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SD-PRIME CORDIAL LABELING OF SUBDIVISION K_4-SNAKE AND RELATED GRAPHS

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ABSTRACT. Let $f : V(G) \to \{1, 2, \dots, |V(G)|\}$ be a bijection, and let us denote $S = f(u) + f(v)$ and $D = |f(u) - f(v)|$ for every edge uv in $E(G)$. Let f' be the induced edge labeling, induced by the vertex labeling f, defined as $f' : E(G) \to \{0, 1\}$ such that for any edge uv in $E(G)$, $f'(uv) = 1$ if $\gcd(S, D) = 1$, and $f'(uv) = 0$ otherwise. Let $e_{f'}(0)$ and $e_{f'}(1)$ be the number of edges labeled with 0 and 1 respectively. f is SD-prime cordial labeling if $|e_{f'}(0) - e_{f'}(1)| \leq 1$ and G is SD-prime cordial graph if it admits SD-prime cordial labeling. In this paper, we have discussed the SD-prime cordial labeling of subdivision of K_4-snake $S(K_4S_n)$, subdivision of double K_4-snake $S(D(K_4S_n))$, subdivision of alternate K_4-snake $S(A(K_4S_n))$ of type 1, 2 and 3, and subdivision of double alternate K_4- snake $S(DA(K_4S_n))$ of type 1, 2 and 3.

Keywords: SD-prime cordial graph, Subdivision of K_4-Snake, Subdivision of Alternate K_4-Snake, Subdivision of Double K_4-Snake, Subdivision of Double Alternate K_4-Snake, $m-$Complete Snake.

AMS Subject Classification: 05C78.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a simple, finite and undirected graph of order $|V(G)|$ and size $|E(G)|$. For standard terminology of Graph Theory, we used [1]. For all detailed survey of graph labeling, we refer [2]. Lau, Chu, Suhadak, Foo and Ng [3] have introduced SD-prime cordial labeling and they proved behaviour of several graphs like path, complete bipartite graph, star, double star, wheel, fan, double fan, ladder and grid. They conjecture that $P_m \times P_n$ is SD-prime cordial, for all $m \geq 2$ and $n \geq 2$. Lau, Shiu, Ng and Jeyanthi [4] give sufficient conditions for a theta graph to have an SD-prime cordial labeling, provide a way to construct new SD-prime cordial graphs from existing ones, and investigate SD-prime cordialness of some general graphs. Lourdusamy and Patrick [5] proved that $S'(K_{1,n}), D_2(K_{1,n}), S(K_{1,n}), DS(K_{1,n}), S'(B_{n,n}), D_2(B_{n,n}), TL_n, DS(B_{n,n}), S(B_{n,n}), CH_n,$ $K_{1,3} \star K_{1,n}, Fl_n, P_n^2, T(P_n), T(C_n), Q_n, A(T_n), P_n \odot K_1, C_n \odot K_1$, J_n and the graph obtained

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by duplication of each vertex and cycle by an edge are SD-prime cordial. Lourdusamy, Wency and Patrick [6] proved that the union of star and path graphs, subdivision of comb graph, subdivision of ladder graph and the graph obtained by attaching star graph at one end of the path are SD-prime cordial graphs. They proved that the union of two SD-prime cordial graphs need not be SD-prime cordial graph. Also, they proved that given a positive integer n, there is SD-prime cordial graph G with n vertices. Thulukkanam, Vijaya Kumar and Thirusangu [7] proved that the extended duplicate graphs of path graph, comb graph, twig graph, star graph, bistar graph and double star graph are SD-prime cordial. Delman, Koilraj and Lawrence Rozario Raj [8] proved that Pl_n graph is SD-prime cordial. Delman, Koilraj and Lawrence Rozario Raj [9] proved that disconnected graphs $G\cup(P_n \odot K_1), G\cup K_{1,n,n}, G\cup PS_n$ and $G\cup P_n$ are SD-prime cordial. Prajapati and Vantiya [10] proved that $T_n (n \neq 3), A(T_n), Q_n, A(Q_n), DT_n, DA(T_n), DQ_n$ and $DA(Q_n)$ are SD-prime cordial. Prajapati and Vantiya [11] proved that $S(T_n), S(A(T_n)), S(Q_n), S(A(Q_n))$ are SD-prime cordial. Prajapati and Vantiya [12] proved that $k-$polygonal snake $S_n(C_k)$ is SD-prime cordial for all integers $k \geq 3, n \geq 2$ (except for $k = n = 3$). Prajapati and Vantiya [14] proved that altrnate $k-$polygonal snake $AS_n(C_k)$ of type-1, 2 and 3 are SD-prime cordial, for all integers $k \geq 3, n \geq 2$. Prajapati and Vantiya [13] proved that double $k-$polygonal snake $D(S_nC_k)$ is SD-prime cordial. Prajapati and Vantiya [15] proved that K_4-snake K_4S_n, (for all $n \neq 2$), double K_4-snake $D(K_4S_n)$, alternate K_4-snake $A(K_4S_n)$ (for all n, except for $n = 2$, if it is of type-I), double alternate K_4-snake $DA(K_4S_n)$ and prism graph Y_n, for $n = 2p$, where p is prime, are SD-prime cordial. In this paper, we investigate the SD-prime cordial labeling of subdivision of K_4-snake $S(K_4S_n)$, subdivision of double K_4-snake $S(D(K_4S_n))$, subdivision of alternate K_4-snake $S(A(K_4S_n))$ of type 1, 2 and 3, and subdivision of double alternate K_4-snake $S(DA(K_4S_n))$ of type 1, 2 and 3.

Notation: Throughout this paper, a path $P_n = u_1, u_2, \ldots, u_n$, and $n \geq 2$.

Definition 1.1. [3] *A bijection* $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ *induces an edge labeling* $f' : E(G) \to \{0, 1\}$ *such that for any edge uv in G,* $f'(uv) = 1$ *if* $\gcd(S, D) = 1$*, and* $f'(uv) = 0$ *otherwise, where* $S = f(u) + f(v)$ *and* $D = |f(u) - f(v)|$*, for every edge uv in* $E(G)$*. The labeling f is called* SD-prime cordial labeling *if* $|e_{f'}(0) - e_{f'}(1)| \leq 1$*. G is called* SD-prime cordial graph *if it admits SD-prime cordial labeling.*

Definition 1.2. [16] *An edge uv is said to be* subdivided *if the edge uv is replaced by the path* $P : uwv$*, where w is the new vertex. The* subdivision graph $S(G)$ *is obtained from graph G by subdividing each edge of G by a vertex.*

Definition 1.3. *An m-*complete snake *is obtained from the path* P_n *by replacing every edge of* P_n *by a complete graph* K_m $(m \geq 3)$*. It is denoted by* K_mS_n*. It is also called* K_m-snake.

