TWMS J. App. and Eng. Math. V.13, N.2, 2023, pp. 591-601

EQUILIBRIUM PROBLEM FOR A THERMOELASTIC KIRCHHOFF–LOVE PLATE WITH A DELAMINATED RIGID INCLUSION

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Abstract. A new variational problem on the equilibrium of a thermoelastic heterogeneous Kirchhoff–Love plate is considered in a domain with a cut. The cut corresponds to an interfacial crack located on a part of the boundary of a rigid inclusion. We suppose that the plate is under the special loads for which the configuration of crack's edges is known a priori. This assumption allows us to rewrite the well known nonpenetration condition for Kirchhoff–Love plates in a refined form, which, in turn, leads to new relations describing the possible mechanical interaction of opposite crack edges. Displacements on the rigid inclusion satisfy special constraints having a linear form. Solvability of the problem is proved, an equivalent differential statement is found.

Keywords: Thermoelastic plate, crack, nonpenetration, variational inequality, differential setting.

AMS Subject Classification: 49J40, 49K20.

1. Introduction

The widespread use of composite parts in industry attracts a growing scientific interest in development of mathematical approaches. As a result, qualitatively new models are being developed, as well as more complex mathematical approaches leading to new formulations of problems. For mathematical challenges with regard to models describing deformation of composite bodies with cracks, the adequacy of chosen models largely depends on methods for setting boundary conditions on crack's curves or surfaces.

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§ Manuscript received: March 02, 2021; accepted: April 03, 2021.

The first author is supported by the Russian Foundation for Basic Research and the Republic of Sakha (Yakutia) (project no. 18-41-140003).

The use of so-called "nonpenetration" boundary conditions in the form of inequalities makes it possible to restrict the range of sought functions based on relations for displacements having a physically clear sense. Nonlinear inequality-type constraints make it possible to describe a mechanical contact of two independent bodies, or opposite crack faces. This approach uses methods of variational inequalities and has been actively developing, see [1–25]. Note that the classical approach of the crack theory (see [26–29]), which implies using boundary conditions in the form of equalities, can lead to physical contradictions for displacements of crack's faces [27]. A wide range of various problems has been studied in the framework of Kirchhoff-Love plates subject to the well-known general nonpenetration condition [2, 16, 17, 18, 19]. An overwhelming majority of results for cracked Kirchhoff–Love plates were obtained for vertical cracks. At the same time, some results were justified for plates with oblique cracks, see, for example, [2, 6, 24].

In this paper, we pay attention to the special case when a certain configuration of plate's edges near a crack is known a priori for an equilibrium state of a plate. This circumstance means that some geometrical features of a possible contact of crack's faces are known, which make it possible to write out boundary conditions in a refined form. Based on these conditions, we define a corresponding set of admissible functions in a suitable Sobolev space. Taking into account of temperature effects can play a significant role in applied problems arising from the issues of operations in the Far North. Within the framework of thermoelastic models of plates, the presence of a delaminated rigid inclusion and taking into account a possible mechanical contact interaction of crack faces (friction forces are not taken into account) determine the novelty of the problem under study. Different qualitative properties of solutions in equilibrium problems for elastic bodies with delaminated rigid inclusions are investigated in [7, 15, 17, 18, 22, 23] and many other papers. Thermoelastic models of plates with cracks have been studied, for example, in [25, 30, 31, 32]. It is well known that the Kirchhoff–Love model is formulated in a two-dimensional domain, while plates are three-dimensional objects. This simplification causes some difficulties in setting boundary conditions that would reflect three-dimensional properties of plates. In particular, an example given in [16] shows that there exist functions with displacements satisfying the general nonpenetration condition, but nevertheless, for which we have a physically unacceptable phenomenon since there is a mutual penetration of opposite crack faces. Therefore, the above-mentioned special cases for plates subject to refined modifications of nonpenetration conditions is a justified branch of the development of the mechanics of deformable solids, see, for example, [33, 34].

A new mathematical model describing an equilibrium of a thermoelastic plate with a crack at the boundary of a rigid bulk inclusion is formulated. The existence of a solution is established for the corresponding variational problem. Assuming that the solution is smooth enough, additional boundary conditions establishing an equivalence of differential and variational formulations are found.

