range of T. Recall that $T \in B(H)$ is said to be paranormal if $T^{\star 2}T^2 - 2\lambda T^\star T + \lambda^2 \geq 0$ for all $\lambda > 0$, and k-paranormal for some integer $k \geq 2$, if $\|Tx\|^k \leq \|T^k x\|$ for all unit vector x in H. An operator T is said to be quasi-class A, if $T^\star|T|^2T \leq T^\star|T^2|T$, where $|T| = (T^\star T)^{\frac{1}{2}}$ is the module of T. Also, T is called k-quasi-paranormal if $T^{*k}(T^{\star 2}T^2 - 2\lambda T^\star T + \lambda^2)T^k \geq 0$. This class is introduced by S. Mecheri in [10], and it is a generalization of the class of quasi-paranormal operators, and it contains the class of quasi-class A operators, see [9]. It is also shown in [4] that quasi-class A operators satisfy Weyl's Theorem. Authors in [3, 13] and [16] introduced the class of parahyponormal and M-parahyponormal operators, and gave several properties for such operators. In the present paper, we introduce a new class of (M, k)-quasi-parahyponormal operators as a generalization of the class of k-quasi-parahyponormal operators. We give the matrix representation, the ascent and the SVEP. Different related properties are also proved.

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[1] Department of Mathematics, Laboratory of Mathematics and Application LMA, Hassiba Benbouali University of Chlef, B.P. 78C, Ouled Fares, 02180, Chlef, Algeria.
e-mail: a.nasli@univ-chlef.dz; ORCID: https://orcid.org/0000-0001-6906-3307.

§ Manuscript received: November 19, 2020; accepted: January 23, 2021.
TWMS Journal of Applied and Engineering Mathematics, Vol.13, No.1

2. Class of k-quasi-parahyponormal operators

Definition 2.1. [13][16] *An operator $T \in B(H)$ is said to be parahyponormal if*

$$(TT^\star)^2 - 2\lambda T^\star T + \lambda^2 \geq 0$$

for all $\lambda > 0$.

Definition 2.2. [13] *An operator $T \in B(H)$ is said to be k-quasi-parahyponormal for some integer k if*

$$T^{\star k}((TT^\star)^2 - 2\lambda T^\star T + \lambda^2)T^k \geq 0$$

for all $\lambda > 0$.

This definition is equivalent to

$$\|T^{k+1}x\|^2 \leq \|T^k x\| \|TT^\star T^k x\|$$

for all $x \in H$. In this section, we give some complement properties for this class of operators.

Theorem 2.1. *Let S be the bilateral weighted shift defined on the usual Hilbert space ℓ_2 by $Se_n = \alpha_n e_{n+1}$, where $(e_n)_n$ is the standard basis, and $(\alpha_n)_n$ is a decreasing complex sequence. Then, S is k-quasi-parahyponormal if and only if*

$$|\alpha_{n+k}| \leq |\alpha_{n+k-1}|^2$$

for all n.

Proof. We have

$$\|S^{k+1}e_n\|^2 \leq \|S^k e_n\| \|SS^\star S^k e_n\|$$

Hence, for all n,

$$\prod_{i=0}^{k} |\alpha_{n+i}|^2 \leq |\alpha_{n+k-1}|^2 \prod_{i=0}^{k-1} |\alpha_{n+i}|^2$$

Thus,

$$|\alpha_{n+k}| \leq |\alpha_{n+k-1}|^2$$

for all n. □

Theorem 2.2. *A unitarily equivalent operator to a k-quasi-parahyponormal operator is also k-quasi-parahyponormal.*

Proof. Let T be a k-quasi-parahyponormal operator, and let $A \in B(H)$ be unitarily equivalent to T. Then, there exists a unitary operator U on H satisfying $A = U^\star TU$. Hence,

$$\begin{aligned}
&A^{\star k}((AA^\star)^2 - 2\lambda A^\star A + \lambda^2)A^k = \\
&= U^\star T^{\star k}U(U^\star(TT^\star)^2U - 2\lambda U^\star T^\star TU + \lambda^2)U^\star T^k U \\
&= U^\star T^{\star k}(TT^\star)^2 T^k U - 2\lambda U^\star T^{\star k} T^\star T T^k U + \lambda^2 U^\star T^{\star k} T^k U \\
&= U^\star T^{\star k}((TT^\star)^2 - 2\lambda T^\star T + \lambda^2)T^k U \geq 0
\end{aligned}$$