Definition 1.4. *A* double m-complete graph *is a graph containing two copies of complete graphs* K_m $(m \geq 3)$ *having exactly one common edge.*

Definition 1.5. *A* double m-complete snake *is obtained from the path* P_n *by replacing every edge of* P_n *by a double m-complete graph in such a way that the edge is replaced by the common edge of double m-complete graph. It is denoted by* $D(K_mS_n)$*. It is also called* double K_m-snake.

Definition 1.6. *An* alternate m-complete snake *is obtained from the path* P_n *by replacing every alternate edge of* P_n *by a complete graph* K_m $(m \geq 3)$*. It is denoted by* $A(K_mS_n)$*.*

It is also called alternate K_m-snake.
Note that, for every m, there are three non-isomorphic alternate m-complete snakes depending on values of n, they are defined as follows:

(1) *An alternate m-complete snake, in which n is even and the edge u_iu_{i+1} of P_n is replaced by a complete graph K_m for every odd i, is said to be an* alternate m-complete snake of type-1. *It is denoted by $A^1(K_mS_n)$.*
(2) *An alternate m-complete snake, in which n is odd and the edge u_iu_{i+1} of P_n is replaced by a complete graph K_m for every odd i, is said to be an* alternate m-complete snake of type-2. *It is denoted by $A^2(K_mS_n)$.*
(3) *An alternate m-complete snake, in which n is even and the edge u_iu_{i+1} of P_n is replaced by a complete graph K_m for every even i, is said to be an* alternate m-complete snake of type-3. *It is denoted by $A^3(K_mS_n)$.*

Definition 1.7. *A* double alternate m-complete snake *is obtained from the path P_n by replacing every alternate edge of P_n by a double m-complete graph in such a way that the edge is replaced by the common edge of double m-complete graph. It is denoted by $DA(K_mS_n)$. It is also called* double alternate K_m-snake.
Note that, for every m, there are three non-isomorphic double alternate m-complete snakes depending on values of n, they are defined as follows:

(1) *A double alternate m-complete snake, in which n is even and the edge u_iu_{i+1} of P_n is replaced by a double m-complete graph for every odd i, is said to be a* double alternate m-complete snake of type-1. *It is denoted by $DA^1(K_mS_n)$.*
(2) *A double alternate m-complete snake, in which n is odd and the edge u_iu_{i+1} of P_n is replaced by a double m-complete graph for every odd i, is said to be a* double alternate m-complete snake of type-2. *It is denoted by $DA^2(K_mS_n)$.*
(3) *A double alternate m-complete snake, in which n is odd and the edge u_iu_{i+1} of P_n is replaced by a double m-complete graph for every odd i, is said to be a* double alternate m-complete snake of type-3. *It is denoted by $DA^3(K_mS_n)$.*

2. Main Results

Theorem 2.1. *The graph $S(K_4S_n)$ is SD-prime cordial.*

Proof. Let $V(S(K_4S_n)) = V(P_n) \cup \{v_i, w_i, v_i', w_i', v_i'', w_i'', w_i''' : 1 \le i \le n-1\}$ and $E(S(K_4S_n)) = \{u_iu_i', u_i'u_{i+1}, u_iv_i', v_i'v_i, u_iv_i'', v_i''w_i, v_iw_i''', w_i'''w_i, v_iw_i'', w_i''u_{i+1}, w_iw_i', w_i'u_{i+1} : 1 \le i \le n-1\}$. Therefore $S(K_4S_n)$ is of order $9n-8$ and size $12n-12$, see the figure 1.

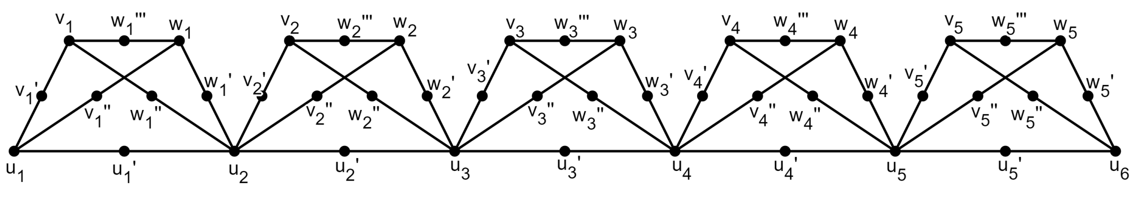

Figure 1. $S(K_4S_6), n = 6$

Define $f : V(S(K_4S_n)) \to \{1, 2, \dots, 9n-8\}$ as follows:

$$f(u_i) = 9i - 8 \quad \text{if } 1 \le i \le n;$$
$$f(v_i) = 9i - 3 \quad \text{if } 1 \le i \le n-1;$$
$$f(w_i) = 9i - 4 \quad \text{if } 1 \le i \le n-1;$$

$$
\begin{aligned}
f(u'_i) &= 9i-2 && \text{if } 1 \le i \le n-1;\\
f(v'_i) &= 9i-6 && \text{if } 1 \le i \le n-1;\\
f(w'_i) &= 9i-1 && \text{if } 1 \le i \le n-1;\\
f(v''_i) &= 9i-5 && \text{if } 1 \le i \le n-1;\\
f(w''_i) &= 9i && \text{if } 1 \le i \le n-1;\\
f(w'''_i) &= 9i-7 && \text{if } 1 \le i \le n-1.
\end{aligned}
$$

Then, the induced edge labeling is:

$$
\begin{aligned}
f^*(u_iv'_i) &= 0 && \text{as } S \text{ and } D \text{ both are multiples of } 2;\\
f^*(v'_iv_i) &= 0 && \text{as } S \text{ and } D \text{ both are multiples of } 3;\\
f^*(v_iw'''_i) &= 0 && \text{as } S \text{ and } D \text{ both are multiples of } 2;\\
f^*(w'''_iw_i) &= 1 && \text{as } D=3 \text{ but } S \text{ is not a multiple of } 3;\\
f^*(w_iw'_i) &= 1 && \text{as } D=3 \text{ but } S \text{ is not a multiple of } 3;\\
f^*(w'_iu_{i+1}) &= 0 && \text{as } S \text{ and } D \text{ both are multiples of } 2;\\
f^*(u_iu'_i) &= 0 && \text{as } S \text{ and } D \text{ both are multiples of } 2;\\
f^*(u'_iu_{i+1}) &= 1 && \text{as } D=3 \text{ but } S \text{ is not a multiple of } 3;\\
f^*(u_iv''_i) &= 1 && \text{as } D=3 \text{ but } S \text{ is not a multiple of } 3;\\
f^*(v''_iw_i) &= 1 && \text{as } D=1;\\
f^*(v_iw''_i) &= 0 && \text{as } S \text{ and } D \text{ both are multiples of } 3;\\
f^*(w''_iu_{i+1}) &= 1 && \text{as } D=1.
\end{aligned}
$$