2. Statement of the problem

Let us formulate an equilibrium problem for an thermoelastic plate containing a bulk rigid inclusion. We consider the case of a delaminated inclusion, when a crack passes through the inclusion's interface. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary Γ. Let the subdomain ω lie strictly inside Ω, i.e. $\overline{\omega} \cap \Gamma = \emptyset$ and has the smooth boundary Σ. We denote by ν the outward unit normal on Σ. Assume that Σ consists of the two following parts γ and $\Sigma\setminus\gamma$ that both are curves with nonzero lengths. In addition, we assume that γ can be extended to Γ so that Ω is splitted into two subdomains Ω_1 and Ω_2 with Lipschitz boundaries $\partial\Omega_1$ and $\partial\Omega_2$ where $meas(\Gamma \cap \partial\Omega_i) > 0,\ i = 1, 2.$

The assumption is sufficient for Korn's inequality to hold in the non-Lipschitz domain $\Omega_\gamma = \Omega\setminus\bar{\gamma}$ [2]. Depending on the direction of the normal $\nu = (\nu_1, \nu_2)$ to γ we will speak about a positive face γ^+ or a negative face γ^- of the curve γ. The jump $[q]$ of the function q on the curve γ is found by the formula $[q] = q|_{\gamma^+} - q|_{\gamma^-}$.

For simplicity, we assume that the thickness $2h$ of the plate is constant and is equal to two, i.e. $h = 1$. We introduce a three-dimensional Cartesian space $\{x_1, x_2, z\}$ such that the set $\{\Omega_\gamma\} \times \{0\} \subset \mathbb{R}^3$ corresponds to the middle plane of the plate. The set $\omega \times [-1, 1]$ is assumed to correspond to a bulk rigid inclusion, i.e. the boundary of the rigid inclusion is defined by the cylindrical surface $\Sigma \times [-1, 1]$. The through interfacial crack locates on the inclusion's boundary and is described by a cylindrical surface defined with the relations $x = (x_1, x_2) \in \gamma, -1 \le z \le 1$, where $|z|$ is the distance to the middle plane. Denote by $\chi = (W, w)$ the vector of mid-plane displacements, where $W = (w_1, w_2)$ are the displacements in the plane $\{x_1, x_2\}$ and w are the displacements along the axis z (deflections).

The temperature field in the plate is denoted by θ. We also need the following set $Q_\gamma = \Omega_\gamma \times (0, T)$, $T > 0$. The strain and integrated stress tensors are denoted by $\varepsilon_{ij} = \varepsilon_{ij}(W)$, $\sigma_{ij} = \sigma_{ij}(W)$, respectively [2]:

$$\sigma_{11} = \varepsilon_{11} + \kappa\varepsilon_{22}, \quad \sigma_{22} = \varepsilon_{22} + \kappa\varepsilon_{11}, \quad \sigma_{12} = (1-\kappa)\varepsilon_{12},$$

$$\varepsilon_{ij}(W) = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right), \quad i, j = 1, 2, \quad x_1 = x,\ x_2 = y,$$

where $\kappa = \mathrm{const}, \quad 0 < \kappa < 1/2$.

In order to describe the possible contact interaction of the crack's edges, for the case of prior knowledge of a certain equilibrium configuration of plate edges near the crack (see fig. 1), we specify the following mutual nonpenetration condition of opposite crack faces [34]

$$[\frac{\partial w}{\partial \nu}] \ge 0, \quad [W]\nu \ge [\frac{\partial w}{\partial \nu}], \quad [w] = 0 \quad \text{on} \quad \gamma^T = \gamma \times (0, T). \tag{1}$$

We should note that the inequality (1) is written for functions χ given in the domain

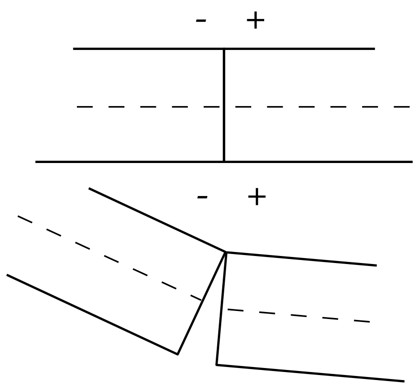

FIGURE 1. An example of crack edges configurations for initial (the upper image) and equilibrium (the lower image) states.