Thus, A is k-quasi-parahyponormal. □

Theorem 2.3. *Let $T \in B(H)$ be a k-quasi-parahyponormal operator, and let $S \in B(H)$ be an isometry. If T commutes with S, then TS is k-quasi-parahyponormal.*

Proof. Since T is k-quasi-parahyponormal,

$$\begin{aligned}
&(TS)^{*k}((TS)(TS)^{\star})2 - 2\lambda(TS)^{\star}TS + \lambda^2)(TS)^k = \\
&= T^{\star k}S^{\star k}\left[STT^{\star}TT^{\star}S^{\star} - 2\lambda T^{\star}T + \lambda^2\right]S^kT^k \\
&= T^{\star k}S^{\star k-1}\left[S^{\star}S(TT^{\star})^2S^{\star}S - 2\lambda S^{\star}T^{\star}TS + \lambda^2 S^{\star}S\right]S^{k-1}T^k \\
&= T^{\star k}S^{\star k-1}\left[(TT^{\star})^2 - 2\lambda T^{\star}T + \lambda^2\right]S^{k-1}T^k \\
&= S^{\star k-1}T^{\star k}\left[(TT^{\star})^2 - 2\lambda T^{\star}T + \lambda^2\right]T^kS^{k-1} \geq 0
\end{aligned}$$

□

Let $T \in B(H)$. Denote by $\mathcal{R}(\sigma(T))$ for the set of all rational analytic functions on the spectrum $\sigma(T)$ of T.

Definition 2.3. [8] *The operator T is said to be n-multicyclic, if there exist n (generating) vectors $x_1, x_2, .., x_n$ in H such that*

$$\bigvee \{g(T)x_i \ , \ 1 \leq i \leq n \ , \ g \in \mathcal{R}(\sigma(T))\} = H$$

We have then

Theorem 2.4. *If T is an n-multicyclic k-quasi-parahyponormal operator, then its restriction on $\overline{ran(T^k)}$ is also n-multicyclic.*

Proof. Put

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on the decomposition $H = \overline{ran(T^k)} \oplus ker(T^{\star k})$. Since $\sigma(T_1) \subset \sigma(T)$ by [13, Theorem 2.3], $\mathcal{R}(\sigma(T_1)) \subset \mathcal{R}(\sigma(T))$. The operator T is n-multicyclic. Then, there exist n generating vectors $x_1, x_2, .., x_n \in H$ for which

$$\bigvee \{g(T)x_i, \ 1 \leq i \leq n, \ g \in \mathcal{R}(\sigma(T))\} = H$$

Put $y_i = T^k x_i$, $1 \leq i \leq n$. Hence,

$$\begin{aligned}
\bigvee \{g(T_1)y_i, \ 1 \leq i \leq n, \ g \in \mathcal{R}(\sigma(T))\} &= \bigvee \left\{g(T_1)T^k x_i, \ 1 \leq i \leq n, \ g \in \mathcal{R}(\sigma(T))\right\} \\
&= \bigvee \left\{g(T)T^k x_i, \ 1 \leq i \leq n, \ g \in \mathcal{R}(\sigma(T))\right\} \\
&= \bigvee \left\{T^k g(T)x_i, \ 1 \leq i \leq n, \ g \in \mathcal{R}(\sigma(T))\right\} \\
&= \overline{ran(T^k)}
\end{aligned}$$

But

$$\bigvee \{g(T_1)y_i, \ 1 \leq i \leq n, \ g \in \mathcal{R}(\sigma(T))\} \subset \bigvee \{g(T_1)y_i, \ 1 \leq i \leq n, \ g \in \mathcal{R}(\sigma(T_1))\}$$

Thus,

$$\overline{ran(T^k)} \subset \bigvee \{g(T_1)y_i, \ 1 \leq i \leq n, \ \ g \in \mathcal{R}(\sigma(T_1))\}$$

Therefore, $\{y_i\}_{i=1}^n$ are n-generating vectors of T_1, and T_1 is n-muticyclic. □

3. Class of (M,k)-quasi-parahyponormal operators

Important properties of (M,k)-quasi-parahyponormal operators are shown in this section. In particular, the matrix representation, the finite ascent and the SVEP of (M,k)-quasi-parahyponormal operators are presented.

