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A NOTE ON THE INTERACTIONS OF NONLINEAR WAVES GOVERNED BY THE GENERALIZED BOUSSINESQ EQUATION

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Abstract. In this work, based on a one dimensional model, the interaction of two solitary waves propagating in opposite directions in a fluid whose equations are governed by the generalized Boussinesq equation, by use of the Poincaré-Lighthill-Kuo (PLK) method. It is shown that bi-directional solitary waves are propagated, and the head-on collision of these two solitons occur. The phase shifts and the trajectories of these two solitons after the collisions are obtained.

Keywords: Head-on Collision, Generalized Boussinesq Equation, Solitary Waves.

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1. Introduction

It is well-known that one of the striking features of solitons is their asymptotic preservation of form when they undergo a collision, as first remarked by Zabusky and Kruskal [10]. The unique effect due to the collision is their phase shift. In a one dimensional system, there are two distinct soliton interactions. The first one is the overtaking collision, and the other is the head-on collision [7]. Because of the multisoliton solutions of the Korteweg-deVries (KdV) equation, the overtaking collision of solitary waves can be studied by the inverse scattering transformation method [2] and Zou and Su [11]. For the numerical analysis of overtaking collisions of solitary waves, it is worth mentioning the works by Li and Sattinger [5] and Haragus et al [3]. However, for a head of collision between two solitary waves, we first examine two solitary waves propagating in opposite directions, and hence we need to employ a suitable asymptotic expansion to solve the original governing equations. Using the extended Poincaré-Lighthill-Kuo(PLK) method, Mirie and Su [7], and Su and Mirie [8], studied the head-on collision of solitary waves in a shallow water theory. Huang and Velarde [4] studied the head-on collision of two concentric cylindrical ion-acoustic waves and obtained the phase shifts of the right and left going waves. In this context, it is worth of mentioning the works by Xue [9] on head -on collision of blood solitary waves, and Demiray [1] on head on collision in solitary waves in fluid-filled elastic tubes.

In the present work, based on a one dimensional model, the interaction of two solitary waves propagating in opposite directions in a fluid whose equations of motion is governed by the generalized Boussinesq equation, is investigated by use of the extended Poincaré-Lighthill-Kuo (PLK) method [2, 4, 7]. It is shown that bi-directional solitary waves are propagated, and the head-on collision of these two waves occur. The phase shifts and trajectories of these two solitons after the collision are obtained.

2. Head-on collision of waves.

In this section, we shall study the interaction of two nonlinear acoustical waves governed by the generalized Boussinesq equation

$$u_{tt} - u_{xx} + u_{xxxx} - \frac{1}{n+1}(u^{n+1})_{xx} = 0, \tag{1}$$

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where $n \geq 1$ is a positive integer, and u is the velocity in the x direction. The dispersion relation of linearized form of the equation (1) may be given by

$$\omega = k(1+k^2)^{1/2}, \tag{2}$$

where ω is the angular frequency and k is the wave number. To study the collision of solitary waves for equation (1), it is convenient to introduce the following stretched coordinates

$$\epsilon^{n/2}(x-t) = \xi + \epsilon^n P_0(\eta,\tau) + \epsilon^{2n} P_1(\xi,\eta,\tau) + ...,$$
$$\epsilon^{n/2}(x+t) = \eta + \epsilon^n Q_0(\xi,\tau) + \epsilon^{2n} Q_1(\xi,\eta,\tau) + ..., \tag{3}$$

where $P_0, Q_0, P_1, Q_1, ...$ are some unknown functions to be determined from the solution. From equation (3), the following operators may be defined

$$\frac{\partial}{\partial x} = \epsilon^{n/2}[(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}) - \epsilon^n(\frac{\partial P_0}{\partial \eta}\frac{\partial}{\partial \xi} + \frac{\partial Q_0}{\partial \xi}\frac{\partial}{\partial \eta}) + ...],$$
$$\frac{\partial}{\partial t} = \epsilon^{n/2}[(-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}) + \epsilon^n(\frac{\partial}{\partial \tau} - \frac{\partial P_0}{\partial \eta}\frac{\partial}{\partial \xi} + \frac{\partial Q_0}{\partial \xi}\frac{\partial}{\partial \eta}) + ...] \quad . \tag{4}$$