Q_γ. In the case when considered functions are defined in Ω_γ, we change γ^T to γ and the nonpenetration condition will be written as:

$$[\frac{\partial w}{\partial \nu}] \ge 0, \quad [W]\nu \ge [\frac{\partial w}{\partial \nu}], \quad [w] = 0 \quad \text{on} \quad \gamma. \tag{2}$$

In addition, we can mention that if condition (2) holds for some function, then this function also satisfies the following well-known general nonpenetration condition for cracks in Kirchhoff–Love plates [3, 2].

$$[W]\nu \geq |[\frac{\partial w}{\partial \nu}]| \quad \text{on} \quad \gamma^T. \tag{3}$$

Due to presence of the rigid inclusion in the plate, restrictions of the functions describing displacements χ to the domain ω satisfy a special kind of relations. We introduce the space which allows us to characterize the properties of the bulk rigid inclusion

$$\begin{aligned} R(\omega) = \{\zeta(x) = (\rho, l) \,|\, \rho(x) = \\ = b(x_2, -x_1) + (c_1, c_2);\ l(x) = a_0 + a_1 x_1 + a_2 x_2, \quad x \in \omega\}, \end{aligned} \tag{4}$$

where $b, c_1, c_2, a_0, a_1, a_2 \in \mathbb{R}$ [3, 7].

Let some initial temperature distribution be given:

$$\theta = \theta_0 \quad \text{at } \ t = 0. \tag{5}$$

On the exterior boundary of the plate, we require the fulfillment of the following conditions:

$$\theta = w = \frac{\partial w}{\partial e} = W = 0 \quad \text{on } \ \Gamma \times (0, T), \tag{6}$$

where e is the external normal vector to Γ. Introduce the Sobolev spaces

$$H^{1,0}(\Omega_\gamma) = \left\{ v \in H^1(\Omega_\gamma) \;\middle|\; v = 0 \ \text{ on } \Gamma \right\},$$

$$H^{2,0}(\Omega_\gamma) = \left\{ v \in H^2(\Omega_\gamma) \;\middle|\; v = \tfrac{\partial v}{\partial e} = 0 \ \text{ on } \Gamma \right\},$$

$$H(\Omega_\gamma) = H^{1,0}(\Omega_\gamma)^2 \times H^{2,0}(\Omega_\gamma).$$

Consider the following sets

$$K = \{\chi = (W, w) \in H(\Omega_\gamma) \,| \quad \chi|_\omega \in R(\omega), \ \chi \ \text{ satisfies } \ (2) \quad \text{a.e. on } \gamma\},$$

$$\mathcal{K} = \{\chi \in L^2(0, T; H(\Omega_\gamma)) \,| \quad \chi(t) \in K \quad \text{ a.e. on } \ (0, T)\}$$

of admissible displacements. We will use the following well-known bilinear forms for Kirchhoff–Love plates

$$B(W, \widetilde{W}) = \langle \sigma_{ij}(W), \varepsilon_{ij}(\widetilde{W}) \rangle,$$

$$b_{\mathcal{Q}}(w, \widetilde{w}) = \int_{\mathcal{Q}} (w_{xx}\widetilde{w}_{xx} + w_{yy}\widetilde{w}_{yy} + \kappa w_{xx}\widetilde{w}_{yy} + \kappa w_{yy}\widetilde{w}_{xx}$$

$$+ 2(1-\kappa) w_{xy}\widetilde{w}_{xy}),$$

where $\langle \cdot\,, \cdot \rangle$ corresponds to the inner product in $L_2(\Omega_\gamma)$, $\mathcal{Q}$ is a subdomain of Ω and lower indexes of functions w, $\widetilde{w}$ refer to the corresponding derivatives [2].

3. Existence of a solution.

Let us introduce the following spaces for sought functions and their components

$$\Xi = \{\theta \in L^2(0, T; H^{1,0}(\Omega_\gamma)) \,|\, \theta_t \in L^2(Q_\gamma)\}$$

equipped with the norm

$$\|\theta\|^2_{\Xi} = \|\theta\|^2_{L^2(0,T;H^{1,0}(\Omega_\gamma))} + \|\theta_t\|^2_{L^2(Q_\gamma)};$$

$$H = H^1(0, T; H(\Omega_\gamma)), \quad U = \Xi \times H.$$

We will assume that $\theta_0 \in H^{1,0}(\Omega_\gamma)$. Properties of Ξ guarantee that an arbitrary $\theta \in \Xi$ has a well-defined trace at $t = 0$; in particular, $\theta(0) \in L^2(\Omega_\gamma)$. The operation of taking a trace acts continuously from Ξ into $L^2(\Omega_\gamma)$. It is easy to show that the following set

$$S = \{(\theta, \chi) \in U \mid \quad \theta(0) = \theta_0 \text{ in } \Omega_\gamma, \quad \chi \in \mathcal{K}\}$$