Definition 3.1. [12][13] *An operator $T \in B(H)$ is said to be M-parahyponormal if there exists $M > 0$, satisfying*

$$M(TT^\star)^2 - 2\lambda T^\star T + \lambda^2 \geq 0$$

Definition 3.2. *An operator $T \in B(H)$ is said to be (M,k)-quasi-parahyponormal for some integer k, if there exists $M > 0$, satisfying*

$$T^{\star k}(M(TT^\star)^2 - 2\lambda T^\star T + \lambda^2)T^k \geq 0$$

This definition is clearly equivalent to

$$\|T^{k+1}x\|^2 \leq \sqrt{M}\|T^k x\|\|TT^\star T^k x\| \tag{1}$$

for all $x \in H$.

Inequality (1) shows that this class of operators is nested with respect to M, i.e., an (M_1,k)-quasi-parahyponormal operator is (M_2,k)-quasi-parahyponormal for $0 < M_1 < M_2$.

Example 3.1. *Let's consider the unilateral weighted right shift on $\ell_2(\mathbb{N})$ defined by $Te_n = \alpha_n e_{n+1}$, where $\alpha_1 = \alpha_3 = \frac{1}{2}, \alpha_2 = \frac{1}{4}$, $\alpha_n = 1$, $n \geq 4$, and (e_n) is the satandard basis of $\ell_2(\mathbb{N})$. By a direct computation, we can show that T is M-quasi-parahyponormal. However,*

$$\|Te_1\|^2 = \frac{1}{4} > \sqrt{M}\|e_1\|\|TT^\star e_1\| = 0$$

which contradicts the inequality (1). Hence, T is not M-parahyponormal.

Example (3.1) shows that the classes of (M,k)-quasi-parahyponormal operators do not coincide.

Proposition 3.1. *The class of (M,k)-quasi-parahyponormal operators is*

1. *Closed for the multiplication by scalars.*
2. *Not convex.*
3. *Not translation invariant.*

Proof. 1. Let T be an (M,k)-quasi-parahyponormal operator, and let α be any complex scalar. For all $x \in H$ we have

$$\begin{aligned}\|(\alpha T)^{k+1}x\|^2 &= |\alpha|^{2k+2}\|T^{k+1}x\|^2 \leq |\alpha|^{2k+2}\sqrt{M}\|T^k x\|\|TT^\star T^k x\| \\ &= \sqrt{M}\|(\alpha T)^k x\|\|(\alpha T)(\alpha T)^\star(\alpha T)^k x\|\end{aligned}$$

Then, αT is also (M,k)-quasi-parahyponormal.

2. On the other hand, operators $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ are M-parahyponormal for $M \geq \frac{1}{4}$, in particular $M = 2$. However, the operator $A = \frac{1}{2}(T+S)$ is not 2-parahyponormal. Indeed, for $x = (0,1) \in \mathbb{C}^2$, we get

$$\|A(0,1)\|^2 = \frac{1}{4} > \sqrt{2}\|(0,1)\|\|AA^\star(0,1)\| = 0$$

This contradicts the inequality (1). Hence, the above class is not convex.

3. For $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\alpha = 1,\ \beta = -1$, and $x = (0,1)$, the operator

$$R = \alpha T + \beta = T - I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

verifies

$$\|R(0,1)\|^2 = 1 > \sqrt{2}\|(0,1)\|\|RR^\star(0,1)\| = 0$$

Hence, R is not 2-parahyponormal. Thus, the considered class is not translation invariant.

□

With the same arguments of Theorems 2.2 and 2.3, we can state the following result

Theorem 3.1. *1. Unitarily equivalent operators to an (M,k)-quasi-parahyponormal operator are also (M,k)-quasi-parahyponormal.*
2. The product TS is (M,k)-quasi-parahyponormal whenever T is (M,k)-quasi-parahyponormal and S is an isometry that commutes with T.