Therefore $e_{f'}(0) = e_{f'}(1) = 6n-6$.
Thus $|e_{f'}(0) - e_{f'}(1)| \le 1$.
Hence $S(K_4S_n)$ is SD-prime cordial. □

Illustration 2.1. *The graph $S(K_4S_6)$ satisfying SD-prime cordial labeling is shown in the figure 2.*

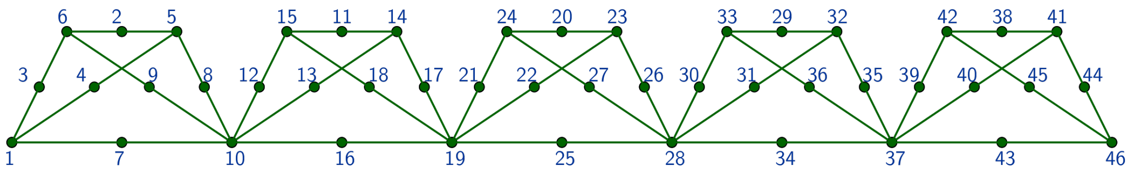

FIGURE 2. SD-Prime Cordial Labeling of $S(K_4S_6), n = 6$

For the general graph $S(K_4S_n)$ one can visit the following GeoGebra applet:
`https://www.geogebra.org/m/zyqxgv5j`

Theorem 2.2. *The graph $S(D(K_4S_n))$ is SD-prime cordial.*

Proof. Let $V(S(D(K_4S_n))) = V(P_n) \cup \{v_i, w_i, v'_i, w'_i, v''_i, w''_i, w'''_i, x_i, y_i, x'_i, y'_i, x''_i, y''_i, y'''_i : 1 \le i \le n-1\}$ and $E(S(D(K_4S_n))) = \{u_iu'_i, u'_iu_{i+1}, u_iv'_i, v'_iv_i, u_iv''_i, v''_iw_i, v_iw'''_i, w'''_iw_i, v_iw''_i, w''_iu_{i+1}, w_iw'_i, w'_iu_{i+1}, u_ix'_i, x'_ix_i, u_ix''_i, x''_iy_i, x_iy'''_i, y'''_iy_i, x_iy''_i, y''_iu_{i+1}, y_iy'_i, y'_iu_{i+1} : 1 \le i \le n-1\}$. Therefore $S(D(K_4S_n))$ is of order $16n-15$ and size $22n-22$, see the figure 3.

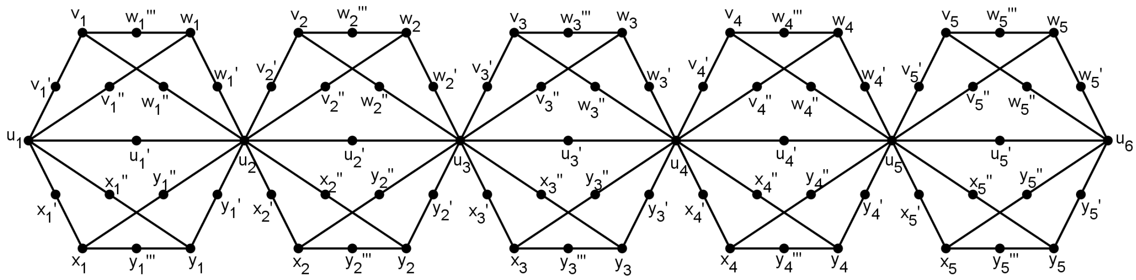

FIGURE 3. $S(D(K_4S_6)), n = 6$

Define $f : V(S(D(K_4S_n))) \to \{1, 2, \ldots, 16n - 15\}$ as follows:

$$
\begin{array}{ll}
f(u_i) = 16i - 15 & \text{if } 1 \le i \le n; \\
f(v_i) = 16i - 7 & \text{if } 1 \le i \le n-1; \\
f(w_i) = 16i - 1 & \text{if } 1 \le i \le n-1; \\
f(u_i') = 16i - 5 & \text{if } 1 \le i \le n-1; \\
f(v_i') = 16i - 13 & \text{if } 1 \le i \le n-1; \\
f(w_i') = 16i & \text{if } 1 \le i \le n-1; \\
f(v_i'') = 16i - 6 & \text{if } 1 \le i \le n-1; \\
f(w_i'') = 16i - 2 & \text{if } 1 \le i \le n-1; \\
f(w_i''') = 16i - 4 & \text{if } 1 \le i \le n-1; \\
f(x_i) = 16i - 14 & \text{if } 1 \le i \le n-1; \\
f(y_i) = 16i - 8 & \text{if } 1 \le i \le n-1; \\
f(x_i') = 16i - 12 & \text{if } 1 \le i \le n-1; \\
f(y_i') = 16i - 3 & \text{if } 1 \le i \le n-1; \\
f(x_i'') = 16i - 10 & \text{if } 1 \le i \le n-1; \\
f(y_i'') = 16i - 11 & \text{if } 1 \le i \le n-1; \\
f(y_i''') = 16i - 9 & \text{if } 1 \le i \le n-1.
\end{array}
$$

Therefore $e_{f'}(1) = e_{f'}(0) = 11n - 11$.
Thus $|e_{f'}(0) - e_{f'}(1)| \le 1$.
Hence $S(D(K_4S_n))$ is SD-prime cordial. □

Illustration 2.2. *The graph $S(D(K_4S_6))$ satisfying SD-prime cordial labeling is shown in the figure 4.*