is convex in U. Consider the following linear and continuous operator $L : U \to U^\star$, with values in the dual space $U^\star$ defined by the formula

$$\{L(\theta, \chi), (\bar{\theta}, \bar{\chi})\} = \int\limits_{Q_\gamma} \left(\theta_t + \delta^2 \frac{\partial}{\partial t}(\operatorname{div} W - \Delta w) \right) \bar{\theta} +$$

$$+ \int\limits_{Q_\gamma} \nabla\theta \nabla\bar{\theta} + \int\limits_0^T (B(W, \widetilde{W}) + b_{\Omega_\gamma}(w, \widetilde{w}) + \delta^2 \langle \theta, \Delta\widetilde{w} \rangle - \delta^2 \langle \theta, \operatorname{div} \widetilde{W} \rangle),$$

where bracket $\{\,\cdot\,,\,\cdot\,\}$ denotes the dual pairing between U and $U^\star$ [25].

Now we can formulate the problem under study. Assume that $f \in L^2(Q_\gamma)$. An element $(\theta, \chi) \in U$ is said to be a solution to the equilibrium problem for the thermoelastic plate with the interfacial crack on the boundary of the rigid inclusion if it satisfies the following variational inequality

$$\{L(\theta, \chi), (\bar{\theta}, \bar{\chi}) - (\theta, \chi)\} \geq \int\limits_{Q_\gamma} f(\bar{\theta} - \theta), \quad (\theta, \chi) \in S \quad \forall\, (\bar{\theta}, \bar{\chi}) \in S. \tag{7}$$

Note that L is pseudo-monotone, but non-coercive in space U [25]. The following result can be proved.

Theorem 3.1. *For δ small enough, there is a solution to problem* (7).

The proof of this statement repeats the steps of reasonings given in [25]. It is expedient to note here that the difference between the considered sets of admissible functions in [25] from K and $\mathcal{K}$ of this paper does not make a significant difference to the course of reasoning.

4. Equivalent differential statement

In this section, we derive equations for describing quasistatic equilibrium for the plate and conditions that are satisfied on γ^T for the solution (θ, χ) of (7). In order to focus on the qualitative properties of the considered model, assume that the parameter $\delta = 1$. In what follows, we will assume that the solution is sufficiently smooth. For brevity, hereafter we denote the quantities W^t, w^t, θ^t by W, w, θ, indicating each time the value of the variable t at which the corresponding relations hold. In order to apply Green's formulas, the both curves Σ and Γ should belong to the class $C^{1,1}$

Substituting into (7) test functions of the form $(\bar{\theta}, \bar{\chi})$, $\bar{\theta} = \theta + \widetilde{\theta}$, $\widetilde{\theta} \in C_0^\infty(Q_\gamma)$, $\bar{\chi} = \chi + \widetilde{\chi}$, $\widetilde{\chi} \in C_0^\infty(\Omega_\gamma \backslash \bar{\omega} \times (0, T))$, we obtain the following equalities

$$\frac{\partial\theta}{\partial t} - \Delta\theta + \frac{\partial}{\partial t}(\operatorname{div} W - \Delta w) = f \quad \text{in} \quad Q_\gamma, \tag{8}$$

$$-\sigma_{ij,j} + \theta_{,i} = 0, \quad i = 1, 2, \quad \text{in} \quad \Omega_\gamma \backslash \bar{\omega} \times (0, T), \tag{9}$$

$$\Delta^2 w + \Delta\theta = 0 \quad \text{in} \quad \Omega_\gamma \backslash \bar{\omega} \times (0, T). \tag{10}$$

Next, we need Green's formulas that are valid for sufficiently smooth functions u and v [2, 25]

$$b_\omega(u,v) = \left\langle M(u), \frac{\partial v}{\partial \nu} \right\rangle_\Sigma - \langle R(u), v\rangle_\Sigma + \langle \Delta^2 u, v\rangle_\omega. \tag{11}$$

Here, the subscripts within the brackets signify that the integration is taken over the domain ω and the boundary Σ respectively. The operators on Σ in the formula (11) are provided by relations:

$$M(u) = \kappa\Delta u + (1-\kappa)\frac{\partial^2 u}{\partial \nu^2}, \quad R(u) = \frac{\partial}{\partial \nu}\Delta u + (1-\kappa)\frac{\partial^3 u}{\partial \nu \partial \tau^2},$$

where $\tau = (-\nu_2, \nu_1)$. For functions of the form $\varphi = (\varphi_1, \varphi_2)$, the following well-known formula holds:

$$\langle \varphi, \nabla u\rangle_\omega = \langle \varphi\nu, u\rangle_\Sigma - \langle \operatorname{div}\varphi, u\rangle_\omega. \tag{12}$$