Remark 3.1. *Assertion (1) of Theorem 3.1 is in general false if the operators are similar to the (M,k)-quasi-parahyponormal operator and not unitarily equivalent.*
Indeed, the bilateral weighted shift T defined on the Hilbert space $\ell_2(\mathbb{Z})$ by

$$Te_n = \begin{cases} e_{n+1}, & n \leq 1 \text{ or } n \geq 3 \\ \sqrt{2}e_3, & n = 2 \end{cases}$$

is M-parahyponormal for $M \geq 8$, by inequality (1). In particuliar it is 8-parahyponormal, and the operator

$$Ue_n = \begin{cases} e_{n+1}, & n \leq 1 \text{ or } n \geq 3 \\ \frac{1}{3}e_3, & n = 2 \end{cases}$$

is invertible and not unitary. Nonetheless, the operator $B = U^{-1}TU$ is not 8-parahyponormal since

$$Be_n = \begin{cases} e_{n+1}, & n \leq 0 \text{ or } n \geq 3 \\ 3\sqrt{2}e_2, & n = 1 \\ \dfrac{1}{3}e_3, & n = 2 \end{cases}$$

and

$$\|Be_1\|^2 = 18 > \sqrt{8}\|e_1\|\|BB^\star e_1\| = \sqrt{8}$$

Theorem 3.2. *Let $T \in B(H)$ be an (M,k)-quasi-parahyponormal operator. If $ran(T^k)$ is dense in H, then T is M-parahyponormal.*

Proof. Let $x \in H$. By the hypothesis, there exists a sequence $(x_n)_n$ in H such that $x = \lim\limits_{n\to\infty} T^k x_n$. Since T is (M,k)-quasi-parahyponormal,

$$\begin{aligned} \sqrt{M}\|x\|\|TT^\star x\| &= \sqrt{M}\|\lim_{n\to\infty}(\|T^k x_n\|\|TT^\star T^k x_n\|) \\ &= \lim_{n\to\infty}\sqrt{M}\|T^k x_n\|\|TT^\star T^k x_n\| \\ &\geq \lim_{n\to\infty}\|T^{k+1}x_n\|^2 \\ &\geq \|\lim_{n\to\infty}T^{k+1}x_n\|^2 \\ &= \|Tx\|^2 \end{aligned}$$

by the continuity of the inner product. Hence, T is M-parahyponormal. □

Corollary 3.1. *Let T be a nonzero (M,k)-quasi-parahyponormal operator but not M-parahyponormal. Then, T admits at least a non trivial closed invariant subspace.*

Proof. Suppose that T has no non trivial closed invariant subspace. Since $T \neq 0$, $ker(T) \neq H$ and $\overline{ran(T)} \neq \{0\}$ are non trivial closed invariant subspaces for T. Thus, we must have $ker(T) = \{0\}$ and $\overline{ran(T)} = H$. By Theorem 3.2, T is M-parahyponormal, which contradicts the hypothesis. □

Note that the existence of nontrivial closed invariant subspaces for operators on Hilbert spaces remains until now, one of the hot open problems in operator theory, see [6, 11] for further details.

Theorem 3.3. *Let $T \in B(H)$ be an (M,k)-quasi-parahyponormal operator. If $\overline{ran(T^k)} \neq H$, then T admits the matrix representation*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $H = \overline{ran(T^k)} \oplus ker(T^{\star k})$. Furthermore, T_1 is M-parahyponormal, T_3 is nilpotent of order k, and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. Since T is (M,k)-quasi-parahyponormal,

$$\langle T^{\star k}(M(TT^\star)^2 - 2\lambda T^\star T + \lambda^2)T^k y, y\rangle \geq 0$$

for all $y \in H$. Hence,

$$\langle (M(TT^\star)^2 - 2\lambda T^\star T + \lambda^2)T^k y, T^k y\rangle \geq 0$$