We further assume that the field quantity u can be expanded into the asymptotic series in ϵ as:

$$u = \epsilon u_1 + \epsilon^{n+1} u_2 + \epsilon^{2n+1} u_3 + ... \tag{5}$$

where $u_1, u_2, u_3, ...$ are functions of the variables ξ, η, τ. Introducing the expansions (4) and (5) into equation (1), and setting the coefficients of alike powers of ϵ equal to zero, the following differential equations are obtained:

$O(\epsilon)$ equation:

$$-4\frac{\partial^2 u_1}{\partial \xi \partial \eta} = 0. \tag{6}$$

$O(\epsilon^{n+1})$ equation:

$$-4\frac{\partial^2 u_2}{\partial \xi \partial \eta} - 2\frac{\partial^2 u_1}{\partial \xi \partial \tau} + 2\frac{\partial^2 u_1}{\partial \eta \partial \tau} + 4\frac{\partial P_0}{\partial \eta}\frac{\partial^2 u_1}{\partial \xi^2} + 4\frac{\partial Q_0}{\partial \xi}\frac{\partial^2 u_1}{\partial \eta^2} + \frac{\partial^4 u_1}{\partial \xi^4} + \frac{\partial^4 u_1}{\partial \eta^4} +$$
$$+4\frac{\partial^4 u_1}{\partial \xi^3 \partial \eta} + 6\frac{\partial^4 u_1}{\partial \xi^2 \partial \eta^2} + 4\frac{\partial^4 u_1}{\partial \xi \partial \eta^3} - \frac{1}{n+1}(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + 2\frac{\partial^2}{\partial \xi \partial \eta})(u_1^{n+1}) = 0. \tag{7}$$

2.1. **Solution of the field equations.** From the solution of equation (6), one obtains

$$u_1 = U(\xi,\tau) + V(\eta,\tau), \tag{8}$$

where $U(\xi,\tau)$ and $V(\eta,\tau)$ are two unknown functions of their arguments whose governing equations will be obtained from the higher order perturbation expansion.

Inserting (8) into equation (7), we have

$$-4\frac{\partial^2 u_2}{\partial \xi \partial \eta} + \frac{\partial}{\partial \xi}[-2\frac{\partial U}{\partial \tau} - U^n\frac{\partial U}{\partial \xi} + \frac{\partial^3 U}{\partial \xi^3}] + \frac{\partial}{\partial \eta}[2\frac{\partial V}{\partial \tau} - V^n\frac{\partial V}{\partial \eta} + \frac{\partial^3 V}{\partial \eta^3}] +$$
$$+\frac{\partial^2}{\partial \xi \partial \eta}[4M(\eta)\frac{\partial U}{\partial \xi} + 4N(\xi)\frac{\partial V}{\partial \eta} - \frac{2}{n+1}(U+V)^{n+1} - \frac{1}{n+1}\sum_{k=2}^{n}\binom{n+1}{k} \times$$
$$\times[\int^{\eta} V^{n+1-k}(\eta')d\eta'\frac{\partial(U^k)}{\partial \xi} + \int^{\xi} U^{n+1-k}(\xi')d\xi'\frac{\partial(V^k)}{\partial \eta}] = 0, \tag{9}$$

where $\begin{pmatrix} m \\ n \end{pmatrix}$ is the Binomial coefficient, $M(\eta)$ and $N(\xi)$ are defined by

$$M(\eta) = P_0 - \frac{1}{4}\int^{\eta} V^n(\eta')d\eta', \quad N(\xi) = Q_0 - \frac{1}{4}\int^{\xi} U^n(\xi')d\xi'. \tag{10}$$