Considering (11), (12) along with the analogous formulas valid for $\Omega_\gamma \backslash \bar{\omega}$, we easily derive the following equalities which hold for the domain Ω_γ and smooth functions vanishing on the outer boundary Γ

$$\langle \varphi, \nabla u\rangle = -[\langle \varphi\nu, u\rangle_\gamma] - \langle \operatorname{div}\varphi, u\rangle_{\Omega_\gamma}, \tag{13}$$

$$\langle \sigma_{ij}(U), \varepsilon_{ij}(V)\rangle = -\langle \sigma_{ij,j}(U), v_i\rangle - \Big[\langle \sigma_\nu(U), V\nu\rangle_\gamma + \langle \sigma_\tau(U), V\tau\rangle_\gamma\Big], \tag{14}$$

where

$$\sigma_\nu(U) = \sigma_{ij}(U)\nu_i\nu_j, \quad \sigma_\tau(U) = (\sigma_\tau^1(U), \sigma_\tau^2(U)) = (\sigma_{1j}(U)\nu_j, \sigma_{2j}(U)\nu_j) - \sigma_\nu(U)\nu,$$

$$V\nu = v_i\nu_i, \quad V\tau = (V_\tau^1, V_\tau^2), \quad v_i = (V\nu)\nu_i + V_\tau^i, \quad i = 1,2;$$

$$b_{\Omega_\gamma}(u,v) = -\Big[\langle M(u), \frac{\partial v}{\partial \nu}\rangle_\gamma\Big] + \Big[\langle R(u), v\rangle_\gamma\Big] + \langle \Delta^2 u, v\rangle. \tag{15}$$

It can be easily seen that substitution of the following test functions $(\bar{\theta}, \chi)$, $(\theta, \widetilde{\chi})$ into (7) yields

$$\int\limits_{Q_\gamma} \left(\frac{\partial \theta}{\partial t} + \frac{\partial}{\partial t}(\operatorname{div} W - \Delta w) - f\right)(\bar{\theta} - \theta) + \tag{16}$$

$$+ \int\limits_{Q_\gamma} \nabla\theta(\nabla\bar{\theta} - \nabla\theta) \geq 0 \quad \forall\, (\bar{\theta}, \chi) \in S,$$

$$\int\limits_0^T \Big(B(W, \widetilde{W} - W) + b_{\Omega_\gamma}(w, \widetilde{w} - w) + \langle \theta, \Delta\widetilde{w} - \Delta w\rangle - \tag{17}$$

$$-\langle \theta, \operatorname{div}\widetilde{W} - \operatorname{div} W\rangle \Big) \geq 0, \quad \forall\, (\theta, \widetilde{\chi}) \in S.$$

Note that summing (16) and (17), we get the relation (7). Using (13) from (16) we find that

$$\int\limits_{\Omega_\gamma} \left(\frac{\partial \theta}{\partial t} + \frac{\partial}{\partial t}(\operatorname{div} W - \Delta w) - f - \triangle\theta\right)\bar{\theta} -$$

$$- \int\limits_\gamma [\frac{\partial \theta}{\partial \nu}\,\bar{\theta}\,] = 0, \quad \forall\, \bar{\theta} \in H^{1,0}(\Omega_\gamma).$$

Thanks to (8) and the arbitrariness of $\bar{\theta} \in H^{1,0}(\Omega_\gamma)$, we have

$$\frac{\partial \theta}{\partial \nu} = 0 \quad \text{on} \ \ \gamma^+, \quad \frac{\partial \theta}{\partial \nu} = 0 \quad \text{on} \ \ \gamma^-,$$

that is

$$\frac{\partial \theta}{\partial \nu} = 0 \quad \text{on} \quad \gamma. \tag{18}$$