Thus, for all $x \in \overline{ran(T^k)}$,

$$\big\langle (M(TT^\star)^2 - 2\lambda T^\star T + \lambda^2)x, x\big\rangle = \big\langle (M(T_1T_1^\star)^2 - 2\lambda T_1^\star T_1 + \lambda^2)x, x\big\rangle \geq 0$$

Consequently, T_1 is M-parahyponormal. Let now P be the orthogonal projection on $\overline{ran(T^k)}$. For all $x = x_1 + x_2, y = y_1 + y_2 \in H$, we have

$$\Big\langle T_3^k x_2, y_2 \Big\rangle = \Big\langle T^k(I-P)x, (I-P)y \Big\rangle = \Big\langle (I-P)x, T^{*k}(I-P)y \Big\rangle = 0$$

Thus, $T_3^k = 0$. Furthermore, $\sigma(T_1) \cup \sigma(T_3) = \sigma(T) \cup \Omega$, where Ω is the union of holes in $\sigma(T)$ which happen to be a subset of $\sigma(T_1) \cap \sigma(T_3)$ by [7, Corollary 7], with $\sigma(T_1) \cap \sigma(T_3)$ has no interior point, and T_3 is nilpotent. Thus, $\sigma(T) = \sigma(T_1) \cup \{0\}$. □

Corollary 3.2. *Let $T \in B(H)$ be (M,k)-quasi-parahyponormal. If the restriction $T_1 = T\big|_{\overline{ran(T^k)}}$ is invertible, then T is similar to the sum of an M-parahyponormal operator and a nilpotent operator.*

Proof. Let

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on } H = \overline{ran(T^k)} \oplus ker(T^{*k})$$

Then, T_1 is M-parahyponormal by Theorem 3.3. Since T_1 is invertible, $0 \notin \sigma(T)$. Hence, $\sigma(T_1) \cap \sigma(T_3) = \emptyset$. By Rosenblum's Corollary [14], [15], there exists $S \in B(H)$ for which $T_1S - ST_3 = T_2$. Thus,

$$\begin{aligned} T &= \begin{pmatrix} I & -S \\ 0 & I \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-1} (T_1 \oplus T_3) \begin{pmatrix} I & S \\ 0 & I \end{pmatrix} \end{aligned}$$

□

Definition 3.3. [1] *An operator T in $B(H)$ is said to have the Single Valued Extension Property, briefly SVEP, at a complex number α, if for each open neighborhood V of α, the unique analytic function $f\colon V \to H$ satisfying*

$$(T-\lambda)f(\lambda)=0$$

for all $\lambda \in V$ is $f \equiv 0$.
Furthermore, T is said to have SVEP if T has SVEP at every complex number.

Definition 3.4. [1] *For $T \in B(H)$, the smallest integer m such that $ker(T^m) = ker(T^{m+1})$ is said to be the ascent of T, and is denoted by $\alpha(T)$. If no such integer exists, we shall write $\alpha(T) = \infty$.*

Definition 3.5. [1] *The smallest integer m such that $ran(T^m) = ran(T^{m+1})$ is said to be the descent of T, and is denoted by $\delta(T)$. If no such integer exists, we set $\delta(T) = \infty$.*

According to [1], if $\alpha(T)$ and $\delta(T)$ are both finite, then $\alpha(T) = \delta(T)$. For more details on these notions, reader can see [1, 2] and [5]. Now, we'll show that the considered operators have finite ascent and SVEP.

Theorem 3.4. *An (M,k)-quasi-parahyponormal operator $T \in B(H)$ has finite ascent.*

Proof. Let $x \in ker(T^{k+1})$. Since T is (M,k)-quasi-parahyponormal operator, there exists $M>0$ such that

$$T^{\star k}(M(TT^\star)^2 - 2\lambda T^\star T + \lambda^2)T^k \geq 0$$

Hence,

$$\langle T^{\star k}(M(TT^\star)^2 T^k x, x\rangle - 2\lambda\langle T^{\star k}T^\star T T^k x, x\rangle + \lambda^2\langle T^{\star k}T^k x, x\rangle \geq 0$$

for all $\lambda > 0$. Thus,

$$M\|TT^\star T^k x\|^2 + \lambda^2\|T^k x\|^2 \geq 0$$

for all $\lambda > 0$. Therefore, $T^k x = 0$, which implies that $x \in ker(T^k)$. This finishes the proof since clearly $ker(T^k) \subset ker(T^{k+1})$. □