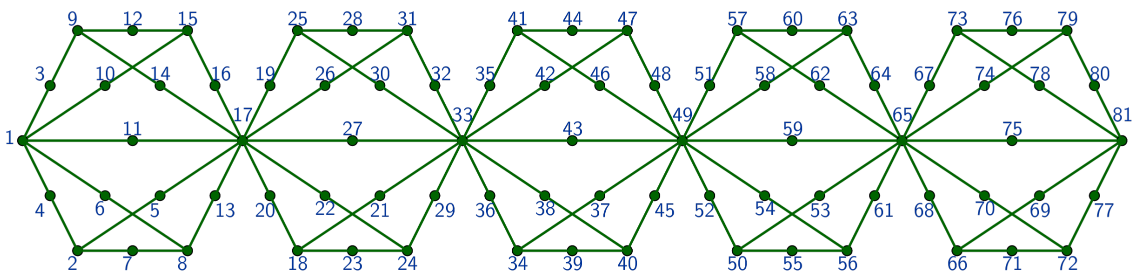

FIGURE 4. SD-Prime Cordial Labeling of $S(D(K_4S_6)), n = 6$

For the general graph $S(D(K_4S_n))$ one can visit the following GeoGebra applet: `https://www.geogebra.org/m/vcjdfrc6`

Theorem 2.3. *The graph $S(A(K_4S_n))$ is SD-prime cordial.*

Proof. **Case-1: $S(A^1(K_4S_n))$:**
In this case, n is even.
Let $V(S(A^1(K_4S_n))) = V(P_n) \cup \{u_i' : 1 \le i \le n-1\} \cup \{v_i, w_i, v_i', w_i', v_i'', w_i'', w_i''' :$ i is odd and $1 \le i \le n-1\}$ and $E(S(A^1(K_4S_n))) = \{u_iu_i', u_i'u_{i+1} : 1 \le i \le n-1\} \cup \{u_iv_i', v_i'v_i, u_iv_i'', v_i''w_i, v_iw_i''', w_i'''w_i, v_iw_i'', w_i''u_{i+1}, w_iw_i', w_i'u_{i+1} : i$ is odd and $1 \le i \le n-1\}$. Therefore $S(A^1(K_4S_n))$ is of order $\frac{11n-2}{2}$ and size $7n-2$, see the figure 5.

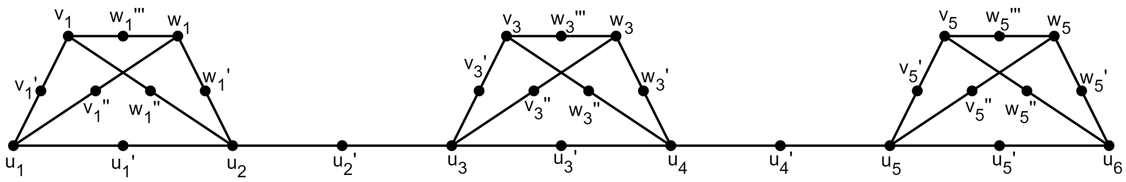

FIGURE 5. $S(A^1(K_4S_6)), n = 6$

Define $f : V(S(A^1(K_4S_n))) \to \{1, 2, \ldots, \frac{11n-2}{2}\}$ as follows:

$$f(u_i) = \begin{cases} \frac{11i-9}{2} & i \equiv 1 \pmod 4 \\ \frac{11i-2}{2} & i \equiv 2 \pmod 4 \\ \frac{11i-11}{2} & i \equiv 3 \pmod 4 \\ \frac{11i-4}{2} & i \equiv 0 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n;$$

$$f(v_i) = \begin{cases} \frac{11i+1}{2} & i \equiv 1 \pmod 4 \\ \frac{11i-7}{2} & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(w_i) = \begin{cases} \frac{11i-1}{2} & i \equiv 1 \pmod 4 \\ \frac{11i-3}{2} & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(u_i') = \begin{cases} \frac{11i+3}{2} & i \equiv 1 \pmod 4 \\ \frac{11i+2}{2} & i \equiv 2 \pmod 4 \\ \frac{11i+5}{2} & i \equiv 3 \pmod 4 \\ \frac{11i}{2} & i \equiv 0 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(v_i') = \frac{11i-5}{2} \quad i \text{ is odd} \qquad \text{if } 1 \le i \le n-1;$$

$$f(w_i') = \begin{cases} \frac{11i+5}{2} & i \equiv 1 \pmod 4 \\ \frac{11i+3}{2} & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(v_i'') = \begin{cases} \frac{11i-3}{2} & i \equiv 1 \pmod 4 \\ \frac{11i+1}{2} & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(w_i'') = \begin{cases} \frac{11i+7}{2} & i \equiv 1 \pmod 4 \\ \frac{11i+9}{2} & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(w_i''') = \begin{cases} \frac{11i-7}{2} & i \equiv 1 \pmod 4 \\ \frac{11i-1}{2} & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1.$$

Therefore $e_{f'}(1) = e_{f'}(0) = \dfrac{7n-2}{2}$.

Case-2: $S(A^2(K_4S_n))$:

In this case, n is odd.

Let $V(S(A^2(K_4S_n))) = V(P_n) \cup \{u_i' : 1 \le i \le n-1\} \cup \{v_i, w_i, v_i', w_i', v_i'', w_i'', w_i''' : i \text{ is odd and } 1 \le i \le n-1\}$ and $E(S(A^2(K_4S_n))) = \{u_iu_i', u_i'u_{i+1} : 1 \le i \le n-1\} \cup \{u_iv_i', v_i'v_i, u_iv_i'', v_i''w_i, v_iw_i''', w_i'''w_i, v_iw_i'', w_i''u_{i+1}, w_iw_i', w_i'u_{i+1} : i \text{ is odd and } 1 \le i \le n-1\}$. Therefore $S(A^2(K_4S_n))$ is of order $\frac{11n-9}{2}$ and size $7n-7$, see the figure 6.

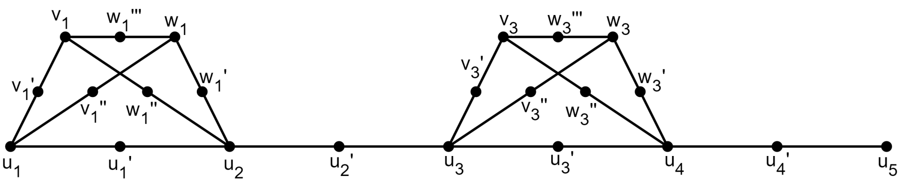

FIGURE 6. $S(A^2(K_4S_5)), n = 5$

Define $f : V(S(A^2(K_4S_n))) \to \{1, 2, \ldots, \frac{11n-9}{2}\}$ as per the case-1 (above).