Integrating the equation (9) with respect to ξ and η we obtain

$$u_2 = F(\xi,\tau) + G(\eta,\tau) - \frac{1}{2}(\frac{\partial U}{\partial \tau} + \frac{1}{2}U^n\frac{\partial U}{\partial \xi} - \frac{1}{2}\frac{\partial^3 U}{\partial \xi^3})\eta + \frac{1}{2}(\frac{\partial V}{\partial \tau} - \frac{1}{2}V^n\frac{\partial V}{\partial \eta} + \frac{1}{2}\frac{\partial^3 V}{\partial \eta^3})\xi +$$
$$+M(\eta)\frac{\partial U}{\partial \xi} + N(\xi)\frac{\partial V}{\partial \eta} - \frac{1}{2(n+1)}(U+V)^{n+1} - \frac{1}{4(n+1)}\sum_{k=2}^{n}\begin{pmatrix} n+1 \\ k \end{pmatrix} \times$$
$$\times[\int^{\eta} V^{n+1-k}(\eta')d\eta'\frac{\partial(U^k)}{\partial \xi} + \int^{\xi} U^{n+1-k}(\xi')d\xi'\frac{\partial(V^k)}{\partial \eta}], \tag{11}$$

where $F(\xi,\tau)$ and $G(\eta,\tau)$ correspond to the homogeneous solution of the differential equation (7).

As is seen from equation (11), u_2 approaches to infinity as $(\xi,\eta) \to \pm\infty$. Therefore, in order to remove the secularity, the coefficients of ξ and η in equation (11) must vanish, i. e.,

$$\frac{\partial U}{\partial \tau} + \frac{1}{2}U^n\frac{\partial U}{\partial \xi} - \frac{1}{2}\frac{\partial^3 U}{\partial \xi^3} = 0, \tag{12}$$

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}V^n\frac{\partial V}{\partial \eta} + \frac{1}{2}\frac{\partial^3 V}{\partial \eta^3} = 0, \tag{13}$$

These evolution equations are two generalized Korteweg-deVries equations.

If the evolution equation for the higher order term, say for $F(\xi,\tau)$, is studied, the resulting governing equation will be of the following form

$$\frac{\partial F}{\partial \tau} + \frac{1}{2}\frac{\partial}{\partial \xi}(U^n F) - \frac{1}{2}\frac{\partial^3 F}{\partial \xi^3} = S_1(U). \tag{14}$$

This is the degenerate(linearized) generalized KdV equation with non-homogeneous term $S_1(U)$ (see, for instance [1]). As is well-known, $\partial U/\partial \xi$ is the solution of the homogeneous equation obtained from (14). Therefore, id $S_1(U)$ contains a term proportional to $\partial U/\partial \xi$, and it causes the secularity in the solution. Based on this explanation, although the terms in equation (11) proportional to $\partial U/\partial \xi$ and $\partial V/\partial \eta$, do not cause any secularity at this order, it might cause to secularity at higher order solutions. In order to remove such secularities, the coefficients $M(\eta)$ and $N(\xi)$ must vanish, which yields

$$P_0 - \frac{1}{4}\int^{\eta} V^n(\eta')d\eta' = 0, \quad Q_0 - \frac{1}{4}\int^{\xi} U^n(\xi')d\xi' = 0. \tag{15}$$

These conditions make it possible to determine the unknown functions $P_0(\eta,\tau)$ and $Q_0(\xi,\tau)$. Then, the final form of the solution for u_2 may be given by

$$u_2 = F(\xi,\tau) + G(\eta,\tau) - \frac{1}{2(n+1)}(U+V)^{n+1} - \frac{1}{4(n+1)}\sum_{k=2}^{n}\begin{pmatrix} n+1 \\ k \end{pmatrix} \times$$
$$\times[\int^{\eta} V^{n+1-k}(\eta')d\eta'\frac{\partial(U^k)}{\partial \xi} + \int^{\xi} U^{n+1-k}(\xi')d\xi'\frac{\partial(V^k)}{\partial \eta}]. \tag{16}$$