From (17) it follows that for a.e. $t \in (0,T)$ the inequality

$$B(W, \bar{W} - W) + b_{\Omega_\gamma}(w, \bar{w} - w) + \tag{19}$$

$$+\langle \theta, \Delta \bar{w} - \Delta w \rangle - \langle \theta, \operatorname{div} \bar{W} - \operatorname{div} W \rangle \geq 0, \quad \forall\, \bar{\chi} \in K$$

holds. Substituting into the last inequality test functions of the form $\bar{\chi} = \chi \pm \widetilde{\chi}$, $\widetilde{\chi} = (\widetilde{W}, \widetilde{w})$, $\widetilde{W} \in H_0^1(\Omega)^2$, $\widetilde{w} \in H_0^2(\Omega)$, we get

$$B(W, \widetilde{W}) + b_{\Omega_\gamma}(w, \widetilde{w}) + \langle \theta, \Delta \widetilde{w} \rangle - \langle \theta, \operatorname{div} \widetilde{W} \rangle = 0. \tag{20}$$

From this, applying integration by parts, we have

$$\int\limits_{\Sigma} \big\{ - M^+(w) \frac{\partial \widetilde{l}}{\partial \nu} + R^+(w)\widetilde{l} - \sigma_\nu^+(W)\widetilde{\rho}\nu - \sigma_\tau^+(W)\widetilde{\rho}_\tau + \widetilde{\rho}\nu[\theta] - \frac{\partial \widetilde{l}}{\partial \nu}[\theta] \big\} = 0,$$

where $\widetilde{w}|_\omega = \widetilde{l}$, $\widetilde{W}|_\omega = \widetilde{\rho}$, $(\widetilde{\rho}, \widetilde{l}) \in R(\omega)$. Hence, since $\widetilde{\chi}$ is arbitrary in $H_0^1(\Omega)^2 \times H_0^2(\Omega)$, we conclude that

$$\int\limits_{\Sigma} \big\{ - M^+(w) \frac{\partial \widetilde{l}}{\partial \nu} + R^+(w)\widetilde{l} - \sigma_\nu^+(W)\widetilde{\rho}\nu - \sigma_\tau^+(W)\widetilde{\rho}_\tau + \tag{21}$$

$$+ \widetilde{\rho}\nu[\theta] - \frac{\partial \widetilde{l}}{\partial \nu}[\theta] \big\} = 0 \quad \forall (\widetilde{\rho}, \widetilde{l}) \in R(\omega).$$

We next take $(\bar{W}, w)$ satisfying

$$[\bar{W}]\nu \geq [\frac{\partial w}{\partial \nu}] \quad \text{on} \quad \gamma$$

into (19) to discover

$$B(W, \bar{W} - W) - \langle \theta, \operatorname{div} \bar{W} - \operatorname{div} W \rangle \geq 0, \quad \forall\, \bar{\chi} \in K. \tag{22}$$

Considering (22) with test functions $\bar{W} = W + \widetilde{W}$, $\widetilde{W} \in H^1(\Omega_\gamma)$, $\widetilde{W}|_\omega = 0$, $\widetilde{W}^+\nu \geq 0$ on γ, we obtain by the Green's formulas (13), (14) the following boundary conditions

$$\sigma_\nu^+(W) - \theta^+ \leq 0, \quad \sigma_\tau^+(W) = 0 \quad \text{on} \ \ \gamma. \tag{23}$$

Substituting $(W, \widetilde{w})$ into (19), we arrive at

$$b_{\Omega_\gamma}(w, \widetilde{w} - w) + \langle \theta, \Delta \widetilde{w} - \Delta w \rangle \geq 0, \tag{24}$$

which holds for all functions $\widetilde{w}$ satisfying conditions $\widetilde{w}|_\omega = a_0 + a_1 x_1 + a_2 x_2$, $x \in \omega$, where $a_0, a_1, a_2 \in \mathbb{R}$,

$$[W]\nu \geq \left[\frac{\partial \widetilde{w}}{\partial \nu}\right] \geq 0 \quad \text{on} \ \ \gamma, \quad \widetilde{w} \in H^{2,0}(\Omega_\psi).$$

In order to analyze (24), we choose test functions of the form $w + \varphi$ with smooth functions φ defined on the domain Ω_γ such that $\operatorname{supp}(\varphi) \subset \mathcal{O}^+(x)$, $\mathcal{O}(x)$ is a neighborhood of some point $x \in \gamma$, and $\mathcal{O}^+(x)$ is a subdomain of $\mathcal{O}(x)$ lying to the side γ^+, $\phi \equiv 0$ in ω, $\frac{\partial \phi^+}{\partial \nu} = 0$, $[\phi] = 0$ on γ. Then, applying (13), (15) and taking into account arbitrariness of φ, we get

$$R(w)^+ = 0 \quad \text{on} \ \ \gamma. \tag{25}$$

Next, choosing test functions $(W, w) + (\widetilde{W}, \widetilde{w})$, $(\widetilde{W}, \widetilde{w}) \in K$ into (19), we have

$$B(W, \widetilde{W}) + b_{\Omega_\gamma}(w, \widetilde{w}) + \langle \theta, \Delta \widetilde{w} \rangle - \langle \theta, \operatorname{div} \widetilde{W} \rangle \geq 0.$$