Therefore $e_{f'}(1) = e_{f'}(0) = \dfrac{7n-7}{2}$.

Case-3: $S(A^3(K_4S_n))$:

In this case, n is even.

Let $V(S(A^3(K_4S_n))) = V(P_n) \cup \{u_i' : 1 \le i \le n-1\} \cup \{v_i, w_i, v_i', w_i', v_i'', w_i'', w_i''' : i \text{ is even and } 1 \le i \le n-1\}$ and $E(S(A^3(K_4S_n))) = \{u_iu_i', u_i'u_{i+1} : 1 \le i \le n-1\} \cup \{u_iv_i', v_i'v_i, u_iv_i'', v_i''w_i, v_iw_i''', w_i'''w_i, v_iw_i'', w_i''u_{i+1}, w_iw_i', w_i'u_{i+1} : i \text{ is even and } 1 \le i \le n-1\}$. Therefore $S(A^3(K_4S_n))$ is of order $\frac{11n-16}{2}$ and size $7n-12$, see the figure 7.

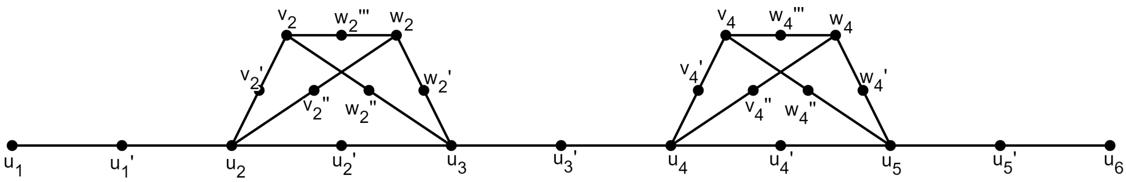

FIGURE 7. $S(A^3(K_4S_6)), n = 6$

Define $f : V(S(A^3(K_4S_n))) \to \{1, 2, \ldots, \frac{11n-16}{2}\}$ as follows:

$$f(u_i) = \begin{cases} 2 & i = 1 \\ \frac{11i-16}{2} & i \equiv 2 \pmod 4 \\ \frac{11i-9}{2} & i \equiv 3 \pmod 4 \\ \frac{11i-18}{2} & i \equiv 0 \pmod 4 \\ \frac{11i-11}{2} & i \equiv 1 \pmod 4 \text{ and } i \ne 1 \end{cases} \qquad \text{if } 1 \le i \le n;$$

$$f(v_i) = \begin{cases} \frac{11i-6}{2} & i \equiv 2 \pmod 4 \\ \frac{11i-14}{2} & i \equiv 0 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(w_i) = \begin{cases} \frac{11i-8}{2} & i \equiv 2 \pmod 4 \\ \frac{11i-10}{2} & i \equiv 0 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(u'_i) = \begin{cases} 1 & i = 1 \\ \frac{11i-4}{2} & i \equiv 2 \pmod 4 \\ \frac{11i-5}{2} & i \equiv 3 \pmod 4 \\ \frac{11i-2}{2} & i \equiv 0 \pmod 4 \\ \frac{11i-7}{2} & i \equiv 1 \pmod 4 \text{ and } i \neq 1 \end{cases} \qquad \text{if } 1 \leq i \leq n-1;$$

$$f(v'_i) = \frac{11i-12}{2} \quad i \text{ is even} \qquad \text{if } 1 \leq i \leq n-1;$$

$$f(w'_i) = \begin{cases} \frac{11i-2}{2} & i \equiv 2 \pmod 4 \\ \frac{11i-4}{2} & i \equiv 0 \pmod 4 \end{cases} \qquad \text{if } 1 \leq i \leq n-1;$$

$$f(v''_i) = \begin{cases} \frac{11i-10}{2} & i \equiv 2 \pmod 4 \\ \frac{11i-6}{2} & i \equiv 0 \pmod 4 \end{cases} \qquad \text{if } 1 \leq i \leq n-1;$$

$$f(w''_i) = \begin{cases} \frac{11i}{2} & i \equiv 2 \pmod 4 \\ \frac{11i+2}{2} & i \equiv 0 \pmod 4 \end{cases} \qquad \text{if } 1 \leq i \leq n-1;$$

$$f(w'''_i) = \begin{cases} \frac{11i-14}{2} & i \equiv 2 \pmod 4 \\ \frac{11i-8}{2} & i \equiv 0 \pmod 4 \end{cases} \qquad \text{if } 1 \leq i \leq n-1.$$

Therefore $e_{f'}(1) = e_{f'}(0) = \frac{7n-12}{2}$.
Thus from all the cases, $|e_{f'}(0) - e_{f'}(1)| \leq 1$.
Hence $S(A(K_4S_n))$ is SD-prime cordial. □

Illustration 2.3. *The graphs $S(A(K_4S_n))$ of types 1, 2 and 3 satisfying SD-prime cordial labeling are shown in the figures 8, 9 and 10.*

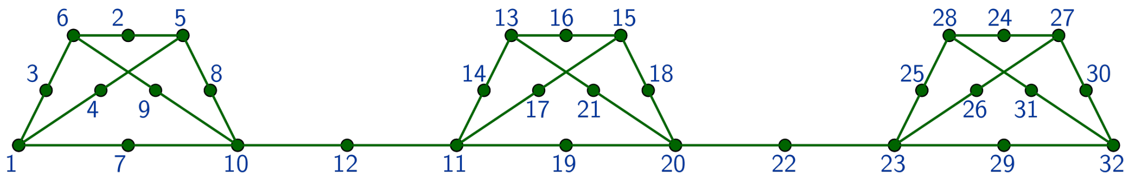

FIGURE 8. SD-Prime Cordial Labeling of $S(A^1(K_4S_6)), n = 6$

For the general graph $S(A^1(K_4S_n))$ one can visit the following GeoGebra applet:
https://www.geogebra.org/m/tmessgeg

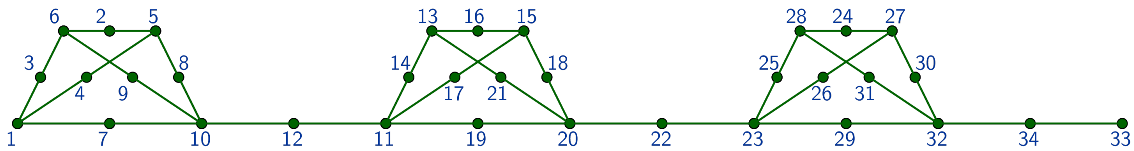

FIGURE 9. SD-Prime Cordial Labeling of $S(A^2(K_4S_7)), n = 7$

For the general graph $S(A^2(K_4S_n))$ one can visit the following GeoGebra applet:
https://www.geogebra.org/m/tmessgeg