As is well-known, the generalized KdV equations given in (12) and (13) assume the solitary wave solution when n is an odd integer, whereas they have a periodic solution when n is an even integer. Therefore, we shall be concerned with the solution when $n = 2m + 1$, where m is a positive integer. In this case, equations (12) and (13) have the following solitary wave solution

$$U = -U_A \mathrm{sech}^{2/2m+1}\zeta_A, \quad \zeta_A = \alpha(\xi + v_A\tau)$$
$$\alpha^2 = \frac{(U_A)^{2m+1}(2m+1)^2}{4(m+1)(2m+3)}, \quad v_A = \frac{(U_A)^{2m+1}}{2(m+1)(2m+3)}, \tag{17}$$

$$V = -U_B \mathrm{sech}^{2/2m+1}\zeta_B, \quad \zeta_B = \beta(\eta + v_B\tau)$$
$$\beta^2 = \frac{(U_B)^{2m+1}(2m+1)^2}{4(m+1)(2m+3)}, \quad v_B = \frac{(U_B)^{2m+1}}{2(m+1)(2m+3)}, \tag{18}$$

where U_A and U_B are the amplitudes of the corresponding solitary waves.

Thus, from equation (15) we obtain the functions P_0 and Q_0 as

$$P_0 = -\frac{(U_B)^{m+1/2}[(m+1)(2m+3)]^{1/2}}{2(2m+1)}\tanh\zeta_B, \tag{19}$$

$$Q_0 = -\frac{(U_A)^{m+1/2}[(m+1)(2m+3)]^{1/2}}{2(2m+1)}\tanh\zeta_A. \tag{20}$$

Hence, up to $O(\epsilon^2)$, the trajectories of two solitary waves for weak head-on collisions are

$$\epsilon^{m+1/2}(x-t) = \xi - \epsilon^{2m+1}\frac{(U_B)^{m+1/2}[(m+1)(2m+3)]^{1/2}}{2(2m+1)}\tanh\zeta_B$$
$$+O(\epsilon^{4m+2}), \quad \epsilon^{m+1/2}(x+t) = \eta$$
$$-\epsilon^{2m+1}\frac{(U_A)^{m+1/2}[(m+1)(2m+3)]^{1/2}}{2(2m+1)}\tanh\zeta_A + O(\epsilon^{4m+2}. \tag{21}$$

To obtain the phase shift after head-on collision of two solitary waves, we shall assume that the solitary waves characterized by U_A and U_B are asymptotically far from each other at the initial time (t=$-\infty$), the solitary wave U_A is at $\xi = 0,\ \eta = -\infty$, and the solitary wave U_B is at $\eta = 0,\ \ \xi = +\infty$, respectively. After the collision (t= $+\infty$), the solitary wave U_B is far to the right of the solitary wave U_A, i.e., the solitary wave U_B is at $\xi = 0,\ \eta = +\infty$ and the solitary wave U_A is at $\eta = 0,\ \xi = -\infty$. Following Su and Mirie [8] and Xue [9], and using the equation (21), the corresponding phase shifts Δ^+ and Δ^- may be obtained as follows.

$$\Delta^+ = \epsilon^{m+1/2}(x-t)|_{\xi=0,\eta=\infty} - \epsilon^{m+1/2}(x-t)|_{\xi=0,\eta=-\infty} =$$
$$= -\epsilon^{2m+1}(U_B)^{m+1/2}\frac{[m+1)(2m+3)]^{1/2}}{(2m+1)}$$
$$\Delta^- = \epsilon^{m+1/2}(x+t)|_{\eta=0,\xi=-\infty} - \epsilon^{m+1/2}(x+t)_{\eta=0,\xi=\infty} =$$
$$= \epsilon^{2m+1}(U_A)^{m+1/2}\frac{[m+1)(2m+3)]^{1/2}}{(2m+1)}. \tag{22}$$

3. Conclusion

By use of the extended PLK method, head-on collisions of two solitary waves propagating in a fluid whose governing equations are characterized by the generalized Boussinesq equation is examined. The result shows that, up to $O(\epsilon^{4m+2})$, the head-on collision of two solitary waves is elastic, and the solitons preserve their original properties after the collision. The leading order analytical phase shifts of head-on collision between two solitary waves are derived explicitly. The higher order corrections may give additional information about the structure of the collision event. This is especially true for large amplitudes.

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