By virtue of (23), (25), the last inequality can be transformed by (13), (15) into the following relation

$$\left[\left\langle M(w) + \theta, \frac{\partial \widetilde{w}}{\partial \nu} \right\rangle_\gamma \right] + \left[\langle \sigma_\nu(W) - \theta, \widetilde{W}\nu \rangle_\gamma \right] \leq 0. \tag{26}$$

Substituting into (26) smooth functions $\widetilde{\chi} = (\widetilde{W}, \widetilde{w})$ defined in Ω_γ, $\widetilde{\chi} = 0$ in ω and having support in $\mathcal{O}^+(x)$, for an arbitrary point $x \in \gamma$, $\partial \widetilde{w}^+ / \partial \nu = \widetilde{W}^+ \nu$ on γ, we infer

$$M^+(w) + \sigma_\nu^+(W) = M^+(w) + \theta^+ + \sigma_\nu^+(W) - \theta^+ \leq 0 \quad \text{ on } \ \gamma. \tag{27}$$

Due to the smoothness of the integrands of (26) in the domain Ω_γ, it can be rewritten as

$$\left[\left\langle M(w) + \theta, \frac{\partial \widetilde{w}}{\partial \nu} \right\rangle_\Sigma \right] + \left[\langle \sigma_\nu(W) - \theta, \widetilde{W}\nu \rangle_\Sigma \right] \leq 0. \tag{28}$$

Further, we represent the last equality as follows,

$$\int\limits_\Sigma \Big\{ (M^+(w) + \theta^+)(\frac{\partial \widetilde{w}^+}{\partial \nu} - \frac{\partial \widetilde{l}}{\partial \nu}) + (\sigma_\nu^+(W) - \theta^+)(\widetilde{W}^+\nu - \widetilde{\rho}\nu) + \tag{29}$$

$$+ M^+(w) \frac{\partial \widetilde{l}}{\partial \nu} + \sigma_\nu^+(W)\widetilde{\rho}\nu + [\theta] \frac{\partial \widetilde{l}}{\partial \nu} - [\theta]\widetilde{\rho}\nu - R(w)^+ l + \sigma_\tau^+ \widetilde{\rho}\nu \Big\} \leq 0,$$

where $\widetilde{w}|_\omega = l$, $\widetilde{W}|_\omega = \widetilde{\rho}$. Since $R(w)^+ = R(w)^-$, $\sigma_\tau^+ = \sigma_\tau^-$ on $\Sigma \backslash \bar{\gamma}$, relations (23), (25) on γ^+ gives us that values $R(w)^+$, σ_τ^+ in (29) are equal to zero on Σ^+. The inequality (28) together with the equality (21) implies

$$\int\limits_\Sigma (M^+(w) + \theta^+)(\frac{\partial \widetilde{w}^+}{\partial \nu} - \frac{\partial \widetilde{l}}{\partial \nu}) + (\sigma_\nu^+(W) - \theta^+)(\widetilde{W}^+\nu - \widetilde{\rho}\nu) \leq 0 \tag{30}$$

for all $(\widetilde{W}, \widetilde{w}) \in K$. Next, we can insert the test functions of the form $(\widetilde{W}, \widetilde{w}) = 0$ $(\widetilde{W}, \widetilde{w}) = (W, w)$ into (30). As a result we have

$$\int\limits_\Sigma (M^+(w) + \theta^+)(\frac{\partial w^+}{\partial \nu} - \frac{\partial l}{\partial \nu}) + (\sigma_\nu^+(W) - \theta^+)(W^+\nu - \rho\nu) \leq 0. \tag{31}$$

Hence, in view of (2), (23), (27) and equalities $w = l$, $W = \rho$ on $\Sigma \backslash \bar{\gamma}$ each term in (31) is non-positive. Therefore, we arrive at

$$(M^+(w) + \theta^+)(\frac{\partial w^+}{\partial \nu} - \frac{\partial l}{\partial \nu}) + (\sigma_\nu^+(W) - \theta^+)(W^+\nu - \rho\nu) = 0 \quad \text{on} \quad \gamma. \tag{32}$$

Let us show that the differential statement consisting of equations (8)–(10), initial and boundary conditions (1), (5), (6), (18), (23), (25), (27), (32), the relation $\chi(t) \in R(\omega)$ for a. e. $t \in (0, T)$ provides the fulfillment of inequality (7).