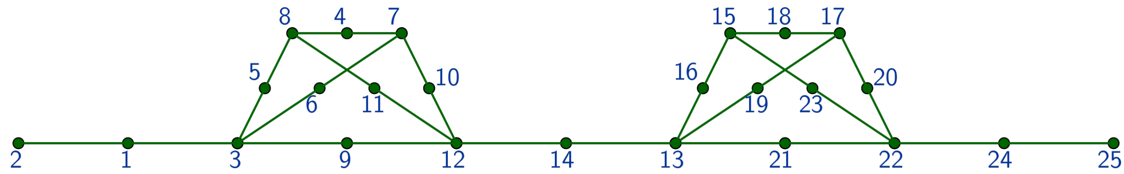

FIGURE 10. SD-Prime Cordial Labeling of $S(A^3(K_4S_6)), n = 6$

For the general graph $S(A^3(K_4S_n))$ one can visit the following GeoGebra applet: *https://www.geogebra.org/m/zve8qbh9*

Theorem 2.4. *The graph $S(DA(K_4S_n))$ is SD-prime cordial.*

Proof. **Case-1: $S(DA^1(K_4S_n))$:**
In this case, n is even.
Let $V(S(DA^1(K_4S_n))) = V(P_n) \cup \{u'_i : 1 \leq i \leq n-1\} \cup \{v_i, w_i, v'_i, w'_i, v''_i, w''_i, w'''_i, x_i, y_i, x'_i, y'_i, x''_i, y''_i, y'''_i : i$ is odd and $1 \leq i \leq n-1\}$ and $E(S(DA^1(K_4S_n))) = \{u_iu'_i, u'_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv'_i, v'_iv_i, u_iv''_i, v''_iw_i, v_iw'''_i, w'''_iw_i, v_iw''_i, w''_iu_{i+1}, w_iw'_i, w'_iu_{i+1}, u_ix'_i, x'_ix_i, u_ix''_i, x''_iy_i, x_iy'''_i, y'''_iy_i, x_iy''_i, y''_iu_{i+1}, y_iy'_i, y'_iu_{i+1} : i$ is odd and $1 \leq i \leq n-1\}$. Therefore $S(DA^1(K_4S_n))$ is of order $9n-1$ and size $12n-2$, see the figure 11.

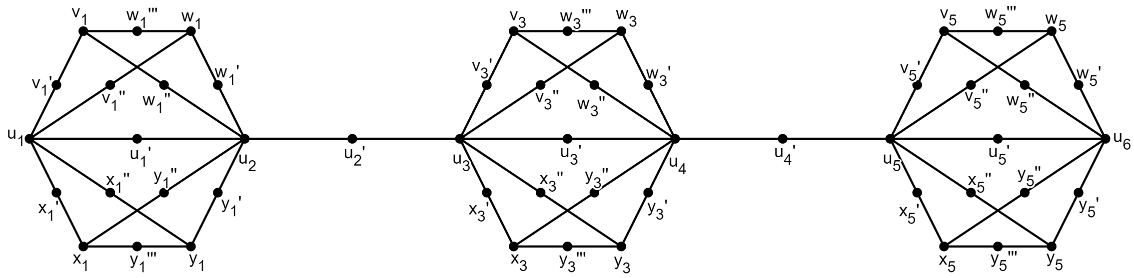

FIGURE 11. $S(DA^1(K_4S_6)), n = 6$

Define $f : V(S(DA^1(K_4S_n))) \to \{1, 2, \ldots, 9n-1\}$ as follows:

$$f(u_i) = \begin{cases} 9i-8 & i \equiv 1 \pmod 4 \\ 9i-1 & i \equiv 2 \pmod 4 \\ 9i-9 & i \equiv 3 \pmod 4 \\ 9i-2 & i \equiv 0 \pmod 4 \end{cases} \qquad \text{if } 1 \leq i \leq n;$$

$$f(u'_i) = \begin{cases} 9i+2 & i \equiv 1 \pmod 4 \\ 9i+1 & i \equiv 2 \pmod 4 \\ 9i-5 & i \equiv 3 \pmod 4 \\ 9i & i \equiv 0 \pmod 4 \end{cases} \qquad \text{if } 1 \leq i \leq n-1;$$

$$f(v_i) = \begin{cases} 9i & i \equiv 1 \pmod 4 \\ 9i-1 & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \leq i \leq n-1;$$

$$f(w_i) = \begin{cases} 9i+6 & i \equiv 1 \pmod 4 \\ 9i+5 & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \leq i \leq n-1;$$

$$f(v'_i) = \begin{cases} 9i-6 & i \equiv 1 \pmod 4 \\ 9i-7 & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \leq i \leq n-1;$$

$$f(w_i') = \begin{cases} 9i+7 & i \equiv 1 \pmod 4 \\ 9i+6 & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(v_i'') = \begin{cases} 9i+1 & i \equiv 1 \pmod 4 \\ 9i & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(w_i'') = \begin{cases} 9i+5 & i \equiv 1 \pmod 4 \\ 9i+4 & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(w_i''') = \begin{cases} 9i+3 & i \equiv 1 \pmod 4 \\ 9i+8 & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(x_i) = \begin{cases} 9i-7 & i \equiv 1 \pmod 4 \\ 9i+2 & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(y_i) = \begin{cases} 9i-1 & i \equiv 1 \pmod 4 \\ 9i-2 & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(x_i') = \begin{cases} 9i-5 & i \equiv 1 \pmod 4 \\ 9i-6 & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(y_i') = \begin{cases} 9i+4 & i \equiv 1 \pmod 4 \\ 9i+3 & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(x_i'') = \begin{cases} 9i-3 & i \equiv 1 \pmod 4 \\ 9i-4 & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(y_i'') = \begin{cases} 9i-4 & i \equiv 1 \pmod 4 \\ 9i-3 & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(y_i''') = \begin{cases} 9i-2 & i \equiv 1 \pmod 4 \\ 9i+1 & i \equiv 3 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1.$$

Therefore $e_{f'}(1) = e_{f'}(0) = 6n-1$.

Case-2: $S(DA^2(K_4S_n))$:

In this case, n is odd.