Corollary 3.3. *(M,k)-quasi-parahyponormal operators have SVEP at 0.*

Proof. Immediate consequence of Theorem 3.4 and [1, Theorem 3.8]. □

Definition 3.6. [1] *For an operator $T \in B(H)$, the local resolvent set of T at a vector $x \in H$, denoted by $\rho_T(x)$, is defined to consist of complex elements z_0 such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in H, for which $(T-z)f(z) = x$.*

Definition 3.7. [1] *The set $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$ is called the local spectrum of T at x.*

We've then the following important result

Theorem 3.5. *Let*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

be an (M,k)-quasi-parahyponormal operator with respect to the decomposition $H = \overline{ran(T^k)} \oplus ker(T^{\star k})$. Then, for all $x = x_1 + x_2 \in H$:

(a) $\sigma_{T_3}(x_2) \subset \sigma_T(x_1 + x_2)$.
(b) $\sigma_{T_1}(x_1) = \sigma_T(x_1 + 0)$.

Proof. a. Let $z_0 \in \rho_T(x_1+x_2)$. Then, there exists a neighborhood U of z_0 and an analytic function $f(z)$ defined on U, with values in H, for which

$$(T-z)f(z) = x,\ z \in U \tag{2}$$

Let $f = f_1 + f_2$ where f_1, f_2 are in the spaces $O(U, \overline{ran(T^k)}), O(U, ker(T^{\star k}))$ respectively, consisting of analytic functions on U with values in H, with respect to the uniform topology [1]. Equality (2) can then be written

$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & T_3 - z \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Then $(T_3 - z)f_2(z) = x_2$, $z \in U$. Hence, $z_0 \in \rho_{T_3}(x_2)$. Thus, (a) holds by passing to the complement.

b. If $z_1 \in \rho_T(x_1+0)$, then there exists a neighborhood V_1 of z_1 and an analytic function g defined on V_1 with values in H verifying

$$(T-z)f(z) = x_1 + 0,\ z \in V_1 \tag{3}$$

Let $g = g_1 + g_2$, where $g_1 \in O(V_1, \overline{ran(T^k)})$, $g_2 \in O(V_1, ker(T^{\star k}))$ are as in (a). From equation (3) we obtain

$$(T_1 - z)g_1(z) + T_2 g_2(z) = x_1 \text{ and } (T_3 - z)g_2(z) = 0,\ z \in V_1$$

Since T_3 is nilpotent by Theorem 3.3, T_3 has SVEP by [1]. Thus, $g_2(z) = 0$. Consequently, $(T_1 - z)g_1(z) = x_1$. Therefore, $z_1 \in \rho_{T_1}(x_1)$, and then $\rho_T(x_1+0) \subset \rho_{T_1}(x_1)$. Thus, $\sigma_{T_1}(x_1) \subset \sigma_T(x_1+0)$.

Now, if $z_2 \in \rho_{T_1}(x_1)$, then, there exists a neighborhood V_2 of z_2 and an analytic function h from V_2 onto H, such that $(T_1 - z)h(z) = x_1$ for all $z \in V_2$. Hence,

$$(T-z)(h(z)+0) = (T_1 - z)h(z) = x_1 = x_1 + 0$$

Thus, $z_2 \in \rho_T(x_1+0)$. □

4. Conclusion

We've shown certain fundamental properties of the given class of operators in the present article. The ascent, the matrix representation, the restriction on invariant subspaces and the n-multicyclicity as well as other considerable properties are established.

Acknowledgement. I would like to present my warm thanks to the referees for their careful and valuable reading of the manuscript. My gratitude is also expressed to the DGRSDT in Algeria for financing the Laboratory of Mathematics and Application LMA in my University.

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Dr. Aissa Nasli Bakir is currently an associate professor in the Department of Mathematics, Hassiba Benbouali University of Chlef, Algeria. He obtained his PhD from Abdelhamid Ibn Badis University of Mostaganem in Algeria in 2010. His area of interest includes functional analysis and operator theory.