Consider first smooth functions $\widetilde{\chi} = (\widetilde{W}, \widetilde{w}) \in K$. Multiply equations (9), (10), taken at a fixed $t \in (0, T)$, by $\widetilde{w}_i - w_i(t)$ and $\widetilde{w} - w(t)$, respectively. Afterwards, we transform the obtained formulas by integration over Ω_γ and application the formulas (13)–(15) along with the boundary conditions (6), (18), (23), (25) and equalities $M(w)^- = 0$, $R(w)^- = 0$, $\sigma_\nu^-(W) = 0$, $\sigma_\tau^-(W) = 0$ on Σ which hold for functions W, w having a definite linear structure in the domain ω. Further, summing the found relations, for a fixed $t \in (0,T)$, we obtain the following equality (within the framework of this section $\delta = 1$)

$$B(W, \widetilde{W} - W) + b_{\Omega_\gamma}(w, \widetilde{w} - w) + \langle \theta, \Delta\widetilde{w} - \Delta w \rangle - \tag{33}$$

$$- \langle \theta, \operatorname{div} \widetilde{W} - \operatorname{div} W \rangle + I_1 + I_2 = 0, \quad \text{where}$$

$$I_1 = -\int_\Sigma \Big\{ (M^+(w) + \theta^+)(\frac{\partial \widetilde{w}^+}{\partial \nu} - \frac{\partial \widetilde{l}}{\partial \nu}) + (\sigma_\nu^+(W) - \theta^+)(\widetilde{W}^+\nu - \widetilde{\rho}\nu) +$$

$$+ M^+(w)\frac{\partial \widetilde{l}}{\partial \nu} + \sigma_\nu^+(W)\widetilde{\rho}\nu + [\theta]\frac{\partial \widetilde{l}}{\partial \nu} - [\theta]\widetilde{\rho}\nu - R(w)^+ l + \sigma_\tau^+ \widetilde{\rho}\nu \Big\},$$

$$I_2 = \int_\Sigma \Big\{ (M^+(w) + \theta^+)(\frac{\partial w^+}{\partial \nu} - \frac{\partial l}{\partial \nu}) + (\sigma_\nu^+(W) - \theta^+)(W^+\nu - \rho\nu) +$$

$$+ M^+(w)\frac{\partial l}{\partial \nu} + \sigma_\nu^+(W)\rho\nu + [\theta]\frac{\partial l}{\partial \nu} - [\theta]\rho\nu - R(w)^+ l + \sigma_\tau^+ \rho\nu \Big\}.$$

One can note that the boundary integral I_2 is equal to zero due to (21), (32). The integral I_1, in virtue of (21), (27), (32), has non-positive value, whence inequality (19) immediately follows. This implies (17). For fixed $t \in (0,T)$, multiplying (8) by $\widetilde{\theta} - \theta(t)$ and integrating again over Ω_γ along with the formulas (13) and conditions (5), (6), (18), we get (16). At this stage, we can apply the approach used in [25], and obtain the inequality (7).

Theorem 4.1. *Assuming that the solution (θ, χ) is sufficiently smooth, the variational problem* (7) *is equivalent to the boundary value problem consisting of the equations* (8)–(10)*, the relation $\chi(t)|_\omega \in R(\omega)$ for $t \in (0,T)$, initial and boundary conditions* (1)*,* (5)*,* (6)*,* (18)*,* (21)*,* (23)*,* (25)*,* (27)*,* (32)*.*

5. Conclusion

The variational problem (7) on the equilibrium of a thermoelastic heterogeneous Kirchhoff–Love cracked plate is studied. It is assumed that the plate has a delaminated rigid inclusion. This means that there is an interfacial crack located on a part of the boundary of the rigid inclusion. In the framework of the assumption that configuration of crack's edges is known a priori, we impose nonpenetration condition in the refined form (1). Also, the presence of a bulk rigid inclusion leads to linear constraints for sought functions in the inclusion's domain ω, see (4). Solvability of the problem is proved, the equivalent differential statement is found.

6. Acknowledgement

The introduction of the paper was written by Galina Semenova with the support of the Ministry of Science and Higher Education of the Russian Federation, supplementary agreement No. 075-02-2020-1543/1, April 29, 2020.

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