Let $V(S(DA^2(K_4S_n))) = V(P_n) \cup \{u_i' : 1 \le i \le n-1\} \cup \{v_i, w_i, v_i', w_i', v_i'', w_i'', w_i''', x_i, y_i, x_i', y_i', x_i'', y_i'', y_i''' : i \text{ is odd and } 1 \le i \le n-1\}$ and $E(S(DA^2(K_4S_n))) = \{u_iu_i', u_i'u_{i+1} : 1 \le i \le n-1\} \cup \{u_iv_i', v_i'v_i, u_iv_i'', v_i''w_i, v_iw_i''', w_i'''w_i, v_iw_i'', w_i''u_{i+1}, w_iw_i', w_i'u_{i+1}, u_ix_i', x_i'x_i, u_ix_i'', x_i''y_i, x_iy_i''', y_i'''y_i, x_iy_i'', y_i''u_{i+1}, y_iy_i', y_i'u_{i+1} : i \text{ is odd and } 1 \le i \le n-1\}$. Therefore $S(DA^2(K_4S_n))$ is of order $9n-8$ and size $12n-12$, see the figure 12.

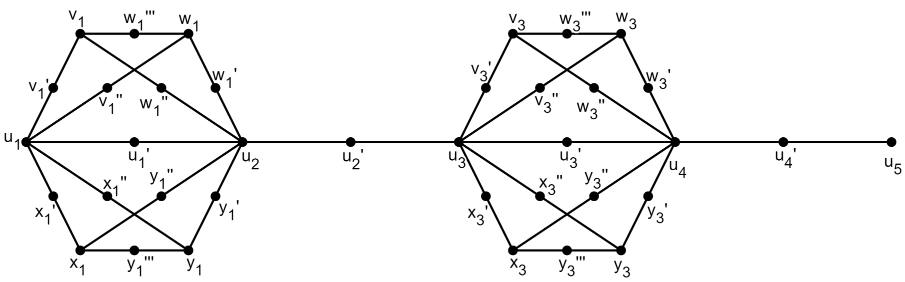

FIGURE 12. $S(DA^2(K_4S_5)), n = 5$

Define $f : V(S(DA^2(K_4S_n))) \to \{1, 2, \ldots, 9n-8\}$ as per the case-1 (above). Therefore $e_{f'}(1) = e_{f'}(0) = 6n-6$.

Case-3: $S(DA^3(K_4S_n))$:

In this case, n is even.

Let $V(S(DA^3(K_4S_n))) = V(P_n) \cup \{u'_i : 1 \le i \le n-1\} \cup \{v_i, w_i, v'_i, w'_i, v''_i, w''_i, w'''_i, x_i, y_i, x'_i, y'_i, x''_i, y''_i, y'''_i : i$ is even and $1 \le i \le n-1\}$ and $E(S(DA^3(K_4S_n))) = \{u_iu'_i, u'_iu_{i+1} : 1 \le i \le n-1\} \cup \{u_iv'_i, v'_iv_i, u_iv''_i, v''_iw_i, v_iw'''_i, w'''_iw_i, v_iw''_i, w''_iu_{i+1}, w_iw'_i, w'_iu_{i+1}, u_ix'_i, x'_ix_i, u_ix''_i, x''_iy_i, x_iy'''_i, y'''_iy_i, x_iy''_i, y''_iu_{i+1}, y_iy'_i, y'_iu_{i+1}$: i is even and $1 \le i \le n-1\}$. Therefore $S(DA^3(K_4S_n))$ is of order $9n-15$ and size $12n-22$, see the figure 13.

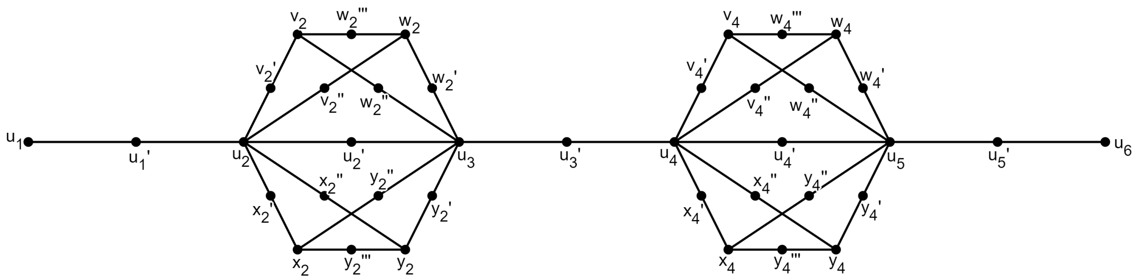

FIGURE 13. $S(DA^3(K_4S_6)), n = 6$

Define $f : V(S(DA^3(K_4S_n))) \to \{1, 2, \ldots, 9n-15\}$ as follows:

$$f(u_i) = \begin{cases} 9i-8 & i \equiv 1 \pmod 4 \\ 9i-16 & i \equiv 2 \pmod 4 \\ 9i-9 & i \equiv 3 \pmod 4 \\ 9i-15 & i \equiv 0 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n;$$

$$f(u'_i) = \begin{cases} 9i-6 & i \equiv 1 \pmod 4 \\ 9i-12 & i \equiv 2 \pmod 4 \\ 9i-7 & i \equiv 3 \pmod 4 \\ 9i-5 & i \equiv 0 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(v_i) = \begin{cases} 9i-8 & i \equiv 2 \pmod 4 \\ 9i-7 & i \equiv 0 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(w_i) = \begin{cases} 9i-2 & i \equiv 2 \pmod 4 \\ 9i-1 & i \equiv 0 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(v'_i) = \begin{cases} 9i-14 & i \equiv 2 \pmod 4 \\ 9i-13 & i \equiv 0 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(w'_i) = \begin{cases} 9i-1 & i \equiv 2 \pmod 4 \\ 9i & i \equiv 0 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(v''_i) = \begin{cases} 9i-7 & i \equiv 2 \pmod 4 \\ 9i-6 & i \equiv 0 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(w''_i) = \begin{cases} 9i-3 & i \equiv 2 \pmod 4 \\ 9i-2 & i \equiv 0 \pmod 4 \end{cases} \qquad \text{if } 1 \le i \le n-1;$$

$$f(w_i''') = \begin{cases} 9i+1 & i \equiv 2 \pmod 4 \\ 9i-4 & i \equiv 0 \pmod 4 \end{cases} \quad \text{if } 1 \le i \le n-1;$$

$$f(x_i) = \begin{cases} 9i-5 & i \equiv 2 \pmod 4 \\ 9i-14 & i \equiv 0 \pmod 4 \end{cases} \quad \text{if } 1 \le i \le n-1;$$

$$f(y_i) = \begin{cases} 9i-9 & i \equiv 2 \pmod 4 \\ 9i-8 & i \equiv 0 \pmod 4 \end{cases} \quad \text{if } 1 \le i \le n-1;$$

$$f(x_i') = \begin{cases} 9i-13 & i \equiv 2 \pmod 4 \\ 9i-12 & i \equiv 0 \pmod 4 \end{cases} \quad \text{if } 1 \le i \le n-1;$$

$$f(y_i') = \begin{cases} 9i-4 & i \equiv 2 \pmod 4 \\ 9i-3 & i \equiv 0 \pmod 4 \end{cases} \quad \text{if } 1 \le i \le n-1;$$

$$f(x_i'') = \begin{cases} 9i-11 & i \equiv 2 \pmod 4 \\ 9i-10 & i \equiv 0 \pmod 4 \end{cases} \quad \text{if } 1 \le i \le n-1;$$

$$f(y_i'') = \begin{cases} 9i-10 & i \equiv 2 \pmod 4 \\ 9i-11 & i \equiv 0 \pmod 4 \end{cases} \quad \text{if } 1 \le i \le n-1;$$

$$f(y_i''') = \begin{cases} 9i-6 & i \equiv 2 \pmod 4 \\ 9i-9 & i \equiv 0 \pmod 4 \end{cases} \quad \text{if } 1 \le i \le n-1.$$

Therefore $e_{f'}(1) = e_{f'}(0) = 6n - 11$.
Thus from all the cases, $|e_{f'}(0) - e_{f'}(1)| \le 1$.
Hence $S(DA(K_4S_n))$ is SD-prime cordial. □

Illustration 2.4. *The graphs $S(DA(K_4S_n))$ of types 1, 2 and 3 satisfying SD-prime cordial labeling are shown in the figures 14, 15 and 16.*

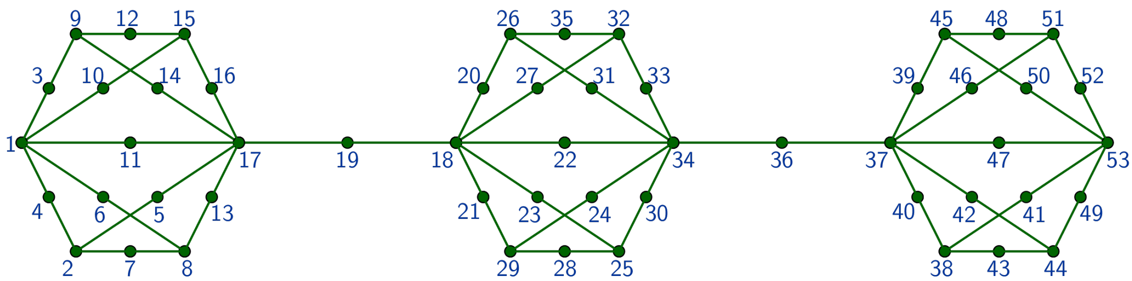

FIGURE 14. SD-Prime Cordial Labeling of $S(DA^1(K_4S_6)), n = 6$

For the general graph $S(DA^1(K_4S_n))$ one can visit the following GeoGebra applet:
https://www.geogebra.org/m/v6jwn7xu

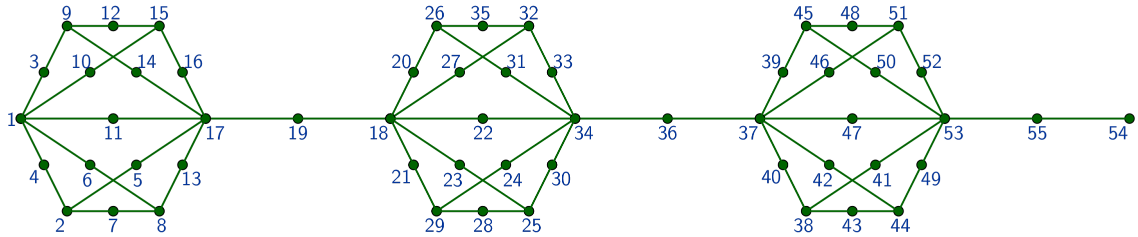

FIGURE 15. SD-Prime Cordial Labeling of $S(DA^2(K_4S_7)), n = 7$

For the general graph $S(DA^2(K_4S_n))$ one can visit the following GeoGebra applet: `https://www.geogebra.org/m/v6jwn7xu`

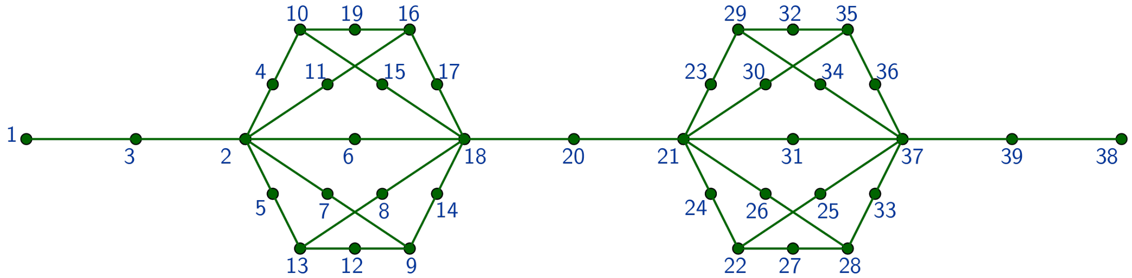

FIGURE 16. SD-Prime Cordial Labeling of $S(DA^3(K_4S_6)), n = 6$

For the general graph $S(DA^3(K_4S_n))$ one can visit the following GeoGebra applet: `https://www.geogebra.org/m/dywtdzwy`

3. Conclusion:

We have proved that subdivision of K_4-snake $S(K_4S_n)$, subdivision of double K_4-snake $S(D(K_4S_n))$, subdivision of alternate K_4-snake $S(A(K_4S_n))$ and subdivision of double alternate K_4- snake graphs $S(DA(K_4S_n))$ are SD-prime cordial.

It is observed that SD-prime cordial labeling gets more difficult, when we have more vertices having large degree. So one can try to find the relation between number of vertices and their degrees, so that the graph will not be SD-prime cordial.

Further investigation can be done for the more general case $S(K_mS_n)$, for arbitrary $m \in N$. But there might be some difficulties for large m, because it will increase the number of vertices having large degree.

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